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ON A MIXED PERSPECTIVE FOR OPERATOR CONVEX FUNCTIONS IN HILBERT SPACES

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Abstract. In this paper we establish some operator superadditivity and monotonicity properties for mappings associated to mixed perspectives of operator concave or operator convex functions. Applications for power functions and logarithm are also provided.

1. Introduction

Let Φ be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_{\Phi}(B,A)$ by setting

$$\mathcal{P}_{\Phi}(B,A):=A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_{\Phi}(B,A)=A\Phi\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset \mathring{I}$.

It is well known that (see [5] and [4] or [6]), if Φ is an *operator convex (concave) function* defined in the positive half-line $(0,\infty)$, then the mapping

$$(B,A)\mapsto \mathcal{P}_{\Phi}(B,A)$$

defined in pairs of positive definite operators, is operator convex (concave), namely we have

$$\mathcal{P}_{\Phi}(\nu B+(1-\nu)D,\nu A+(1-\nu)C)\leq(\geq)\,\nu\mathcal{P}_{\Phi}(B,A)+(1-\nu)\mathcal{P}_{\Phi}(D,C) \tag{1.1}$$

for any positive invertible operators A, B, C, D and $\nu\in[0,1]$.

In the recent paper [1] we established the following reverse inequality for the perspective $\mathcal{P}_{\Phi}(B,A)$.

Let $\Phi:[m,M]\to\mathbb{R}$ be a *convex function* on the real interval $[m,M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA\leq B\leq MA, \tag{1.2}$$

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then we have

$$0 \leq \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A) \tag{1.3}$$
$$\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right)$$
$$\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]A.$$

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq \Phi''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.4}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.2) is valid with $[m,M] \subset \mathring{J}$, then we have the following result as well [2]

$$\frac{1}{2}d\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right) \tag{1.5}$$
$$\leq \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A)$$
$$\leq \frac{1}{2}D\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right).$$

If $d>0$, then the first inequality in (1.5) is better than the same inequality in (1.3).

In the recent paper [3] we showed that, for Φ an operator concave function defined in the positive half-line and for any positive invertible operators A, B, C, D we have

$$\mathcal{P}_{\Phi}(B+D,A+C) \geq \mathcal{P}_{\Phi}(B,A)+\mathcal{P}_{\Phi}(D,C), \tag{1.6}$$

i.e. $\mathcal{P}_{\Phi}$ is operator superadditive as a function of *pairs of positive operators.* In addition, if Φ is nonnegative in the positive half-line and $A>C$ and $B>D$, then

$$\mathcal{P}_{\Phi}(B,A) \geq \mathcal{P}_{\Phi}(D,C), \tag{1.7}$$

i.e. $\mathcal{P}_{\Phi}$ is operator monotonic nondecreasing as a function of *pairs of positive operators.*

Moreover, for any positive invertible operators A, B, C, D such that

$$K(C,D) \geq (A,B) \geq k(C,D) \tag{1.8}$$

namely, $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants k, K, we have

$$K\mathcal{P}_{\Phi}(D,C) \geq \mathcal{P}_{\Phi}(B,A) \geq k\mathcal{P}_{\Phi}(D,C). \tag{1.9}$$

Motivated by the above results, in this paper we introduce a mixed perspective associated to a given vector $x \in H$, $\|x\|=1$, an operator convex function and two invertible positive operators and investigate its main properties. As examples, some vector inequalities and norm inequalities for power and logarithmic functions are also provided.

2. The Main Results

We define, for the continuos function $\Phi : (0,\infty) \to \mathbb{R}$, for the positive operator B and the positive invertible operator A and the vector $x \in H$, $\|x\| = 1$ the following mixed perspective

$$\mathcal{S}_\Phi (B, A)(x) := A^{1/2}\Phi\left(\langle Bx, x\rangle A^{-1}\right) A^{1/2}.$$

Theorem 1. *Let $\Phi : (0,\infty) \to \mathbb{R}$ be an operator convex (concave) function. Then for any $x \in H$, $\|x\| = 1$ the function $\mathcal{S}_\Phi(\cdot,\cdot)(x)$ is jointly operator convex (concave).*

Proof. Let $B_1,\ B_2 > 0$ and the positive invertible operators $A_1,\ A_2$. For $\lambda \in [0,1]$ let put

$$B := \lambda B_1 + (1-\lambda) B_2,\ \ A = \lambda A_1 + (1-\lambda) A_2.$$

Consider the operators

$$X = (\lambda A_1)^{1/2} A^{-1/2} \text{ and } Y = ((1-\lambda)A_2)^{1/2} A^{-1/2}.$$

