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GRÜSS' TYPE INEQUALITIES FOR POSITIVE LINEAR MAPS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities of Grüss' type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces, are given. Applications for power function and logarithm are provided as well.

1. Introduction

In 1935, G. Grüss [24] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ, Φ, γ, Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [27, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two n-tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the book [27]. For other related results see the papers [1]-[3], [4]-[6], [7]-[9], [11]-[21], [23], [31], [33] and the references therein.

In the recent paper [15] we obtained the following result for continuous functions of selfadjoint operators in complex Hilbert spaces:

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Theorem 1. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that the spectrum* $\operatorname{Sp}(A)\subseteq[m,M]$ *for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\theta:=\min_{t\in[m,M]}f(t)$ and $\Theta:=\max_{t\in[m,M]}f(t)$ then*

$$\begin{aligned}(1.4)\quad &\Big|\langle f(A)g(A)y,y\rangle-\langle f(A)y,y\rangle\langle g(A)x,x\rangle\\ &\qquad-\frac{\theta+\Theta}{2}\left[\langle g(A)y,y\rangle-\langle g(A)x,x\rangle\right]\Big|\\ &\le\frac{1}{2}(\Theta-\theta)\left[\|g(A)y\|^2+\langle g(A)x,x\rangle^2-2\langle g(A)x,x\rangle\langle g(A)y,y\rangle\right]^{1/2}\end{aligned}$$

for any $x,\ y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have

$$\begin{aligned}(1.5)\quad &|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\\ &\le\frac{1}{2}(\Theta-\theta)\left[\|g(A)x\|^2-\langle g(A)x,x\rangle^2\right]^{1/2}\le\frac{1}{4}(\Theta-\theta)(\Psi-\psi)\end{aligned}$$

for each $x\in H$ with $\|x\|=1$, where $\psi:=\min_{t\in[m,M]}g(t)$ and $\Psi:=\max_{t\in[m,M]}g(t)$.

For other related results see the recent monograph [18].

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^{+}(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [10] (see also [25, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A+\mu B)=\lambda\Phi(A)+\mu\Phi(B)$$

for any $\lambda,\ \mu\in\mathbb{C}$ and $A,\ B\in\mathcal{B}(H)$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A\in\mathcal{B}^{+}(H)$ then $\Phi(A)\in\mathcal{B}^{+}(K)$. We write $\Phi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(K)]$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H)=1_K$. We write $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A\le B\text{ implies }\Phi(A)\le\Phi(B)$$

and preserves the adjoint operation $\Phi(A^*)=\Phi(A)^*$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $\alpha 1_H\le A\le\beta 1_H$, then $\alpha 1_K\le\Phi(A)\le\beta 1_K$.

Let $P_j\in\mathcal{B}(H)$, $j=1,...,k$ be contractions with

$$\sum_{j=1}^{k}P_j^*P_j=1_H. \tag{1.6}$$

The map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ defined by [25]

$$\Phi(A):=\sum_{j=1}^{k}P_j^*AP_j$$

is a *normalized positive linear map* on $\mathcal{B}(H)$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

Motivated by the above results, we establish in this paper some inequalities of Grüss' type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces. Applications for power function and logarithm are provided as well.

2. The Main Results

Now, for γ, $\Gamma \in \mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [22])

$$\begin{aligned}&\bar{U}_I(\gamma,\Gamma)\\&:=\left\{g : I \to \mathbb{C} \,|\, \operatorname{Re}\left[(\Gamma - g(t))\left(\overline{g(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for almost every } t \in I\right\}\end{aligned}$$

and

$$\bar{\Delta}_I(\gamma,\Gamma) := \left\{g : I \to \mathbb{C} \,|\, \left|g(t) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in I\right\}.$$

The following representation result may be stated [22].

