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A WEAKEN VERSION OF DAVIS-CHOI-JENSEN'S INEQUALITY FOR NORMALISED POSITIVE LINEAR MAPS

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Abstract. In this paper we show that the celebrated Davis-Choi-Jensen's inequality for normalised positive linear maps can be extended in a weaken form for convex functions. A reverse inequality and applications for important instances of convex (concave) functions are also given.

1. Introduction

The following result that provides an vector operator version for the Jensen inequality is well known, see for instance [6] or [7, p. 5]:

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{1.1}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the *Hölder-McCarthy inequality* [5]: Let A be a selfadjoint positive operator on a Hilbert space H, then

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

In [2] (see also [3, p. 16]) we obtained the following additive reverse of (1.1):

Theorem 2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subset \mathring{I}$, *then*

$$(0 \leq)\, \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

This is a generalization of the scalar discrete inequality obtained in [4]. For other reverse inequalities of this type see [3, p. 16].

The following particular cases are of interest: If A is a selfadjoint operator on H, then we have the inequality:

$$(0 \leq)\, \langle \exp(A)x, x\rangle - \exp(\langle Ax, x\rangle) \leq \langle A\exp(A)x, x\rangle - \langle Ax, x\rangle \langle \exp(A)x, x\rangle, \tag{1.3}$$

for each $x \in H$ with $\|x\| = 1$.

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Let A be a positive definite operator on the Hilbert space H. Then we have the following inequality for the logarithm:

$$(0\leq)\ln\left(\langle Ax,x\rangle\right)-\langle\ln\left(A\right)x,x\rangle\leq\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1, \tag{1.4}$$

for each $x\in H$ with $\|x\|=1$.

If $p\geq 1$ and A is a positive operator on H, then

$$(0\leq)\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\leq p\left[\langle A^{p}x,x\rangle-\langle Ax,x\rangle\left\langle A^{p-1}x,x\right\rangle\right], \tag{1.5}$$

for each $x\in H$ with $\|x\|=1$. If A is positive definite, then the inequality (1.5) also holds for $p<0$. If $0<p<1$ and A is a positive definite operator then the reverse inequality also holds

$$(0\leq)\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\leq p\left[\langle Ax,x\rangle\cdot\left\langle A^{p-1}x,x\right\rangle-\langle A^{p}x,x\rangle\right], \tag{1.6}$$

for each $x\in H$ with $\|x\|=1$.

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_{h}(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^{+}(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [7, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi\left(\lambda A+\mu B\right)=\lambda\Phi\left(A\right)+\mu\Phi\left(B\right)$$

for any λ, $\mu\in\mathbb{C}$ and A, $B\in\mathcal{B}(H)$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A\in\mathcal{B}^{+}(H)$ then $\Phi(A)\in\mathcal{B}^{+}(K)$. We write $\Phi\in\mathfrak{P}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_{H})=1_{K}$. We write $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A\leq B\text{ implies }\Phi\left(A\right)\leq\Phi\left(B\right)$$

and preserves the adjoint operation $\Phi(A^{*})=\Phi(A)^{*}$. If $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and $\alpha 1_{H}\leq A\leq\beta 1_{H}$, then $\alpha 1_{K}\leq\Phi(A)\leq\beta 1_{K}$.

If the map $\Psi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear, positive and $\Psi(1_{H})\in\mathcal{B}^{++}(K)$ then by putting $\Phi=\Psi^{-1/2}(1_{H})\Psi\Psi^{-1/2}(1_{H})$ we get that $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq(\geq)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right)$$

for all $\lambda\in[0,1]$ and for every selfadjoint operators A, $B\in\mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known:

Theorem 3 (Davis-Choi-Jensen's Inequality)**.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I and $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f\left(\Phi\left(A\right)\right)\leq\Phi\left(f\left(A\right)\right). \tag{1.7}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.7) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{1.8}$$

It is obvious that, by (1.7) we have the vector inequality

$$\langle f(\Phi(A))y, y\rangle \leq \langle \Phi(f(A))y, y\rangle \tag{1.9}$$

for any $y \in K$. By (1.1) we also have

$$f(\langle \Phi(A)y, y\rangle) \leq \langle f(\Phi(A))y, y\rangle \tag{1.10}$$

for any $y \in K$, $\|y\| = 1$. Therefore, for an *operator convex function* on I we have

$$f(\langle \Phi(A)y, y\rangle) \leq \langle f(\Phi(A))y, y\rangle \leq \langle \Phi(f(A))y, y\rangle \tag{1.11}$$

for any $y \in K$, $\|y\| = 1$.