These satisfy the equality

$$\begin{aligned} X^*X + Y^*Y &= A^{-1/2}\lambda A_1 A^{-1/2} + A^{-1/2}(1-\lambda)A_2A^{-1/2} \\ &= A^{-1/2}\left(\lambda A_1 + (1-\lambda) A_2\right) A^{-1/2} = 1. \end{aligned}$$

Observe that

$$\begin{aligned} &X^*\left(\langle B_1x, x\rangle A_1^{-1}\right) X + Y^*\left(\langle B_2x, x\rangle A_2^{-1}\right) Y \\ &= A^{-1/2}(\lambda A_1)^{1/2}\left(\langle B_1x, x\rangle A_1^{-1}\right)(\lambda A_1)^{1/2}A^{-1/2} \\ &+ A^{-1/2}((1-\lambda)A_2)^{1/2}\left(\langle B_2x, x\rangle A_2^{-1}\right)((1-\lambda)A_2)^{1/2}A^{-1/2} \\ &= \langle B_1x, x\rangle A^{-1/2}(\lambda A_1)^{1/2}A_1^{-1}(\lambda A_1)^{1/2}A^{-1/2} \\ &+ \langle B_2x, x\rangle A^{-1/2}((1-\lambda)A_2)^{1/2}A_2^{-1}((1-\lambda)A_2)^{1/2}A^{-1/2} \\ &= \lambda\langle B_1x, x\rangle A^{-1} + (1-\lambda)\langle B_2x, x\rangle A^{-1} = \langle Bx, x\rangle A^{-1} \end{aligned}$$

for any $x \in H$, $\|x\| = 1$ and $\lambda \in [0,1]$.

By Hansen-Pedersen inequality [7] for operator convex functions, we have

$$\begin{aligned} &\mathcal{S}_\Phi\left(\lambda B_1 + (1-\lambda) B_2, \lambda A_1 + (1-\lambda) A_2\right)(x) \\ &= \mathcal{S}_\Phi(B, A)(x) = A^{1/2}\Phi\left(\langle Bx, x\rangle A^{-1}\right) A^{1/2} \\ &= A^{1/2}\Phi\left(X^*\left(\langle B_1x, x\rangle A_1^{-1}\right) X + Y^*\left(\langle B_2x, x\rangle A_2^{-1}\right) Y\right) A^{1/2} \\ &\le A^{1/2}\left[X^*\Phi\left(\langle B_1x, x\rangle A_1^{-1}\right) X + Y^*\Phi\left(\langle B_2x, x\rangle A_2^{-1}\right) Y\right] A^{1/2} \\ &= A^{1/2}\Big[A^{-1/2}(\lambda A_1)^{1/2}\Phi\left(\langle B_1x, x\rangle A_1^{-1}\right)(\lambda A_1)^{1/2}A^{-1/2} \\ &+ A^{-1/2}((1-\lambda)A_2)^{1/2}\Phi\left(\langle B_2x, x\rangle A_2^{-1}\right)((1-\lambda)A_2)^{1/2}A^{-1/2}\Big] A^{1/2} \\ &= \lambda A_1^{1/2}\Phi\left(\langle B_1x, x\rangle A_1^{-1}\right) A_1^{1/2} + (1-\lambda)A_2^{1/2}\Phi\left(\langle B_2x, x\rangle A_2^{-1}\right) A_2^{1/2} \\ &= \lambda\mathcal{S}_\Phi(B_1, A_1)(x) + (1-\lambda)\mathcal{S}_\Phi(B_2, A_2)(x) \end{aligned}$$

for any $x \in H$, $\|x\| = 1$ and $\lambda \in [0,1]$.

The case of operator concave functions goes likewise and the proof is completed. □

Corollary 1. *Let Φ be an operator convex (concave) function defined in the positive half-line, then for any $x \in H, \|x\| = 1$ and any positive invertible operators A, B, C, D we have*

$$\mathcal{S}_\Phi (B + D, A + C)(x) \leq (\geq) \mathcal{S}_\Phi (B, A)(x) + \mathcal{S}_\Phi (D, C)(x), \tag{2.1}$$

i.e. $\mathcal{S}_\Phi (\cdot, \cdot)(x)$ is operator subadditive (superadditive) as a function of pairs of positive operators.

In addition, if Φ is nonnegative and operator concave in the positive half-line and $A > C$ and $B > D$, then

$$\mathcal{S}_\Phi (B, A)(x) \geq \mathcal{S}_\Phi (D, C)(x), \tag{2.2}$$

i.e. $\mathcal{S}_\Phi (\cdot, \cdot)(x)$ is operator monotonic nondecreasing as a function of pairs of positive operators.