Proposition 1. *For any γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_I(\gamma,\Gamma)$ and $\bar{\Delta}_I(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_I(\gamma,\Gamma) = \bar{\Delta}_I(\gamma,\Gamma). \tag{2.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma - z)(\bar{z} - \gamma)] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}[(\Gamma - z)(\bar{z} - \gamma)]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned}\bar{U}_I(\gamma,\Gamma) = \{g : I \to \mathbb{C} \mid\ &(\operatorname{Re}\Gamma - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\gamma)\\&+ (\operatorname{Im}\Gamma - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for a.e. } t \in I\}.\end{aligned} \tag{2.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_I(\gamma,\Gamma) := \{g : I \to \mathbb{C} \mid\ &\operatorname{Re}(\Gamma) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\gamma)\\&\text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in I\}.\end{aligned} \tag{2.3}$$

One can easily observe that $\bar{S}_I(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_I(\phi,\Phi) \subseteq \bar{U}_I(\phi,\Phi). \tag{2.4}$$

We have:

Theorem 2. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ with $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars $m<M$ and $\Phi\in\mathcal{P}_N(B(H),B(K))$. If f and g are continuous on $[m,M]$ and $f\in\bar{\Delta}_I(\gamma,\Gamma)$ for some γ, $\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, then for every x, $y\in H$ with $\|x\|=\|y\|=1$ we have*

$$\begin{aligned}(2.5)\quad &\Big|\langle\Phi(f(A)g(A))y,y\rangle-\langle\Phi(g(A))x,x\rangle\langle\Phi(f(A))y,y\rangle\\&\quad+\frac{\gamma+\Gamma}{2}\left[\langle\Phi(g(A))x,x\rangle-\langle\Phi(g(A))y,y\rangle\right]\Big|\\&\le\frac{1}{2}|\Gamma-\gamma|\langle\Phi(|g(A)-\langle\Phi(g(A))x,x\rangle 1_H|)y,y\rangle\\&\le\frac{1}{2}|\Gamma-\gamma|\left[\left\langle\Phi\left(g^2(A)\right)y,y\right\rangle-2\langle\Phi(g(A))x,x\rangle\langle\Phi(g(A))y,y\rangle+\langle\Phi(g(A))x,x\rangle^2\right]^{\frac{1}{2}}.\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.6)\quad &|\langle\Phi(f(A)g(A))x,x\rangle-\langle\Phi(g(A))x,x\rangle\langle\Phi(f(A))x,x\rangle|\\&\le\frac{1}{2}|\Gamma-\gamma|\langle\Phi(|g(A)-\langle\Phi(g(A))x,x\rangle 1_H|)x,x\rangle\\&\le\frac{1}{2}|\Gamma-\gamma|\left[\left\langle\Phi\left(g^2(A)\right)x,x\right\rangle-\langle\Phi(g(A))x,x\rangle^2\right]^{1/2},\end{aligned}$$

for every $x\in H$ with $\|x\|=1$.

Proof. First, observe that, for each $\lambda\in\mathbb{C}$, $\Phi\in\mathcal{P}_N(B(H),B(K))$ and x, $y\in H$ with $\|x\|=\|y\|=1$ we have

$$\begin{aligned}&\Phi\left[(f(A)-\lambda 1_H)(g(A)-\langle\Phi(g(A))x,x\rangle 1_H)\right]\\&=\Phi\left[f(A)g(A)+\lambda\langle\Phi(g(A))x,x\rangle 1_H-\langle\Phi(g(A))x,x\rangle f(A)-\lambda g(A)\right]\\&=\Phi(f(A)g(A))+\lambda\langle\Phi(g(A))x,x\rangle 1_K\\&-\langle\Phi(g(A))x,x\rangle\Phi(f(A))-\lambda\Phi(g(A))\end{aligned}$$

and

$$\begin{aligned}(2.7)\quad &\langle\Phi\left[(f(A)-\lambda 1_H)(g(A)-\langle\Phi(g(A))x,x\rangle 1_H)\right]y,y\rangle\\&=\langle\Phi(f(A)g(A))y,y\rangle-\langle\Phi(g(A))x,x\rangle\langle\Phi(f(A))y,y\rangle\\&+\lambda\left[\langle\Phi(g(A))x,x\rangle-\langle\Phi(g(A))y,y\rangle\right].\end{aligned}$$

