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OPERATOR QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED TO DAVIS-CHOI-JENSEN'S INEQUALITY FOR POSITIVE MAPS

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ABSTRACT. In this paper we establish some operator quasilinearity properties of some functionals associated to Davis-Choi-Jensen's inequality for positive maps and operator convex (concave) functions. Applications for power function and logarithm are also provided.

1. INTRODUCTION

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [18, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \le B \text{ implies } \Phi(A) \le \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \le A \le \beta 1_H$, then $\alpha 1_K \le \Phi(A) \le \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \le (\ge)(1-\lambda) f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

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The following Jensen's type result is well known [1]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \leq \Psi(f(A)). \tag{1.2}$$

We define by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is positive invertible operator in K and define the functional $\mathbf{F} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\mathbf{F}_{f,A}(\Psi) = \Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H),$$

where $f : I \to \mathbb{R}$ is an operator convex (concave) function on the interval I and A is a selfadjoint operator whose spectrum is contained in I.

In this paper we establish some operator quasilinearity properties of some functionals associated to Davis-Choi-Jensen's inequality (1.2) for positive maps and operator convex (concave) functions. Applications for power function and logarithm are also provided.

2. The Main Results

The following result holds:

Theorem 2. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, then*

$$\mathbf{F}_{f,A}(\Psi_1 + \Psi_2) \leq (\geq)\, \mathbf{F}_{f,A}(\Psi_1) + \mathbf{F}_{f,A}(\Psi_2), \tag{2.1}$$

namely $\mathbf{F}_{f,A}$ is operator subadditive (superadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. We give the proof for the case of operator convex functions. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ then $\Psi_1 + \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and we have

$$\begin{aligned} \mathbf{F}_{f,A}(\Psi_1 + \Psi_2) &= (\Psi_1 + \Psi_2)^{1/2}(1_H) \\ &\cdot f\left((\Psi_1 + \Psi_2)^{-1/2}(1_H)(\Psi_1 + \Psi_2)(A)(\Psi_1 + \Psi_2)^{-1/2}(1_H)\right) \\ &\cdot (\Psi_1 + \Psi_2)^{1/2}(1_H) \end{aligned} \tag{2.2}$$

where by " $\cdot$ " we understand above the usual operator multiplication.

Observe that

$$(2.3)\quad \begin{aligned} &(\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1+\Psi_2)(A)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1(A)+\Psi_2(A))(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_1(A)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &+(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_2(A)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H)\left(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H)\right)\\ &\cdot\Psi_1^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &+(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H)\left(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H)\right)\\ &\cdot\Psi_2^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H). \end{aligned}$$

If we denote by

$$V:=\Psi_1^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)\ \text{ and }\ U:=\Psi_2^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)$$

then

$$V^*:=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H)\ \text{ and }\ U^*:=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H).$$

Also, we have

$$\begin{aligned} V^*V+U^*U&=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_1(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &+(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_2(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1(1_H)+\Psi_2(1_H))(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1+\Psi_2)(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)=1_K \end{aligned}$$

and (2.3) may be written as

$$\begin{aligned} &(\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1+\Psi_2)(A)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &=V^*\left(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H)\right)V\\ &+U^*\left(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H)\right)U \end{aligned}$$

and by taking f and using Hansen-Pedersen-Jensen's inequality for operator convex functions, we have

$$(2.4)\quad \begin{aligned} &f\left((\Psi_1+\Psi_2)^{-1/2}(1_H)(\Psi_1+\Psi_2)(A)(\Psi_1+\Psi_2)^{-1/2}(1_H)\right)\\ &\le V^*f\left(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H)\right)V\\ &+U^*f\left(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H)\right)U\\ &=(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_1^{1/2}(1_H)f\left(\Psi_1^{-1/2}(1_H)\Psi_1(A)\Psi_1^{-1/2}(1_H)\right)\\ &\cdot\Psi_1^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H)\\ &+(\Psi_1+\Psi_2)^{-1/2}(1_H)\Psi_2^{1/2}(1_H)f\left(\Psi_2^{-1/2}(1_H)\Psi_2(A)\Psi_2^{-1/2}(1_H)\right)\\ &\cdot\Psi_2^{1/2}(1_H)(\Psi_1+\Psi_2)^{-1/2}(1_H). \end{aligned}$$