It is then natural to ask the following question:

Does the inequality between the first and last term in (1.11) remains valid in the more general case of convex functions defined on the interval I ?

A positive answer to this question and some reverse inequalities are provided below. Some applications for important instances of convex (concave) functions are also given.

2. A Jensen's Type Inequality

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $t, s \in \mathring{I}$ and $t < s$, then $f'_-(t) \leq f'_+(t) \leq f'_-(s) \leq f'_+(s)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(t) \geq f(a) + (t-a)\varphi(a) \text{ for any } t, a \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(t) \leq \varphi(t) \leq f'_+(t) \text{ for any } t \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

We have:

Theorem 4. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ a normalised positive linear map. Then for any selfadjoint operator A whose spectrum* $\mathrm{Sp}(A)$ *is contained in I we have*

$$f(\langle \Phi(A)y, y\rangle) \leq \langle \Phi(f(A))y, y\rangle \tag{2.2}$$

for any $y \in K$, $\|y\| = 1$.

Proof. Let m, M with $m < M$ and such that $\operatorname{Sp}(A) \subseteq [m, M] \subset I$. Then $m1_H \le A \le M1_H$ and since $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have that $m1_K \le \Phi(A) \le M1_K$ showing that $\langle \Phi(A) y, y\rangle \in [m, M]$ for any $y \in K$, $\|y\| = 1$.

By the gradient inequality (2.1) we have for $a = \langle \Phi(A) y, y\rangle \in [m, M]$ that

$$f(t) \ge f(\langle \Phi(A) y, y\rangle) + (t - \langle \Phi(A) y, y\rangle) \varphi(\langle \Phi(A) y, y\rangle)$$

for any $t \in I$.

Using the continuous functional calculus for the operator A we have for a fixed $y \in K$ with $\|y\| = 1$ that

$$f(A) \ge f(\langle \Phi(A) y, y\rangle) 1_H + \varphi(\langle \Phi(A) y, y\rangle)(A - \langle \Phi(A) y, y\rangle 1_H). \tag{2.3}$$

Since $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (2.3) we get

$$\Phi(f(A)) \ge f(\langle \Phi(A) y, y\rangle) 1_K + \varphi(\langle \Phi(A) y, y\rangle)(\Phi(A) - \langle \Phi(A) y, y\rangle 1_K) \tag{2.4}$$

for any $y \in K$ with $\|y\| = 1$.

This inequality is of interest in itself.

Taking the inner product in (2.4) we have for any $y \in K$ with $\|y\| = 1$ that

$$\begin{aligned}
&\langle \Phi(f(A)) y, y\rangle \\
&\ge f(\langle \Phi(A) y, y\rangle)\|y\|^2 + \varphi(\langle \Phi(A) y, y\rangle)\left(\langle \Phi(A) y, y\rangle - \langle \Phi(A) y, y\rangle \|y\|^2\right) \\
&= f(\langle \Phi(A) y, y\rangle)
\end{aligned}$$

and the inequality (2.2) is proved. □

Corollary 1. *Let* $f : I \to \mathbb{R}$ *be a convex function on the interval* I *and* $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ *with* $\Psi(1_H) \in \mathcal{B}^{++}(K)$. *Then for any selfadjoint operator* A *whose spectrum* $\operatorname{Sp}(A)$ *is contained in* I *we have*

$$f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right) \le \frac{\langle \Psi(f(A)) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} \tag{2.5}$$

for any $v \in K$ *with* $v \ne 0$.

Proof. If we write the inequality (2.2) for $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we have

$$\begin{aligned}
&f\left(\left\langle \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) y, y\right\rangle\right) \\
&\le \left\langle \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H) y, y\right\rangle
\end{aligned} \tag{2.6}$$

for any $y \in K$, $\|y\| = 1$.