If we take in (2.7) $\lambda=\frac{\gamma+\Gamma}{2}$, then we get

$$\begin{aligned}(2.8)\quad &\left\langle\Phi\left[\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)(g(A)-\langle\Phi(g(A))x,x\rangle 1_H)\right]y,y\right\rangle\\&=\langle\Phi(f(A)g(A))y,y\rangle-\langle\Phi(g(A))x,x\rangle\langle\Phi(f(A))y,y\rangle\\&+\frac{\gamma+\Gamma}{2}\left[\langle\Phi(g(A))x,x\rangle-\langle\Phi(g(A))y,y\rangle\right],\end{aligned}$$

for any x, $y\in H$ with $\|x\|=\|y\|=1$.

Since A is selfadjoint, then by the continuous functional calculus for A we have that the operator

$$\left(f(A)-\frac{\gamma+\Gamma}{2}1_H\right)(g(A)-\langle\Phi(g(A))x,x\rangle 1_H)$$

is also selfadjoint for any $x \in H$ with $\|x\| = 1$.

We know that for any selfadjoint operator B we have by the Jensen's inequality for positive maps [19]:

$$|\langle \Phi(B) y, y\rangle| \le \langle \Phi(|B|) y, y\rangle$$

for any $y \in H$ with $\|y\| = 1$.

Using this property and (2.8) we then have

$$\begin{aligned}
&\Big|\langle \Phi(f(A) g(A)) y, y\rangle - \langle \Phi(g(A)) x, x\rangle \langle \Phi(f(A)) y, y\rangle \\
&+\frac{\gamma+\Gamma}{2}\left[\langle \Phi(g(A)) x, x\rangle - \langle \Phi(g(A)) y, y\rangle\right]\Big| \\
&= \left|\left\langle \Phi\left[\left(f(A) - \frac{\gamma+\Gamma}{2} 1_H\right)\left(g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right)\right] y, y\right\rangle\right| \\
&\le \left\langle \Phi\left[\left|\left(f(A) - \frac{\gamma+\Gamma}{2} 1_H\right)\left(g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right)\right|\right] y, y\right\rangle
\end{aligned} \tag{2.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Since A is selfadjoint and $f \in \bar{\Delta}_I(\gamma, \Gamma)$, then by the continuous functional calculus for operator A we have

$$\left|f(A) - \frac{\gamma+\Gamma}{2} 1_H\right| \le \frac{1}{2}|\Gamma - \gamma| 1_H,$$

which implies that

$$\begin{aligned}
&\left|\left(f(A) - \frac{\gamma+\Gamma}{2} 1_H\right)\left(g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right)\right| \\
&= \left|\left(f(A) - \frac{\gamma+\Gamma}{2} 1_H\right)\right| \left|g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right| \\
&\le \frac{1}{2}|\Gamma - \gamma| \left|g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right|
\end{aligned} \tag{2.10}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the map Φ in the inequality (2.10), we get

$$\begin{aligned}
&\Phi\left[\left|\left(f(A) - \frac{\gamma+\Gamma}{2} 1_H\right)\left(g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right)\right|\right] \\
&\le \frac{1}{2}|\Gamma - \gamma| \Phi\left(\left|g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right|\right),
\end{aligned} \tag{2.11}$$

for any $x \in H$ with $\|x\| = 1$.