Finally, by multiplying both sides of (2.4) by $(\Psi_1+\Psi_2)^{1/2}(1_H)$ we get

$$\begin{aligned}\mathbf{F}_{f,A}(\Psi_1+\Psi_2) &\le \Psi_1^{1/2}(1_H)\, f\left(\Psi_1^{-1/2}(1_H)\,\Psi_1(A)\,\Psi_1^{-1/2}(1_H)\right)\Psi_1^{1/2}(1_H)\\ &\quad+\Psi_2^{1/2}(1_H)\, f\left(\Psi_2^{-1/2}(1_H)\,\Psi_2(A)\,\Psi_2^{-1/2}(1_H)\right)\Psi_2^{1/2}(1_H)\\ &=\mathbf{F}_{f,A}(\Psi_1)+\mathbf{F}_{f,A}(\Psi_2)\end{aligned}$$

and the proof is concluded. □

Corollary 1. *Let $f: I\to\mathbb{R}$ be an operator convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ and $\lambda\in[0,1]$, then*

$$\mathbf{F}_{f,A}((1-\lambda)\Psi_1+\lambda\Psi_2)\le(\ge)(1-\lambda)\mathbf{F}_{f,A}(\Psi_1)+\lambda\mathbf{F}_{f,A}(\Psi_2), \tag{2.5}$$

namely $\mathbf{F}_{f,A}$ is operator convex (concave) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

Proof. If Ψ_1, $\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ and $\lambda\in[0,1]$, then $(1-\lambda)\Psi_1+\lambda\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ and we have by (2.1) that

$$\begin{aligned}\mathbf{F}_{f,A}((1-\lambda)\Psi_1+\lambda\Psi_2) &\le (\ge)\,\mathbf{F}_{f,A}((1-\lambda)\Psi_1)+\mathbf{F}_{f,A}(\lambda\Psi_2)\\ &= (1-\lambda)\mathbf{F}_{f,A}(\Psi_1)+\lambda\mathbf{F}_{f,A}(\Psi_2)\end{aligned}$$

since $\mathbf{F}_{f,A}$ is positive homogeneous on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, namely

$$\mathbf{F}_{f,A}(\alpha\Psi)=\alpha\mathbf{F}_{f,A}(\Psi)$$

for any $\alpha>0$ and $\Psi\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. □

For Ψ_1, $\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ we denote that $\Psi_2\succ_I\Psi_1$ if $\Psi_2-\Psi_1\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. This means that $\Psi_2-\Psi_1$ is a linear positive functional and $\Psi_2(1_H)-\Psi_1(1_H)\in\mathcal{B}^{++}(K)$.

Corollary 2. *Let $f: I\to[0,\infty)$ be an operator concave function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ_1, $\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2\succ_I\Psi_1$ then*

$$\mathbf{F}_{f,A}(\Psi_2)\ge\mathbf{F}_{f,A}(\Psi_1), \tag{2.6}$$

namely $\mathbf{F}_{f,A}$ is operator monotonic nondecreasing in the order "$\succ_I$" of $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

Proof. Let Ψ_1, $\Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2\succ_I\Psi_1$, then by (2.1) we have

$$\mathbf{F}_{f,A}(\Psi_2)=\mathbf{F}_{f,A}(\Psi_1+\Psi_2-\Psi_1)\ge\mathbf{F}_{f,A}(\Psi_1)+\mathbf{F}_{f,A}(\Psi_2-\Psi_1)$$

implying that

$$\mathbf{F}_{f,A}(\Psi_2)-\mathbf{F}_{f,A}(\Psi_1)\ge\mathbf{F}_{f,A}(\Psi_2-\Psi_1).$$

Since f is positive and $\Psi_{2,1}:=\Psi_2-\Psi_1\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_{2,1}(1_H)=\Psi_2(1_H)-\Psi_1(1_H)\in\mathcal{B}^{++}(K)$ it follows that

$$f\left(\Psi_{2,1}^{-1/2}(1_H)\,\Psi_{2,1}(A)\,\Psi_{2,1}^{-1/2}(1_H)\right)\ge 0$$

and by multiplying both sides by $\Psi_{2,1}^{1/2}(1_H)$ we get that $\mathbf{F}_{f,A}(\Psi_2-\Psi_1)\ge 0$ and the inequality (2.6) is proved. □

We have

Corollary 3. *Let $f : I \to [0, \infty)$ be an operator concave function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$ then*

$$T\mathbf{F}_{f,A}(\Upsilon) \geq \mathbf{F}_{f,A}(\Psi) \geq t\mathbf{F}_{f,A}(\Upsilon). \tag{2.7}$$

The proof follows by (2.6) on taking first $\Psi_2 = T\Upsilon$, $\Psi_1 = \Psi$ and then $\Psi_2 = \Psi$, $\Psi_1 = t\Upsilon$ and by the positive homogeneity of $\mathbf{F}_{f,A}$.