Proof. Let $x \in H$ with $\|x\| = 1$. First, we observe that $\mathcal{S}_\Phi (\cdot, \cdot)(x)$ is positive homogeneous as a function of pairs of positive operators, namely

$$\mathcal{S}_\Phi (\alpha B, \alpha A)(x) := \alpha \mathcal{S}_\Phi (B, A)(x)$$

for any $\alpha > 0$ and any pair of positive operators (B, A).

By Theorem 1 we have for any positive invertible operators A, B, C, D that

$$\begin{aligned}
\mathcal{S}_\Phi (B + D, A + C)(x) &= \mathcal{S}_\Phi \left(2\frac{B + D}{2}, 2\frac{A + C}{2} \right)(x) \\
&= 2\mathcal{S}_\Phi \left(\frac{B + D}{2}, \frac{A + C}{2} \right)(x) \\
&\leq (\geq) 2\frac{\mathcal{S}_\Phi (B, A)(x) + \mathcal{S}_\Phi (D, C)(x)}{2} \\
&= \mathcal{S}_\Phi (B, A)(x) + \mathcal{S}_\Phi (D, C)(x)
\end{aligned}$$

and the inequality (2.1) is proved.

If $A > C > 0$ and $B > D > 0$ and Φ is operator concave, then by (2.1) we have

$$\begin{aligned}
\mathcal{S}_\Phi (B, A)(x) &= \mathcal{S}_\Phi (B - D + D, A - C + C)(x) \\
&\geq \mathcal{S}_\Phi (B - D, A - C)(x) + \mathcal{S}_\Phi (D, C)(x)
\end{aligned}$$

giving that

$$\mathcal{S}_\Phi (B, A)(x) - \mathcal{S}_\Phi (D, C)(x) \geq \mathcal{S}_\Phi (B - D, A - C)(x).$$

Since Φ is positive and $A > C$ and $B > D$ then

$$\mathcal{S}_\Phi (B - D, A - C)(x) \geq 0$$

and the inequality (2.2) is proved. □

Corollary 2. *Let Φ be a nonnegative and operator concave function defined in the positive half-line then for any positive invertible operators A, B, C, D such that*

$$K(C, D) \geq (A, B) \geq k(C, D) \tag{2.3}$$

namely, $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants k, K. Then for any $x \in H, \|x\| = 1$ we have

$$K\mathcal{S}_\Phi (D, C)(x) \geq \mathcal{S}_\Phi (B, A)(x) \geq k\mathcal{S}_\Phi (D, C)(x). \tag{2.4}$$

Proof. We have by (2.2) that

$$K\mathcal{S}_{\Phi}(D,C)(x)=\mathcal{S}_{\Phi}(KD,KC)(x)\geq\mathcal{S}_{\Phi}(B,A)(x),$$

which proves the first inequality in (2.4).

The second inequality goes likewise and the corollary is proved. □

If $0<m_1I\leq A\leq M_1I$ and $m_2I\leq B\leq M_2I$ for some real $m_2<M_2$ then obviously

$$m_2\leq\langle Bx,x\rangle\leq M_2$$

for any $\|x\|=1$ and

$$M_1^{-1}I\leq A^{-1}\leq m_1^{-1}I,$$

which implies that

$$m_2M_1^{-1}I\leq\langle Bx,x\rangle A^{-1}\leq M_2m_1^{-1}I,$$

for any $\|x\|=1$.

That is equivalent to

$$m_2M_1^{-1}A\leq\langle Bx,x\rangle I\leq M_2m_1^{-1}A,$$

for any $\|x\|=1$.

Motivated by these observations we can introduce the following set of pairs of operators.

Let $x\in H$ with $\|x\|=1$. Now, if we have m, M with $M>m$ then we can define $\mathcal{G}(m,M)(x)$ as the following set of pairs of operators (B,A) such that A is positive invertible, B is selfadjoint and satisfy

$$mA\leq\langle Bx,x\rangle I\leq MA. \tag{2.5}$$

We observe that, if (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$ then also $(B,A)+(D,C)$, $\alpha(B,A)$ and $\nu(B,A)+(1-\nu)(D,C)\in\mathcal{G}(m,M)(x)$ for any $\alpha>0$ and $\nu\in[0,1]$ meaning that $\mathcal{G}(m,M)(x)$ is a *cone* and, *a fortiori* a convex set for any $x\in H$ with $\|x\|=1$.

We observe that the condition (2.5) is equivalent to

$$mI\leq\langle Bx,x\rangle A^{-1}\leq MI,$$

any $x\in H$ with $\|x\|=1$.

We also notice that if $0<m_1I\leq A\leq M_1I$ and $m_2I\leq B\leq M_2I$ for some real $m_2<M_2$ then we can take in (2.5) $m=m_2M_1^{-1}$ and $M=M_2m_1^{-1}$.