By Kadison's inequality we have

$$\begin{aligned}
&\Phi^2\left(\left|g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right|\right) \\
&\le \Phi\left(\left|g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right|^2\right) \\
&= \Phi\left(\left(g(A) - \langle \Phi(g(A)) x, x\rangle 1_H\right)^2\right) \\
&= \Phi\left(g^2(A) - 2\langle \Phi(g(A)) x, x\rangle g(A) + \langle \Phi(g(A)) x, x\rangle^2 1_H\right) \\
&= \Phi\left(g^2(A)\right) - 2\langle \Phi(g(A)) x, x\rangle \Phi(g(A)) + \langle \Phi(g(A)) x, x\rangle^2 1_K
\end{aligned} \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the square root in (2.12) we get

$$\begin{aligned}
&\Phi\left(\left|g\left(A\right)-\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle 1_{H}\right|\right)\\
&\leq\left[\Phi\left(g^{2}\left(A\right)\right)-2\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle \Phi\left(g\left(A\right)\right)+\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle ^{2}1_{K}\right]^{1/2}
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

By taking the inner product in this inequality, we get

$$\begin{aligned}
&\left\langle \Phi\left(\left|g\left(A\right)-\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle 1_{H}\right|\right)y,y\right\rangle \qquad (2.13)\\
&\leq\left\langle \left[\Phi\left(g^{2}\left(A\right)\right)-2\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle \Phi\left(g\left(A\right)\right)+\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle ^{2}1_{K}\right]^{1/2}y,y\right\rangle \\
&\leq\left\langle \left[\Phi\left(g^{2}\left(A\right)\right)-2\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle \Phi\left(g\left(A\right)\right)+\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle ^{2}1_{K}\right]y,y\right\rangle ^{1/2},
\end{aligned}$$

for any $x,\ y\in H$ with $\|x\|=\|y\|=1$, where for the last inequality we used the well known Hölder-McCarthy inequality.

By (2.11) and (2.13) we then have

$$\begin{aligned}
&\left\langle \Phi\left[\left|\left(f\left(A\right)-\frac{\gamma+\Gamma}{2}1_{H}\right)\left(g\left(A\right)-\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle 1_{H}\right)\right|\right]y,y\right\rangle \\
&\qquad\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left\langle \Phi\left(\left|g\left(A\right)-\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle 1_{H}\right|\right)y,y\right\rangle \\
&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left\langle \left[\Phi\left(g^{2}\left(A\right)\right)-2\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle \Phi\left(g\left(A\right)\right)+\left\langle \Phi\left(g\left(A\right)\right)x,x\right\rangle ^{2}1_{K}\right]y,y\right\rangle ^{1/2}
\end{aligned}$$

for any $x,\ y\in H$ with $\|x\|=\|y\|=1$, which together with (2.9) produces the desired result (2.5). $\square$

Corollary 2. *1Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ with $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars $m<M$ and $\Phi\in\mathcal{P}_{N}(B(H),B(K))$. If f is continuous on $[m,M]$ and $f\in\bar{\Delta}_{I}(\gamma,\Gamma)$ for some $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, then for every $x\in H$ with $\|x\|=1$ we have*

$$\begin{aligned}
0&\leq\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle ^{2} \qquad (2.14)\\
&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left\langle \Phi\left(\left|f\left(A\right)-\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle 1_{H}\right|\right)x,x\right\rangle \\
&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left[\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle ^{2}\right]^{1/2}\\
&\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}.
\end{aligned}$$

Proof. From the inequality (2.6) we have for $g=f$ the second and third inequality in (2.14).

Since we showed that

$$\begin{aligned}
0&\leq\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle ^{2}\\
&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\left[\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle ^{2}\right]^{1/2}
\end{aligned}$$

for every $x\in H$ with $\|x\|=1$, then we get

$$\left[\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle ^{2}\right]^{1/2}\leq\frac{1}{2}\left|\Gamma-\gamma\right|$$

for every $x \in H$ with $\|x\| = 1$, which proves the last part of (2.14). $\square$