We consider now the functional $\mathbf{J}_{f,A} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ defined by

$$\begin{aligned} \mathbf{J}_{f,A}(\Psi) & \\ &:= \Psi(f(A)) - \mathbf{F}_{f,A}(\Psi) \\ &= \Psi(f(A)) - \Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H). \end{aligned} \tag{2.8}$$

We can state the following result:

Theorem 3. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in I. Then the functional $\mathbf{J}_{f,A}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, it is positive homogeneous and concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$. $\mathbf{J}_{f,A}$ is also superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.*

Proof. We consider only the operator convex case. The positivity of $\mathbf{J}_{f,A}$ on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ is equivalent to Davis-Choi-Jensen's inequality for general positive linear maps (1.2). The positive homogeneity follows by the same property of $\mathbf{F}_{f,A}$ and the definition of $\mathbf{J}_{f,A}$.

If Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0, 1]$, then by Corollary 1 we have

$$\begin{aligned} &\mathbf{J}_{f,A}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &= ((1-\lambda)\Psi_1 + \lambda\Psi_2)(f(A)) - \mathbf{F}_{f,A}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &\geq (1-\lambda)\Psi_1(f(A)) + \lambda\Psi_2(f(A)) - (1-\lambda)\mathbf{F}_{f,A}(\Psi_1) - \lambda\mathbf{F}_{f,A}(\Psi_2) \\ &= (1-\lambda)[\Psi_1(f(A)) - \mathbf{F}_{f,A}(\Psi_1)] + \lambda[\Psi_2(f(A)) - \mathbf{F}_{f,A}(\Psi_2)] \\ &= (1-\lambda)\mathbf{J}_{f,A}(\Psi_1) + \lambda\mathbf{J}_{f,A}(\Psi_2) \end{aligned}$$

that proves the operator concavity of $\mathbf{J}_{f,A}$.

The operator superadditivity follows in a similar way and we omit the details. □

Corollary 4. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$ then*

$$T\mathbf{J}_{f,A}(\Upsilon) \geq \mathbf{J}_{f,A}(\Psi) \geq t\mathbf{J}_{f,A}(\Upsilon) \tag{2.9}$$

or, equivalently,

$$\begin{aligned} T(\Upsilon(f(A)) - \mathbf{F}_{f,A}(\Upsilon)) &\geq \Psi(f(A)) - \mathbf{F}_{f,A}(\Psi) \\ &\geq t(\Upsilon(f(A)) - \mathbf{F}_{f,A}(\Upsilon)) \geq 0. \end{aligned} \tag{2.10}$$

The inequality (2.10) has been obtained in [5] in an equivalent form for operator concave function f and normalised functionals Ψ and Υ.

Now, assume that A a selfadjoint operator whose spectrum is contained in $[m, M]$ for some real constants $M > m$. If f is convex, then for any $t \in [m, M]$ we have

$$f(t) \leq \frac{(M-t)f(m) + (t-m)f(M)}{M-m}. \tag{2.11}$$

If A a selfadjoint operator whose spectrum is contained in $[m,M]$, then $m1_H \le A \le M1_H$ and by taking the map Ψ we get $m\Psi(1_H) \le \Psi(A) \le M\Psi(1_H)$ for $\Psi \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. This is equivalent to

$$m1_K \le \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) \le M1_K.$$

By using the continuous functional calculus, we have by (2.11) that

$$\begin{aligned} & f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \\ & \le \frac{1}{M-m}\left[f(m)\left(M1_K - \Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\right] \\ & + \frac{1}{M-m}\left[f(M)\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) - m1_K\right)\right]. \end{aligned}$$

By multiplying this inequality both sides with $\Psi^{1/2}(1_H)$ we get the inequality

$$\mathbf{F}_{f,A}(\Psi) \le \mathbf{T}_{f,A}(\Psi), \tag{2.12}$$

where

$$\mathbf{T}_{f,A}(\Psi) := \frac{f(m)(M\Psi(1_H) - \Psi(A)) + f(M)(\Psi(A) - m\Psi(1_H))}{M-m}$$

is a trapezoidal type functional. We observe that $\mathbf{T}_{f,A}$ is *additive* and *positive homogeneous* on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