Remark 1. *It is obvious that if $\Phi:[m,M]\to\mathbb{R}$ is an operator convex (concave) function, then for any $x\in H,\|x\|=1$ the function $\mathcal{S}_{\Phi}(\cdot,\cdot)(x)$ is jointly operator convex (concave) on $\mathcal{G}(m,M)(x)$. If Φ is operator concave and nonnegative on $[m,M]$, then $\mathcal{S}_{\Phi}(\cdot,\cdot)(x)$ is operator superadditive and monotonic nondecreasing on $\mathcal{G}(m,M)(x)$.*

Let $\Phi:[m,M]\to\mathbb{R}$ be a continuous *convex function* on the real interval $[m,M]$. Then for any $t\in[m,M]$ we have by the convexity of Φ that

$$\frac{(M-t)\Phi(m)+(t-m)\Phi(M)}{M-m}\geq\Phi(t). \tag{2.6}$$

Let $x \in H$ with $\|x\| = 1$. If A is positive invertible, B is selfadjoint and satisfy (2.5) then by (2.6) we have

$$\begin{aligned} &\frac{\Phi(m)\left(MI - \langle Bx, x\rangle A^{-1}\right) + \Phi(M)\left(\langle Bx, x\rangle A^{-1} - mI\right)}{M - m} \\ &\geq \Phi\left(\langle Bx, x\rangle A^{-1}\right) \end{aligned} \tag{2.7}$$

and by multiplying both sides of (2.7) by $A^{1/2}$ we get

$$\frac{\Phi(m)(MA - \langle Bx, x\rangle I) + (\Phi(M)\langle Bx, x\rangle I - mA)}{M - m} \geq \mathcal{S}_\Phi(B, A)(x) \tag{2.8}$$

for any $(B, A) \in \mathcal{G}(m, M)(x)$.

We define the mapping $\mathcal{F}_\Phi(\cdot, \cdot)(x)$ on $\mathcal{G}(m, M)(x)$ by

$$\begin{aligned} \mathcal{F}_\Phi(B, A)(x) := &\frac{\Phi(m)(MA - \langle Bx, x\rangle I) + \Phi(M)(\langle Bx, x\rangle I - mA)}{M - m} \\ &- \mathcal{S}_\Phi(B, A)(x). \end{aligned}$$

From the above considerations, we see that, if $\Phi : [m, M] \to \mathbb{R}$ is a continuous *convex function* on the real interval $[m, M]$, then for $x \in H$ with $\|x\| = 1$ we have

$$\mathcal{F}_\Phi(B, A)(x) \geq 0 \tag{2.9}$$

for any $(B, A) \in \mathcal{G}(m, M)(x)$.

We have:

Theorem 2. *Let Φ be an operator convex function defined on the interval $[m, M]$. Then the mapping $\mathcal{F}_\Phi(\cdot, \cdot)(x)$ is nonnegative, positive homogeneous, operator concave, operator superadditive and operator monotonic nondecreasing on $\mathcal{G}(m, M)(x)$.*

Proof. If Φ is an operator convex function defined on the interval $[m, M]$, then it is convex and by (2.9) we have that $\mathcal{F}_\Phi(\cdot, \cdot)(x)$ is nonnegative on $\mathcal{G}(m, M)(x)$ in the operator order.

If $\alpha > 0$ and $(B, A) \in \mathcal{G}(m, M)(x)$ then

$$\begin{aligned} &\mathcal{F}_\Phi(\alpha B, \alpha A)(x) \\ &= \frac{\Phi(m)(M\alpha A - \alpha\langle Bx, x\rangle I) + \Phi(M)(\alpha\langle Bx, x\rangle I - m\alpha A)}{M - m} - \mathcal{S}_\Phi(\alpha B, \alpha A)(x) \\ &= \frac{\alpha}{M - m}[\Phi(m)(MA - \langle Bx, x\rangle I) + \Phi(M)(\langle Bx, x\rangle I - mA)] - \alpha\mathcal{S}_\Phi(B, A)(x) \\ &= \alpha\mathcal{F}_\Phi(B, A)(x) \end{aligned}$$

that proves the positive homogeneity of $\mathcal{F}_\Phi(\cdot, \cdot)(x)$.