Corollary 3. *Let A be a selfadjoint operator on the Hilbert space $(H; \langle .,. \rangle)$ with $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars $m < M$ and $\Phi \in \mathcal{P}_N(B(H), B(K))$. If f and g are continuous on $[m, M]$ and $f \in \bar{\Delta}_I(\gamma, \Gamma)$, $g \in \bar{\Delta}_I(\delta, \Delta)$ for some γ, Γ, δ, $\Delta \in \mathbb{C}$, $\gamma \neq \Gamma$, $\delta \neq \Delta$, then for every $x \in H$ with $\|x\| = 1$ we have*

$$
\begin{aligned}
(2.15) \qquad & \left|\langle \Phi(f(A)g(A))x, x\rangle - \langle \Phi(g(A))x, x\rangle \langle \Phi(f(A))x, x\rangle\right| \\
&\leq \frac{1}{2}|\Gamma - \gamma| \langle \Phi(|g(A) - \langle \Phi(g(A))x, x\rangle 1_H|)x, x\rangle \\
&\leq \frac{1}{2}|\Gamma - \gamma| \left[\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle - \langle \Phi(g(A))x, x\rangle^2\right]^{1/2} \\
&\leq \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|.
\end{aligned}
$$

for every $x \in H$ with $\|x\| = 1$.

Remark 1. *If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (2.15) we get*

$$
\begin{aligned}
&\left|\left\langle \Psi^{-1/2}(1_H)\Psi(f(A)g(A))\Psi^{-1/2}(1_H)x, x\right\rangle\right. \\
&\left.-\left\langle \Psi^{-1/2}(1_H)\Psi(g(A))\Psi^{-1/2}(1_H)x, x\right\rangle \left\langle \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H)x, x\right\rangle\right| \\
&\qquad\leq \frac{1}{2}|\Gamma - \gamma| \\
&\times\left\langle \Psi^{-1/2}(1_H)\Psi\left(\left|g(A) - \left\langle \Psi^{-1/2}(1_H)\Psi(g(A))\Psi^{-1/2}(1_H)x, x\right\rangle 1_H\right|\right)\Psi^{-1/2}(1_H)x, x\right\rangle \\
&\qquad\leq \frac{1}{2}|\Gamma - \gamma| \\
&\times\left[\left\langle \Psi^{-1/2}(1_H)\Psi\left(g^2(A)\right)\Psi^{-1/2}(1_H)x, x\right\rangle - \left\langle \Psi^{-1/2}(1_H)\Psi(g(A))\Psi^{-1/2}(1_H)x, x\right\rangle^2\right]^{1/2} \\
&\qquad\qquad\leq \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|
\end{aligned}
$$

for every $x \in H$ with $\|x\| = 1$.

If in this inequality we take

$$
x = \frac{1}{\left\|\Psi^{1/2}(1_H)v\right\|}\Psi^{1/2}(1_H)v,
$$

where $v \in K$ with $v \neq 0$, then we get

$$
\begin{aligned}
(2.16) \qquad & \left|\frac{\langle \Psi(f(A)g(A))v, v\rangle}{\langle \Psi(1_H)v, v\rangle} - \frac{\langle \Psi(f(A))v, v\rangle}{\langle \Psi(1_H)v, v\rangle}\frac{\langle \Psi(g(A))v, v\rangle}{\langle \Psi(1_H)v, v\rangle}\right| \\
&\leq \frac{1}{2}|\Gamma - \gamma| \frac{\left\langle \Psi\left(\left|g(A) - \frac{\langle \Psi(g(A))v, v\rangle}{\langle \Psi(1_H)v, v\rangle}1_H\right|\right)v, v\right\rangle}{\langle \Psi(1_H)v, v\rangle} \\
&\leq \frac{1}{2}|\Gamma - \gamma| \left[\frac{\left\langle \Psi\left(g^2(A)\right)v, v\right\rangle}{\langle \Psi(1_H)v, v\rangle} - \left(\frac{\langle \Psi(g(A))v, v\rangle}{\langle \Psi(1_H)v, v\rangle}\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|,
\end{aligned}
$$

for any $v \in K$ with $v \neq 0$.