We define the functional $\mathbf{D}_{f,A} : \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)] \to \mathcal{B}(K)$

$$\begin{aligned} \mathbf{D}_{f,A}(\Psi) &:= \mathbf{T}_{f,A}(\Psi) - \mathbf{F}_{f,A}(\Psi) \\ &= \frac{f(m)(M\Psi(1_H) - \Psi(A)) + f(M)(\Psi(A) - m\Psi(1_H))}{M-m} \\ &- \Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H). \end{aligned} \tag{2.13}$$

We observe that if f is convex (concave) on $[m,M]$ and $m1_H \le A \le M1_H$, then

$$\mathbf{D}_{f,A}(\Psi) \ge (\le)\, 0 \text{ for any } \Psi \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)].$$

We have:

Theorem 4. *Let $f : I \to \mathbb{R}$ be an operator convex (concave) function on the interval $[m,M]$ and A a selfadjoint operator whose spectrum is contained in $[m,M]$. Then the functional $\mathbf{D}_{f,A}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, it is positive homogeneous and operator concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. $\mathbf{D}_{f,A}$ is also operator superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.*

The proof is similar to the one from Theorem 3 and we omit the details.

Corollary 5. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and A a selfadjoint operator whose spectrum is contained in I. If Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$ then*

$$T\mathbf{D}_{f,A}(\Upsilon) \ge \mathbf{D}_{f,A}(\Psi) \ge t\mathbf{D}_{f,A}(\Upsilon) \tag{2.14}$$

or, equivalently,

$$
\begin{aligned}
(2.15)\quad & T\left[\frac{f(m)\left(M\Upsilon(1_H)-\Upsilon(A)\right)+f(M)\left(\Upsilon(A)-m\Upsilon(1_H)\right)}{M-m}-\mathbf{F}_{f,A}(\Upsilon)\right]\\
&\geq \frac{f(m)\left(M\Psi(1_H)-\Psi(A)\right)+f(M)\left(\Psi(A)-m\Psi(1_H)\right)}{M-m}-\mathbf{F}_{f,A}(\Psi)\\
&\geq t\left[\frac{f(m)\left(M\Upsilon(1_H)-\Upsilon(A)\right)+f(M)\left(\Upsilon(A)-m\Upsilon(1_H)\right)}{M-m}-\mathbf{F}_{f,A}(\Upsilon)\right]\\
&\geq 0.
\end{aligned}
$$

3. Examples for Power Function and Logarithm

It is well known that the function $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(x) = x^\nu$ for $\nu \in (0,1)$ is operator concave and positive on $[0,\infty)$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ defined by

$$\mathbf{F}_{\nu,A}(\Psi) = \Psi^{1/2}(1_H)\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)^\nu \Psi^{1/2}(1_H)$$

where A is a positive operator on H.

Assume that C, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators [12]

$$C\nabla_\nu B := (1-\nu)C + \nu B,$$

the *weighted operator arithmetic mean* and

$$C\sharp_\nu B := C^{1/2}\left(C^{-1/2}BC^{-1/2}\right)^\nu C^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we write $C\nabla B$ and $C\sharp B$ for brevity, respectively.

The definition $C\sharp_\nu B$ can be extended accordingly for any real number ν.

Using this notation, we observe that

$$\mathbf{F}_{\nu,A}(\Psi) = \Psi(1_H)\sharp_\nu\Psi(A). \tag{3.1}$$

In particular, for $\nu = \frac{1}{2}$ we have

$$\mathbf{F}_{\frac{1}{2},A}(\Psi) = \Psi(1_H)\sharp\Psi(A).$$

Using the results from the previous section for the operator concave function f_ν we have that $\mathbf{F}_{\nu,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order " $\succ_I$ " and we have the inequality

$$T\Upsilon(1_H)\sharp_\nu\Upsilon(A) \geq \Psi(1_H)\sharp_\nu\Psi(A) \geq t\Upsilon(1_H)\sharp_\nu\Upsilon(A), \tag{3.2}$$

where Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$.

We notice that the operator concavity, superadditivity and monotonicity may be also derived from the corresponding properties of weighted operator geometric mean, see [18, p. 146].