Let (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$ and $\nu\in[0,1]$. Then we have

$$\begin{aligned}
&\mathcal{F}_{\Phi}((\nu B+(1-\nu)D,\nu A+(1-\nu)C))(x)\\
&=\frac{1}{M-m}[\Phi(m)(M(\nu A+(1-\nu)C)-\langle(\nu B+(1-\nu)D)x,x\rangle I)\\
&+\Phi(M)(\langle(\nu B+(1-\nu)D)x,x\rangle I-m(\nu A+(1-\nu)C))]\\
&-\mathcal{S}_{\Phi}((\nu B+(1-\nu)D,\nu A+(1-\nu)C))(x)\\
&=\frac{\nu}{M-m}[\Phi(m)(MA-\langle Bx,x\rangle I)+\Phi(M)(\langle Bx,x\rangle I-mA)]\\
&+\frac{1-\nu}{M-m}[\Phi(m)(MC-\langle Dx,x\rangle I)+\Phi(M)(\langle Dx,x\rangle I-mC)]\\
&-\mathcal{S}_{\Phi}((\nu B+(1-\nu)D,\nu A+(1-\nu)C))(x)\\
&=:U(x).
\end{aligned}$$

By Remark 1 we also have that

$$-\mathcal{S}_{\Phi}((\nu B+(1-\nu)D,\nu A+(1-\nu)C))(x)\geq-\nu\mathcal{S}_{\Phi}(B,A)(x)-(1-\nu)\mathcal{S}_{\Phi}(D,C)(x)$$

for any (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$ and $\nu\in[0,1]$.

Therefore

$$\begin{aligned}
U(x)&\geq\frac{\nu}{M-m}[\Phi(m)(MA-\langle Bx,x\rangle I)+\Phi(M)(\langle Bx,x\rangle I-mA)]\\
&-\nu\mathcal{S}_{\Phi}(B,A)(x)\\
&+\frac{1-\nu}{M-m}[\Phi(m)(MC-\langle Dx,x\rangle I)+\Phi(M)(\langle Dx,x\rangle I-mC)]\\
&-(1-\nu)\mathcal{S}_{\Phi}(D,C)(x)\\
&=\nu\mathcal{F}_{\Phi}(B,A)(x)+(1-\nu)\mathcal{F}_{\Phi}(D,C)(x)
\end{aligned}$$

for any (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$ and $\nu\in[0,1]$, showing that $\mathcal{F}_{\Phi}(\cdot,\cdot)(x)$ is operator concave on $\mathcal{G}(m,M)(x)$.

If (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$, then by the above properties we have

$$\mathcal{F}_{\Phi}(B+D,A+C)(x)\geq\mathcal{F}_{\Phi}(B,A)(x)+\mathcal{F}_{\Phi}(D,C)(x),$$

which proves the operator superadditivity of $\mathcal{F}_{\Phi}(\cdot,\cdot)(x)$ on $\mathcal{G}(m,M)(x)$.

The operator monotonicity of $\mathcal{F}_{\Phi}(\cdot,\cdot)(x)$ follows in a similar way as in the proof of Corollary 1 and the details are omitted. □

Corollary 3. *Let Φ be an operator convex function defined on the interval $[m,M]$ and (B,A), $(D,C)\in\mathcal{G}(m,M)(x)$. If there exists some positive constants k, K such that $KC\geq A\geq kC$ and $KD\geq B\geq kD$, then we have the inequalities*

(2.10)
$$\begin{aligned}
&K\left\{\frac{\Phi(m)(MC-\langle Dx,x\rangle I)+\Phi(M)(\langle Dx,x\rangle I-mC)}{M-m}-\mathcal{S}_{\Phi}(D,C)(x)\right\}\\
&\geq\frac{\Phi(m)(MA-\langle Bx,x\rangle I)+\Phi(M)(\langle Bx,x\rangle I-mA)}{M-m}-\mathcal{S}_{\Phi}(B,A)\\
&\geq k\left\{\frac{\Phi(m)(MC-\langle Dx,x\rangle I)+\Phi(M)(\langle Dx,x\rangle I-mC)}{M-m}-\mathcal{S}_{\Phi}(D,C)(x)\right\}\\
&\geq 0.
\end{aligned}$$

For $x \in H$ with $\|x\| = 1$, for the continuos function $\Phi : (0, \infty) \to \mathbb{R}$ and for the positive operator B and the positive invertible operator A we consider the *scalar-valued functional* defined by

$$\begin{aligned}\sigma_{\Phi}(B, A)(x) &:= \langle \mathcal{S}_{\Phi}(B, A)(x)x, x\rangle = \left\langle A^{1/2}\Phi\left(\langle Bx, x\rangle A^{-1}\right)A^{1/2}x, x\right\rangle \\ &= \left\langle \Phi\left(\langle Bx, x\rangle A^{-1}\right)A^{1/2}x, A^{1/2}x\right\rangle.\end{aligned} \tag{2.11}$$

By Theorem 1, if $\Phi : (0, \infty) \to \mathbb{R}$ is an operator convex (concave) function, then $\sigma_{\Phi}(\cdot, \cdot)(x)$ *is jointly operator convex (concave).*