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with the property (1.6). With the assumptions in Corollary 3 we have by (2.15), for the normalised positive linear map $\Phi(A) := \sum_{j=1}^{k} P_j^* A P_j$, that

$$\begin{aligned}
&\left|\sum_{j=1}^{k}\left\langle P_j^* f(A) g(A) P_j x, x\right\rangle - \sum_{j=1}^{k}\left\langle P_j^* g(A) P_j x, x\right\rangle \sum_{j=1}^{k}\left\langle P_j^* f(A) P_j x, x\right\rangle\right| \\
&\leq \frac{1}{2}|\Gamma - \gamma| \left\langle \sum_{j=1}^{k} P_j^* \left| g(A) - \left\langle \sum_{\ell=1}^{k} P_\ell^* g(A) P_\ell x, x\right\rangle 1_H \right| P_j x, x\right\rangle \\
&\leq \frac{1}{2}|\Gamma - \gamma| \left[\sum_{j=1}^{k}\left\langle P_j^* g^2(A) P_j x, x\right\rangle - \left(\sum_{j=1}^{k}\left\langle P_j^* g(A) P_j x, x\right\rangle\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|
\end{aligned} \tag{2.17}$$

for every $x \in H$ with $\|x\| = 1$.

Corollary 4. *With the assumptions of Corollary 3 we have the norm inequality*

$$\|\Phi(f(A) g(A))\| \leq \|\Phi(f(A))\| \|\Phi(g(A))\| + \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|. \tag{2.18}$$

Proof. By the inequality (3.1) and the triangle inequality we have

$$\begin{aligned}
&|\langle \Phi(f(A) g(A)) x, x\rangle| - |\langle \Phi(g(A)) x, x\rangle| |\langle \Phi(f(A)) x, x\rangle| \\
&\leq |\langle \Phi(f(A) g(A)) x, x\rangle - \langle \Phi(g(A)) x, x\rangle \langle \Phi(f(A)) x, x\rangle| \\
&\leq \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|
\end{aligned}$$

for every $x \in H$ with $\|x\| = 1$, which implies that

$$\begin{aligned}
&|\langle \Phi(f(A) g(A)) x, x\rangle| \\
&\leq |\langle \Phi(g(A)) x, x\rangle| |\langle \Phi(f(A)) x, x\rangle| + \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|
\end{aligned} \tag{2.19}$$

for every $x \in H$ with $\|x\| = 1$.

By taking the supremum in (2.19) we get

$$\begin{aligned}
&\|\Phi(f(A) g(A))\| \\
&= \sup_{\|x\|=1} |\langle \Phi(f(A) g(A)) x, x\rangle| \\
&\leq \sup_{\|x\|=1} \{|\langle \Phi(g(A)) x, x\rangle| |\langle \Phi(f(A)) x, x\rangle|\} + \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta| \\
&\leq \sup_{\|x\|=1} |\langle \Phi(g(A)) x, x\rangle| \sup_{\|x\|=1} |\langle \Phi(f(A)) x, x\rangle| + \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta| \\
&= \|\Phi(f(A))\| \|\Phi(g(A))\| + \frac{1}{4}|\Gamma - \gamma| |\Delta - \delta|
\end{aligned}$$

and the inequality (2.18) is proved. □

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with the property (1.6). With the assumptions in Corollary 3 we have, by (2.18) for the normalised positive linear map $\Phi(A) := \sum_{j=1}^{k} P_j^* A P_j$, that

$$\left\| \sum_{j=1}^{k} P_j^* f(A) g(A) P_j \right\| \leq \left\| \sum_{j=1}^{k} P_j^* f(A) P_j \right\| \left\| \sum_{j=1}^{k} P_j^* g(A) P_j \right\| + \frac{1}{4} |\Gamma - \gamma| |\Delta - \delta| . \tag{2.20}$$