Now, let $v \in K$ with $v \ne 0$ and take $y = \frac{1}{\|\Psi^{1/2}(1_H)v\|}\Psi^{1/2}(1_H) v$ in (2.6) to get

$$\begin{aligned}
&f\left(\left\langle \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\frac{\Psi^{1/2}(1_H) v}{\|\Psi^{1/2}(1_H) v\|}, \frac{\Psi^{1/2}(1_H) v}{\|\Psi^{1/2}(1_H) v\|}\right\rangle\right) \\
&\le \left\langle \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H)\frac{\Psi^{1/2}(1_H) v}{\|\Psi^{1/2}(1_H) v\|}, \frac{\Psi^{1/2}(1_H) v}{\|\Psi^{1/2}(1_H) v\|}\right\rangle
\end{aligned}$$

that is equivalent to

$$f\left(\left\langle \frac{\Psi(A)v}{\left\|\Psi^{1/2}(1_H)v\right\|}, \frac{v}{\left\|\Psi^{1/2}(1_H)v\right\|}\right\rangle\right) \le \left\langle \frac{\Psi(f(A))v}{\left\|\Psi^{1/2}(1_H)v\right\|}, \frac{v}{\left\|\Psi^{1/2}(1_H)v\right\|}\right\rangle$$

and since

$$\left\|\Psi^{1/2}(1_H)v\right\|^2 = \langle \Psi(1_H)v, v\rangle$$

for $v \in K$ with $v \neq 0$, then we obtain the desired inequality (2.5). □

By taking some example of fundamental convex (concave) functions, we can state the following results:

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a normalised positive linear map.

(i) If A is a selfadjoint operator on H and $r \ge 1$, then we have

$$|\langle \Phi(A)y, y\rangle|^r \le \langle \Phi(|A|^r)y, y\rangle \tag{2.7}$$

and in particular

$$|\langle \Phi(A)y, y\rangle| \le \langle \Phi(|A|)y, y\rangle \tag{2.8}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\|\Phi(A)\|^r \le \|\Phi(|A|^r)\|. \tag{2.9}$$

(ii) If A is a positive operator on a Hilbert space H, then for any $p \ge 1$ ($p \in (0,1)$) we have

$$\langle \Phi(A)y, y\rangle^p \le (\ge) \langle \Phi(A^p)y, y\rangle \tag{2.10}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\|\Phi(A)\|^p \le (\ge) \|\Phi(A^p)\|. \tag{2.11}$$

If A is a positive definite operator on a Hilbert space H, then for any $p < 0$ we have

$$\langle \Phi(A)y, y\rangle^p \le \langle \Phi(A^p)y, y\rangle \tag{2.12}$$

for all $y \in K$, $\|y\| = 1$.

(iii) If A is a selfadjoint operator on H then we have

$$\exp(\langle \Phi(A)y, y\rangle) \le \langle \Phi(\exp(A))y, y\rangle \tag{2.13}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\exp(\|\Phi(A)\|) \le \|\Phi(\exp(A))\|. \tag{2.14}$$

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with

$$\sum_{j=1}^{k} P_j^* P_j = 1_H. \tag{2.15}$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [7]

$$\Phi(A) := \sum_{j=1}^{k} P_j^* A P_j$$

is a normalized positive linear map on $\mathcal{B}(H)$. Therefore, if $f: I \to \mathbb{R}$ be a convex function on the interval I and A is selfadjoint operator whose spectrum $\operatorname{Sp}(A)$ is contained in I, we have by (2.2) that

$$f\left(\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right) \leq \left\langle \sum_{j=1}^{k} P_j^* f(A) P_j y, y\right\rangle \tag{2.16}$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.16), then we recapture Jensen's inequality (1.1).

We then have for any selfadjoint operator A and $r \geq 1$ that

$$\left|\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right|^r \leq \left\langle \sum_{j=1}^{k} P_j^* |A|^r P_j y, y\right\rangle \tag{2.17}$$

and

$$\exp\left(\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right) \leq \left\langle \sum_{j=1}^{k} P_j^* (\exp A) P_j y, y\right\rangle \tag{2.18}$$

for all $y \in K$, $\|y\| = 1$. In the case $r = 1$ we have

$$\left|\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right| \leq \left\langle \sum_{j=1}^{k} P_j^* |A| P_j y, y\right\rangle. \tag{2.19}$$