If Φ is an operator concave function defined in the positive half-line then $\sigma_{\Phi}(\cdot, \cdot)(x)$ is superadditive as a function of pairs of positive operators. In addition, if Φ is nonnegative in the positive half-line, then $\sigma_{\Phi}(\cdot, \cdot)(x)$ is *monotonic nondecreasing as a function of pairs of positive operators.*

If if Φ is nonnegative in the positive half-line, $(B, A), (D, C) \in \mathcal{G}(m, M)(x)$ $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants k, K, then we have

$$\begin{aligned}& K\left\langle \Phi\left(\langle Dx, x\rangle C^{-1}\right)C^{1/2}x, C^{1/2}x\right\rangle \\ &\geq \left\langle \Phi\left(\langle Bx, x\rangle A^{-1}\right)A^{1/2}x, A^{1/2}x\right\rangle \\ &\geq kK\left\langle \Phi\left(\langle Dx, x\rangle C^{-1}\right)C^{1/2}x, C^{1/2}x\right\rangle.\end{aligned} \tag{2.12}$$

Similar results may be stated for the functional $\sigma_{\Phi}(\cdot, \cdot)(x)$ in the case when the function $\Phi : [m, M] \to \mathbb{R}$ and $(B, A) \in \mathcal{G}(m, M)(x)$.

We can also define the functional $\phi_{\Phi}(\cdot, \cdot)(x)$ on $\mathcal{G}(m, M)(x)$ by

$$\begin{aligned}&\phi_{\Phi}(B, A)(x) \\ &:= \langle \mathcal{F}_{\Phi}(B, A)(x)x, x\rangle \\ &= \frac{\Phi(m)\langle MA - Bx, x\rangle + \Phi(M)\langle (B - mA)x, x\rangle}{M - m} - \left\langle \Phi\left(\langle Bx, x\rangle A^{-1}\right)A^{1/2}x, A^{1/2}x\right\rangle.\end{aligned}$$

Let Φ be an operator convex function defined on the interval $[m, M]$. Then by Theorem 2 the functional $\phi_{\Phi}(\cdot, \cdot)(x)$ is *nonnegative, positive homogeneous, concave, superadditive and monotonic nondecreasing* on $\mathcal{G}(m, M)(x)$.

Let $(B, A), (D, C) \in \mathcal{G}(m, M)(x)$. In addition, if there exists some positive constants k, K such that $KC \geq A \geq kC$ and $KD \geq B \geq kD$, then we have the inequalities

$$\begin{aligned}& K\left\{\frac{\Phi(m)\langle (MC - D)x, x\rangle + \Phi(M)\langle (D - mC)x, x\rangle}{M - m}\right. \\ &\left. - \left\langle \Phi\left(\langle Dx, x\rangle C^{-1}\right)C^{1/2}x, C^{1/2}x\right\rangle\right\} \\ &\geq \frac{\Phi(m)\langle (MA - B)x, x\rangle + \Phi(M)\langle (B - mA)x, x\rangle}{M - m} \\ &- \left\langle \Phi\left(\langle Bx, x\rangle A^{-1}\right)A^{1/2}x, A^{1/2}x\right\rangle \\ &\geq k\left\{\frac{\Phi(m)\langle (MC - D)x, x\rangle + \Phi(M)\langle (D - mC)x, x\rangle}{M - m}\right. \\ &\left. - \left\langle \Phi\left(\langle Dx, x\rangle C^{-1}\right)C^{1/2}x, C^{1/2}x\right\rangle\right\}.\end{aligned} \tag{2.13}$$

3. Some Examples

For $\nu \in (0,1)$ $(\nu \in [-1,0] \cup [1,2])$ the power function $\Phi_\nu : (0,\infty) \to [0,\infty)$, $\Phi_\nu(x) = x^\nu$ is operator concave (convex) and positive on $(0,\infty)$.

Let $B_1, B_2 > 0$ and the positive invertible operators A_1, A_2. Then by Theorem 1 we have for any $\lambda \in [0,1]$ and $y \in H$, $y \neq 0$ that

$$\begin{aligned} &\langle [\lambda B_1 + (1-\lambda) B_2] y, y\rangle^\nu (\lambda A_1 + (1-\lambda) A_2)^{1-\nu} \\ &\geq (\leq) \lambda \langle B_1 y, y\rangle^\nu A_1^{1-\nu} + (1-\lambda) \langle B_2 y, y\rangle^\nu A_2^{1-\nu} \end{aligned} \tag{3.1}$$

and

$$\langle (B_1 + B_2) y, y\rangle^\nu (A_1 + A_2)^{1-\nu} \geq (\leq) \langle B_1 y, y\rangle^\nu A_1^{1-\nu} + \langle B_2 y, y\rangle^\nu A_2^{1-\nu}. \tag{3.2}$$