3. Some Examples

Let $\Phi \in \mathcal{P}_N(B(H), B(K))$. For real valued functions f and g that are continuous on $[m, M]$, by putting $\theta := \min_{t \in [m,M]} f(t)$, $\Theta := \max_{t \in [m,M]} f(t)$, $\psi := \min_{t \in [m,M]} g(t)$ and $\Psi := \max_{t \in [m,M]} g(t)$ then we have by (2.15) that

$$\begin{aligned} &|\langle \Phi(f(A) g(A)) x, x \rangle - \langle \Phi(g(A)) x, x \rangle \langle \Phi(f(A)) x, x \rangle| \\ &\leq \frac{1}{2} (\Theta - \theta) \langle \Phi(|g(A) - \langle \Phi(g(A)) x, x \rangle 1_H|) x, x \rangle \\ &\leq \frac{1}{2} (\Theta - \theta) \left[\langle \Phi(g^2(A)) x, x \rangle - \langle \Phi(g(A)) x, x \rangle^2 \right]^{1/2} \\ &\leq \frac{1}{4} (\Theta - \theta)(\Psi - \psi) \end{aligned} \tag{3.1}$$

for every $x \in H$ with $\|x\| = 1$, where A is a selfadjoint operator on the Hilbert space $(H; \langle ., . \rangle)$ with $\mathrm{Sp}(A) \subseteq [m, M]$.

We say that the functions f, $g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition $(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$ for each t, $s \in [a, b]$.

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

Let $\Phi \in \mathcal{P}_N(B(H), B(K))$ and A is a selfadjoint operator on the Hilbert space $(H; \langle ., . \rangle)$ with $\mathrm{Sp}(A) \subseteq [m, M]$. If f, $g : [m, M] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[m, M]$, then we have the Čebyšev type inequality [30]

$$\langle \Phi(f(A) g(A)) x, x \rangle \geq (\leq) \langle \Phi(f(A)) x, x \rangle \langle \Phi(g(A)) x, x \rangle \tag{3.2}$$

for every $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $m < M$. If A is positive $(m \geq 0)$ and p, $q > 0$, then

$$\begin{aligned} &(0 \leq) \langle \Phi(A^{p+q}) x, x \rangle - \langle \Phi(A^p) x, x \rangle \langle \Phi(A^q) x, x \rangle \\ &\leq \frac{1}{2} (M^p - m^p) \langle \Phi(|A^q - \langle \Phi(A^q) x, x \rangle 1_H|) x, x \rangle \\ &\leq \frac{1}{2} (M^p - m^p) \left[\langle \Phi(A^{2q}) x, x \rangle - \langle \Phi(A^q) x, x \rangle^2 \right]^{1/2} \\ &\leq \frac{1}{4} (M^p - m^p)(M^q - m^q) \end{aligned} \tag{3.3}$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and p, $q < 0$, then

$$
\begin{aligned}
(3.4)\qquad & (0 \leq) \left\langle \Phi\left(A^{p+q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle \left\langle \Phi\left(A^{q}\right) x, x\right\rangle \\
& \leq \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left\langle \Phi\left(\left|A^{q} - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \\
& \leq \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\left\langle \Phi\left(A^{2q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle^{2}\right]^{1/2} \\
& \leq \frac{1}{4} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}}
\end{aligned}
$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p < 0$, $q > 0$ then

$$
\begin{aligned}
(3.5)\qquad & (0 \leq) \left\langle \Phi\left(A^{p}\right) x, x\right\rangle \left\langle \Phi\left(A^{q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{p+q}\right) x, x\right\rangle \\
& \leq \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left\langle \Phi\left(\left|A^{q} - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \\
& \leq \frac{1}{2} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left[\left\langle \Phi\left(A^{2q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle^{2}\right]^{1/2} \\
& \leq \frac{1}{4} \frac{M^{-p} - m^{-p}}{M^{-p} m^{-p}} \left(M^{q} - m^{q}\right)
\end{aligned}
$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p > 0$, $q < 0$ then