If we consider the functional

$$\mathbf{J}_{\nu,A}(\Psi) := \Psi(A^\nu) - \Psi(1_H)\sharp_\nu\Psi(A), \tag{3.3}$$

then $\mathbf{J}_{\nu,A}$ is *negative, operator convex and operator subadditive* on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

Also, if $0 < m1_H \le A \le M1_H$, then we can consider the functional

$$\mathbf{D}_{\nu,A}(\Psi) := \frac{m^{\nu}(M\Psi(1_H) - \Psi(A)) + M^{\nu}(\Psi(A) - m\Psi(1_H))}{M-m} - \Psi(1_H)\sharp_{\nu}\Psi(A) \tag{3.4}$$

and from the above section we conclude that $\mathbf{D}_{\nu,A}$ is *negative, operator convex and operator subadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Now, consider the function $\Phi_p(t) = t^p$ that is operator convex on $(0,\infty)$ if either $1 \le p \le 2$ or $-1 \le p \le 0$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ defined by

$$\begin{aligned}\mathbf{F}_{p,A}(\Psi) &= \Psi^{1/2}(1_H)\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)^p\Psi^{1/2}(1_H)\\ &= \Psi(1_H)\sharp_p\Psi(A),\end{aligned}$$

where A is a positive definite operator on H.

In particular, we have

$$\begin{aligned}\mathbf{F}_{2,A}(\Psi) &= \Psi^{1/2}(1_H)\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)^2\Psi^{1/2}(1_H)\\ &= \Psi(A)\Psi^{-1}(1_H)\Psi(A)\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_{-1,A}(\Psi) &= \Psi^{1/2}(1_H)\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)^{-1}\Psi^{1/2}(1_H)\\ &= \Psi(1_H)\Psi^{-1}(A)\Psi(1_H).\end{aligned}$$

From the above section we can infer that $\mathbf{F}_{p,A}$ is *positive, operator convex and subadditive* on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

For Ψ_1, $\Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ we have by the properties of $\mathbf{F}_{p,A}$ that the scalar valued function

$$\rho_{p,A}(\Psi) := \|\mathbf{F}_{p,A}(\Psi)\| = \|\Psi(1_H)\sharp_p\Psi(A)\|$$

is subadditive and positive homogeneous on $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$.

Consider the functional

$$\mathbf{J}_{p,A}(\Psi) := \Psi(A^p) - \Psi(1_H)\sharp_p\Psi(A). \tag{3.5}$$

If A is positive definite and either $1 \le p \le 2$ or $-1 \le p \le 0$, then the functional $\mathbf{J}_{p,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order " $\succ_I$ " and we have the inequality

$$\begin{aligned}T[\Upsilon(A^p) - \Upsilon(1_H)\sharp_p\Upsilon(A)] &\ge \Psi(A^p) - \Psi(1_H)\sharp_p\Psi(A)\\ &\ge t[\Upsilon(A^p) - \Upsilon(1_H)\sharp_p\Upsilon(A)] \ge 0\end{aligned} \tag{3.6}$$

where Ψ, $\Upsilon \in \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$.

Also, if $0 < m1_H \le A \le M1_H$, then by considering the functional

$$\mathbf{D}_{p,A}(\Psi) := \frac{m^{p}(M\Psi(1_H) - \Psi(A)) + M^{p}(\Psi(A) - m\Psi(1_H))}{M-m} - \Psi(1_H)\sharp_{p}\Psi(A) \tag{3.7}$$

we have that $\mathbf{D}_{p,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order " $\succ_I$ " and we have the inequality

$$\begin{aligned}
&T\left[\frac{m^p(M\Upsilon(1_H)-\Upsilon(A))+M^p(\Upsilon(A)-m\Upsilon(1_H))}{M-m}-\Upsilon(1_H)\sharp_p\Upsilon(A)\right]\\
&\geq\frac{m^p(M\Psi(1_H)-\Psi(A))+M^p(\Psi(A)-m\Psi(1_H))}{M-m}-\Psi(1_H)\sharp_p\Psi(A)\\
&\geq t\left[\frac{m^p(M\Upsilon(1_H)-\Upsilon(A))+M^p(\Upsilon(A)-m\Upsilon(1_H))}{M-m}-\Upsilon(1_H)\sharp_p\Upsilon(A)\right]\\
&\geq 0,
\end{aligned}$$

where Ψ, $\Upsilon\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, t, $T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$.

It is well known that the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$ is operator concave on $(0,\infty)$. We consider the functional on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ defined by

$$\mathbf{F}_{\ln,A}(\Psi)=\Psi^{1/2}(1_H)\ln\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H)$$

where A is a positive definite operator on H.