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequalities

$$\left\|\sum_{j=1}^{k} P_j^* A P_j\right\|^r \leq \left\|\sum_{j=1}^{k} P_j^* |A|^r P_j\right\|, \quad r \geq 1 \tag{2.20}$$

and

$$\exp\left(\left\|\sum_{j=1}^{k} P_j^* A P_j\right\|\right) \leq \left\|\sum_{j=1}^{k} P_j^* (\exp A) P_j\right\|. \tag{2.21}$$

In the case $r = 1$ we have

$$\left\|\sum_{j=1}^{k} P_j^* A P_j\right\| \leq \left\|\sum_{j=1}^{k} P_j^* |A|^r P_j\right\|. \tag{2.22}$$

If A is a positive operator on a Hilbert space H, then for any $p \in (-\infty, 0) \cup [1, \infty)$ ($p \in (0, 1)$) we have by (2.16) for power function that

$$\left\langle \sum_{j=1}^{k} P_j^* A P_j y, y\right\rangle^p \leq (\geq) \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y\right\rangle \tag{2.23}$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.23), then we recapture Hölder-McCarthy's inequality.

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequality

$$\left\| \sum_{j=1}^{k} P_j^* A P_j \right\|^p \leq (\geq) \left\| \sum_{j=1}^{k} P_j^* A^p P_j \right\|, \tag{2.24}$$

where $p \geq 1$ $(p \in (0,1))$.

3. A Reverse Inequality

We have:

Theorem 5. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a normalised positive linear map and A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subset \mathring{I}$, then*

$$\begin{aligned} 0 &\leq \langle \Phi(f(A)) y, y\rangle - f(\langle \Phi(A) y, y\rangle) \\ &\leq \langle \Phi(Af'(A)) y, y\rangle - \langle \Phi(A) y, y\rangle \langle \Phi(f'(A)) y, y\rangle \end{aligned} \tag{3.1}$$

for any $y \in K$, $\|y\| = 1$.

Proof. From the gradient inequality (2.1) we have

$$f(t) \geq f(s) + (t - s) f'(s) \tag{3.2}$$

for any $t, s \in \mathring{I}$.

Let $y \in K$, $\|y\| = 1$. If we take in (3.2) $t = \langle \Phi(A) y, y\rangle \in \mathring{I}$, then we get

$$f(\langle \Phi(A) y, y\rangle) \geq f(s) + (\langle \Phi(A) y, y\rangle - s) f'(s)$$

for any $s \in \mathring{I}$ that can be written as

$$(s - \langle \Phi(A) y, y\rangle) f'(s) \geq f(s) - f(\langle \Phi(A) y, y\rangle)$$

for any $s \in \mathring{I}$.

Let $y \in K$, $\|y\| = 1$. Using the continuous functional calculus for the operator A we have

$$Af'(A) - \langle \Phi(A) y, y\rangle f'(A) \geq f(A) - f(\langle \Phi(A) y, y\rangle) 1_H. \tag{3.3}$$

Since $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (3.3) we have

$$\Phi(Af'(A)) - \langle \Phi(A) y, y\rangle \Phi(f'(A)) \geq \Phi(f(A)) - f(\langle \Phi(A) y, y\rangle) 1_K, \tag{3.4}$$

for any $y \in K$, $\|y\| = 1$.

This is an inequality of interest in itself.

Taking the inner product in (3.4) we obtain the desired result (3.1). □

Corollary 2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$ and A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subset \mathring{I}$, then*

$$\begin{aligned} 0 &\leq \frac{\langle \Psi(f(A)) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} - f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right) \\ &\leq \frac{\langle \Psi(Af'(A)) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} - \frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} \frac{\langle \Psi(f'(A)) v, v\rangle}{\langle \Psi(1_H) v, v\rangle} \end{aligned} \tag{3.5}$$

for any $v \in K$ *with* $v \neq 0$.

The proof follows from the inequality (3.1) by a similar argument to the one from the proof of Corollary 1 and the details are omitted.

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a normalised positive linear map.

(i) If A is a selfadjoint operator on H, then we have

$$\begin{aligned} 0 &\leq \langle \Phi(\exp(A)) y, y\rangle - \exp(\langle \Phi(A) y, y\rangle) \\ &\leq \langle \Phi(A\exp(A)) y, y\rangle - \langle \Phi(A) y, y\rangle \langle \Phi(\exp(A)) y, y\rangle \end{aligned} \tag{3.6}$$

for all $y \in K$, $\|y\| = 1$.