In particular, we have for $\nu = 2$ that

$$\begin{aligned} &\langle [\lambda B_1 + (1-\lambda) B_2] y, y\rangle^2 (\lambda A_1 + (1-\lambda) A_2)^{-1} \\ &\leq \lambda \langle B_1 y, y\rangle^2 A_1^{-1} + (1-\lambda) \langle B_2 y, y\rangle^2 A_2^{-1} \end{aligned} \tag{3.3}$$

and

$$\langle (B_1 + B_2) y, y\rangle^2 (A_1 + A_2)^{-1} \leq \langle B_1 y, y\rangle^2 A_1^{-1} + \langle B_2 y, y\rangle^2 A_2^{-1}. \tag{3.4}$$

Also, for $\nu = -1$ we have that

$$\begin{aligned} &\langle [\lambda B_1 + (1-\lambda) B_2] y, y\rangle^{-1} (\lambda A_1 + (1-\lambda) A_2)^{2} \\ &\leq \lambda \langle B_1 y, y\rangle^{-1} A_1^{2} + (1-\lambda) \langle B_2 y, y\rangle^{-1} A_2^{2} \end{aligned} \tag{3.5}$$

and

$$\langle (B_1 + B_2) y, y\rangle^{-1} (A_1 + A_2)^{2} \leq \langle B_1 y, y\rangle^{-1} A_1^{2} + \langle B_2 y, y\rangle^{-1} A_2^{2}. \tag{3.6}$$

By (3.1) and (3.2) for $\nu = \frac{1}{2}$ we get

$$\begin{aligned} &\langle [\lambda B_1 + (1-\lambda) B_2] y, y\rangle^{1/2} (\lambda A_1 + (1-\lambda) A_2)^{1/2} \\ &\geq \lambda \langle B_1 y, y\rangle^{1/2} A_1^{1/2} + (1-\lambda) \langle B_2 y, y\rangle^{1/2} A_2^{1/2} \end{aligned} \tag{3.7}$$

and

$$\langle (B_1 + B_2) y, y\rangle^{1/2} (A_1 + A_2)^{1/2} \geq \langle B_1 y, y\rangle^{1/2} A_1^{1/2} + \langle B_2 y, y\rangle^{1/2} A_2^{1/2}. \tag{3.8}$$

If $\nu \in [1,2]$, then by taking the operator norm in (3.1) we obtain

$$\begin{aligned} &\langle (\lambda B_1 + (1-\lambda) B_2) y, y\rangle^\nu \left\| (\lambda A_1 + (1-\lambda) A_2)^{1-\nu} \right\| \\ &\leq \lambda \langle B_1 y, y\rangle^\nu \left\| A_1^{1-\nu} \right\| + (1-\lambda) \langle B_2 y, y\rangle^\nu \left\| A_2^{1-\nu} \right\| \end{aligned}$$

for any $y \in H$, $y \neq 0$.

Taking the supremum in this inequality over $y \in H$, $\|y\| = 1$,we then get the norm inequality

$$\begin{aligned} &\|\lambda B_1 + (1-\lambda) B_2\|^\nu \left\| (\lambda A_1 + (1-\lambda) A_2)^{1-\nu} \right\| \\ &\leq \lambda \|B_1\|^\nu \left\| A_1^{1-\nu} \right\| + (1-\lambda) \|B_2\|^\nu \left\| A_2^{1-\nu} \right\|, \end{aligned}$$

meaning that the functional $\vartheta(A,B) := \|B\|^\nu \left\| A^{1-\nu} \right\|$ is *jointly convex* for $\nu \in [1,2]$.

Let $x\in H$ with $\|x\|=1$ and $(B,A)\,,(D,C)\in\mathcal{G}\,(m,M)\,(x)\,.$ In addition, if there exists some positive constants $k,$ K such that $KC\geq A\geq kC$ and $KD\geq B\geq kD,$ then by Theorem 2 we have for $\nu\in[-1,0]\cup[1,2]$ that

$$\begin{aligned}K\Bigg\{&\frac{m^{\nu}\langle(MC-D)\,x,x\rangle+M^{\nu}\langle(D-mC)\,x,x\rangle}{M-m}\\&-\langle Dx,x\rangle^{\nu}\left\langle C^{1-\nu}x,x\right\rangle\Bigg\}\\&\geq\frac{m^{\nu}\langle(MA-B)\,x,x\rangle+M^{\nu}\langle(B-mA)\,x,x\rangle}{M-m}\\&-\langle Bx,x\rangle^{\nu}\left\langle A^{1-\nu}x,x\right\rangle\\&\geq k\Bigg\{\frac{m^{\nu}\langle(MC-D)\,x,x\rangle+M^{\nu}\langle(D-mC)\,x,x\rangle}{M-m}\\&-\langle Dx,x\rangle^{\nu}\left\langle C^{1-\nu}x,x\right\rangle\Bigg\}\,.\end{aligned}\tag{3.9}$$

The logarithmic function $\Phi\,(t)=\ln t$ is operator concave function on $(0,\infty)\,.$ For the positive operator $B,$ the positive invertible operator A and the vector $x\in H,\|x\|=1$ we have

$$\begin{aligned}\mathcal{S}_{\ln}\,(B,A)\,(x)&:=A^{1/2}\ln\left(\langle Bx,x\rangle A^{-1}\right)A^{1/2}=(\ln\langle Bx,x\rangle)\,A-A\ln A\\&=(\ln\langle Bx,x\rangle)\,A+\eta\,(A)\end{aligned}\tag{3.10}$$

where $\eta\,(A):=-A\ln A$ is the J. von Neumann operator entropy considered by Nakamura and Umegaki in [8].

Using Theorem 1 we conclude that for any positive invertible operators $B_1,$ $B_2,$ $A_1,$ $A_2,$ $x\in H,\|x\|=1$ and $\lambda\in[0,1]$ we have

$$\begin{aligned}&\eta\,(\lambda A_1+(1-\lambda)\,A_2)-\lambda\eta\,(A_1)-(1-\lambda)\,\eta\,(A_2)\\&\geq\lambda\,(\ln\langle B_1x,x\rangle)\,A_1+(1-\lambda)\,(\ln\langle B_2x,x\rangle)\,A_2\\&-(\ln\langle(\lambda B_1+(1-\lambda)\,B_2)\,x,x\rangle)\,(\lambda A_1+(1-\lambda)\,A_2)\,.\end{aligned}\tag{3.11}$$

In particular, we have

$$\begin{aligned}&\eta\,(A_1+A_2)-\eta\,(A_1)-\eta\,(A_2)\\&\geq(\ln\langle B_1x,x\rangle)\,A_1+(\ln\langle B_2x,x\rangle)\,A_2-(\ln\langle(B_1+B_2)\,x,x\rangle)\,(A_1+A_2)\,.\end{aligned}\tag{3.12}$$

The function $\Phi\,(t)=t\ln t$ is operator convex function on $(0,\infty)\,.$ For the positive operator $B,$ the positive invertible operator A and the vector $x\in H,\|x\|=1$ we have

$$\begin{aligned}\mathcal{S}_{(\cdot)\ln(\cdot)}\,(B,A)\,(x)&=A^{1/2}\left[\langle Bx,x\rangle A^{-1}\ln\left(\langle Bx,x\rangle A^{-1}\right)\right]A^{1/2}\\&=A^{1/2}\left[\langle Bx,x\rangle A^{-1}\left(\ln\langle Bx,x\rangle I+\ln A^{-1}\right)\right]A^{1/2}\\&=\langle Bx,x\rangle\,(\ln\langle Bx,x\rangle I-\ln A)\,.\end{aligned}$$

Using Theorem 1 we conclude that for any positive invertible operators $B_1,$ $B_2,$ $A_1,$ $A_2,$ $x\in H,\|x\|=1$ and $\lambda\in[0,1]$ we have

$$\begin{aligned}&\lambda\langle B_1x,x\rangle\ln A_1+(1-\lambda)\langle B_2x,x\rangle\ln A_2\\&-\langle(\lambda B_1+(1-\lambda)\,B_2)\,x,x\rangle\ln(\lambda A_1+(1-\lambda)\,A_2)\\&\leq\lambda\langle B_1x,x\rangle\ln\langle B_1x,x\rangle I+(1-\lambda)\langle B_2x,x\rangle\ln\langle B_2x,x\rangle I\\&-\langle(\lambda B_1+(1-\lambda)\,B_2)\,x,x\rangle\ln\langle(\lambda B_1+(1-\lambda)\,B_2)\,x,x\rangle I.\end{aligned}\tag{3.13}$$

In particular, we have

$$\begin{aligned}
&\langle B_1 x, x\rangle \ln A_1 + \langle B_2 x, x\rangle \ln A_2 - \langle (B_1 + B_2)\, x, x\rangle \ln (A_1 + A_2) \\
&\leq \langle B_1 x, x\rangle \ln \langle B_1 x, x\rangle\, I + \langle B_2 x, x\rangle \ln \langle B_2 x, x\rangle\, I \\
&- \langle (B_1 + B_2)\, x, x\rangle \ln \langle (B_1 + B_2)\, x, x\rangle\, I.
\end{aligned} \tag{3.14}$$

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