Kamei and Fujii [8], [9] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B):=A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2}, \tag{3.8}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [17].

If $B\geq A$ and A is positive and invertible, then $A^{-1/2}BA^{-1/2}\geq I$ and by the continuous functional calculus we have $\ln\left(A^{-1/2}BA^{-1/2}\right)\geq 0$, which implies by multiplying both sides with $A^{1/2}$ that $S(A|B)\geq 0$.

For some recent results on relative operator entropy see [3]-[4], [10]-[11] and [13]-[14].

Using the relative operator entropy notation, we have

$$\mathbf{F}_{\ln,A}(\Psi)=S(\Psi(1_H)|\Psi(A)),$$

where A is a positive definite operator on H and $\Psi\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

By using the properties established in the previous section applied for the operator concave function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$, we have that $\mathbf{F}_{\ln,A}$ is *operator concave and operator superadditive* on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. These properties may be also derived from the corresponding properties of the relative operator entropy, see for instance [18, p. 153].

Moreover, if $\Psi\succ_I\Upsilon$ then

$$\mathbf{F}_{\ln,A}(\Psi)-\mathbf{F}_{\ln,A}(\Upsilon)\geq\mathbf{F}_{\ln,A}(\Psi-\Upsilon) \tag{3.9}$$

and, in addition, if $\Psi(A)+\Upsilon(1_H)\geq\Upsilon(A)+\Psi(1_H)$ then

$$\mathbf{F}_{\ln,A}(\Psi)\geq\mathbf{F}_{\ln,A}(\Upsilon). \tag{3.10}$$

The function $f(t)=-\ln t$, $t>0$ is operator convex. If we consider now the functional

$$\mathbf{J}_{-\ln,A}(\Psi):=S(\Psi(1_H)|\Psi(A))-\Psi(\ln(A)), \tag{3.11}$$

then from the above section we can infer that $\mathbf{J}_{-\ln,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order " $\succ_I$ " and we have the inequality

$$T\left(S\left(\Upsilon\left(1_H\right)|\Upsilon\left(A\right)\right)-\Upsilon\left(\ln\left(A\right)\right)\right)\geq S\left(\Psi\left(1_H\right)|\Psi\left(A\right)\right)-\Psi\left(\ln\left(A\right)\right)$$
$$\geq t\left(S\left(\Upsilon\left(1_H\right)|\Upsilon\left(A\right)\right)-\Upsilon\left(\ln\left(A\right)\right)\right)\geq 0 \tag{3.12}$$

provided that $\Psi,\ \Upsilon\in\mathfrak{P}_I\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$, $t,\ T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$.

Consider also the functional

$$\mathbf{D}_{-\ln,A}\left(\Psi\right):=S\left(\Psi\left(1_H\right)|\Psi\left(A\right)\right)$$
$$-\frac{\ln m(M\Psi\left(1_H\right)-\Psi\left(A\right))+\ln M\left(\Psi\left(A\right)-m\Psi\left(1_H\right)\right)}{M-m} \tag{3.13}$$

for $\Psi\in\mathfrak{P}_I\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$. Therefore, we have that $\mathbf{D}_{-\ln,A}$ is *positive, operator concave, operator superadditive, operator monotonic nondecreasing* in the order " $\succ_I$ " and we have the inequality

$$T\left[S\left(\Upsilon\left(1_H\right)|\Upsilon\left(A\right)\right)-\frac{\ln m(M\Upsilon\left(1_H\right)-\Upsilon\left(A\right))+\ln M\left(\Upsilon\left(A\right)-m\Upsilon\left(1_H\right)\right)}{M-m}\right]$$
$$\geq S\left(\Psi\left(1_H\right)|\Psi\left(A\right)\right)-\frac{\ln m(M\Psi\left(1_H\right)-\Psi\left(A\right))+\ln M\left(\Psi\left(A\right)-m\Psi\left(1_H\right)\right)}{M-m}$$
$$\geq t\left[S\left(\Upsilon\left(1_H\right)|\Upsilon\left(A\right)\right)-\frac{\ln m(M\Upsilon\left(1_H\right)-\Upsilon\left(A\right))+\ln M\left(\Upsilon\left(A\right)-m\Upsilon\left(1_H\right)\right)}{M-m}\right]$$
$$\geq 0$$

provided that $\Psi,\ \Upsilon\in\mathfrak{P}_I\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$, $t,\ T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$.

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