(ii) If A is a positive (positive definite) operator on a Hilbert space H, then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$\begin{aligned} 0 &\leq \langle \Phi(A^p) y, y\rangle - \langle \Phi(A) y, y\rangle^p \\ &\leq p\left[\langle \Phi(A^p) y, y\rangle - \langle \Phi(A) y, y\rangle \left\langle \Phi\left(A^{p-1}\right) y, y\right\rangle\right] \end{aligned} \tag{3.7}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive operator on a Hilbert space H, then for any $p \in (0,1)$ we have

$$\begin{aligned} 0 &\leq \langle \Phi(A) y, y\rangle^p - \langle \Phi(A^p) y, y\rangle \\ &\leq p\left[\langle \Phi(A) y, y\rangle \left\langle \Phi\left(A^{p-1}\right) y, y\right\rangle - \langle \Phi(A^p) y, y\rangle\right] \end{aligned} \tag{3.8}$$

for all $y \in K$, $\|y\| = 1$.

(iii) If A is a positive definite operator on a Hilbert space H, then

$$0 \leq \ln(\langle \Phi(A) y, y\rangle) - \langle \Phi(\ln A) y, y\rangle \leq \langle \Phi(A) y, y\rangle \left\langle \Phi\left(A^{-1}\right) y, y\right\rangle - 1 \tag{3.9}$$

for all $y \in K$, $\|y\| = 1$.

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with the property (2.15). If $f : I \to \mathbb{R}$ is a convex function on the interval I and A is selfadjoint operator whose spectrum $\mathrm{Sp}(A)$ is contained in I, then we have by (3.1) that

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^{k} P_j^* f(A) P_j y, y\right\rangle - f\left(\sum_{j=1}^{k} \left\langle P_j^* A P_j y, y\right\rangle\right) \\ &\leq \left\langle \sum_{j=1}^{k} P_j^* A f'(A) P_j y, y\right\rangle - \left\langle \sum_{j=1}^{k} P_j^* A P_j y, y\right\rangle \left\langle \sum_{j=1}^{k} P_j^* f'(A) P_j y, y\right\rangle \end{aligned} \tag{3.10}$$

for all $y \in K$, $\|y\| = 1$. This is a generalization of (1.2).

In particular, if A is a selfadjoint operator on H, then we have

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^{k} P_j^* \exp(A) P_j y, y\right\rangle - \exp\left(\sum_{j=1}^{k} \left\langle P_j^* A P_j y, y\right\rangle\right) \\ &\leq \left\langle \sum_{j=1}^{k} P_j^* A \exp(A) P_j y, y\right\rangle - \left\langle \sum_{j=1}^{k} P_j^* A P_j y, y\right\rangle \left\langle \sum_{j=1}^{k} P_j^* \exp P_j y, y\right\rangle \end{aligned} \tag{3.11}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive (positive definite) operator on a Hilbert space H, then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$0 \leq \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle - \left(\sum_{j=1}^{k} \langle P_j^* A P_j y, y \rangle \right)^p \leq p \left[\left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle - \left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^{k} P_j^* A^{p-1} P_j y, y \right\rangle \right], \tag{3.12}$$

for all $y \in K$, $\|y\| = 1$. However, when $p \in (0, 1)$ and A is a positive, then

$$0 \leq \left(\sum_{j=1}^{k} \langle P_j^* A P_j y, y \rangle \right)^p - \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle \leq p \left[\left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^{k} P_j^* A^{p-1} P_j y, y \right\rangle - \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle \right], \tag{3.13}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive definite operator on H, then

$$0 \leq \ln \left(\sum_{j=1}^{k} \langle P_j^* A P_j y, y \rangle \right) - \left\langle \sum_{j=1}^{k} P_j^* (\ln A) P_j y, y \right\rangle \leq \left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^{k} P_j^* A^{-1} P_j y, y \right\rangle - 1 \tag{3.14}$$

for all $y \in K$, $\|y\| = 1$.

These inequalities generalize the corresponding results from (1.4)-(1.6).

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