$$
\begin{aligned}
(3.6)\qquad & (0 \leq) \left\langle \Phi\left(A^{p}\right) x, x\right\rangle \left\langle \Phi\left(A^{q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{p+q}\right) x, x\right\rangle \\
& \leq \frac{1}{2} \left(M^{p} - m^{p}\right) \left\langle \Phi\left(\left|A^{q} - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \\
& \leq \frac{1}{2} \left(M^{p} - m^{p}\right) \left[\left\langle \Phi\left(A^{2q}\right) x, x\right\rangle - \left\langle \Phi\left(A^{q}\right) x, x\right\rangle^{2}\right]^{1/2} \\
& \leq \frac{1}{4} \left(M^{p} - m^{p}\right) \frac{M^{-q} - m^{-q}}{M^{-q} m^{-q}}
\end{aligned}
$$

for each $x \in H$ with $\|x\| = 1$.

We notice that the positivity of the quantities in the left hand side of the above inequalities (3.3)-(3.6) follows from the inequality (3.2).

The following particular cases when one function is the power while the second is the logarithm are of interest as well:

Let A be a positive definite operator with $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars $0 < m < M$. If $p > 0$, then for each $x \in H$ with $\|x\| = 1$

$$
\begin{aligned}
&(0 \leq) \left\langle \Phi\left(A^{p} \ln A\right) x, x\right\rangle - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle \left\langle \Phi(\ln A) x, x\right\rangle \\
&\leq \begin{cases} \frac{1}{2}\left(M^{p}-m^{p}\right)\left\langle \Phi\left(\left|\ln A - \left\langle \Phi(\ln A) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \\ \\ \ln \sqrt{\frac{M}{m}} \left\langle \Phi\left(\left|A^{p} - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\left(M^{p}-m^{p}\right)\left[\left\langle \Phi\left(\ln^{2}(A)\right) x, x\right\rangle - \left\langle \Phi(\ln A) x, x\right\rangle^{2}\right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \left[\left\langle \Phi\left(A^{2p}\right) x, x\right\rangle - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle^{2}\right]^{1/2} \end{cases} \\
&\leq \frac{1}{2}\left(M^{p}-m^{p}\right) \ln \sqrt{\frac{M}{m}}.
\end{aligned} \tag{3.7}
$$

If $p < 0$, then for each $x \in H$ with $\|x\| = 1$

$$
\begin{aligned}
&(0 \leq) \left\langle \Phi\left(A^{p}\right) x, x\right\rangle \left\langle \Phi(\ln A) x, x\right\rangle - \left\langle \Phi\left(A^{p} \ln A\right) x, x\right\rangle \\
&\leq \begin{cases} \frac{1}{2} \frac{M^{-p}-m^{-p}}{M^{-p} m^{-p}} \left\langle \Phi\left(\left|\ln A - \left\langle \Phi(\ln A) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \\ \\ \ln \sqrt{\frac{M}{m}} \left\langle \Phi\left(\left|A^{p} - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle 1_{H}\right|\right) x, x\right\rangle \end{cases} \\
&\leq \begin{cases} \frac{1}{2} \frac{M^{-p}-m^{-p}}{M^{-p} m^{-p}} \left[\left\langle \Phi\left(\ln^{2}(A)\right) x, x\right\rangle - \left\langle \Phi(\ln A) x, x\right\rangle^{2}\right]^{1/2} \\ \\ \ln \sqrt{\frac{M}{m}} \left[\left\langle \Phi\left(A^{2p}\right) x, x\right\rangle - \left\langle \Phi\left(A^{p}\right) x, x\right\rangle^{2}\right]^{1/2} \end{cases} \\
&\leq \frac{1}{2} \frac{M^{-p}-m^{-p}}{M^{-p} m^{-p}} \ln \sqrt{\frac{M}{m}}.
\end{aligned} \tag{3.8}
$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa