Received 24/04/16

SEVERAL REFINEMENTS OF POWER SERIES INEQUALITIES AND APPLICATIONS

LOREDANA CIURDARIU

ABSTRACT. The aim of this paper is to present some applications of several new Young-type inequalities given by Alzer, H., Fonseca, C. M. and Kovacec, A. for power series using a method given in a paper of Ibrahim, A., Dragomir S. S., and Darus M. Also some applications to special functions are given.

1. Introduction

The famous Young's inequality, as a classical result, state that:

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

when a and b are positive numbers, $a \neq b$ and $\nu \in (0,1)$.

There are many interesting generalizations of this well-known inequality and its reverse, see for example [6, 7, 1] and references therein.

As in [1], we consider $A_{\nu}(a,b) = \nu a + (1-\nu)b$, and $G_{\nu}(a,b) = a^{\nu}b^{1-\nu}$.

More recently, in [1] are given new results which extend many generalizations of Young's inequality given before. This result is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah, [6], [7].

We recall the results in order to use them in the next section.

The following inequality will be a very important tool in the demonstration of our next theorems where are given some improvements of inequalities given in [5], using the method given in [5].

Theorem 1. *Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

We consider as in [5], an analytic function defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Date: April 20, 2016.

2000 *Mathematics Subject Classification.* 26D20.

Key words and phrases. Young inequality, power series,

RGMIA Res. Rep. Coll. 19 (2016), Art. 64

with real coefficients and convergent on the unit disk $D(0,R),\ R>0$. Let $f_A(z)$ is a new power series defined by $\sum_{n=0}^{\infty}|a|_n z^n$ where $a_n=|a_n|sgn(a_n)$ where $sgn(x)$ is the real signum function as in [5]. The power series $f_A(z)$ has the same radius of convergence as the original power series $f(z)$.

2. **The Young-type inequalities for power series**

If we take in Theorem 1, $\lambda=n\in\mathbf{N}^*$, $\nu=\frac{1}{p}$, $\tau=\frac{1}{p_1}$, a^p instead of a and b^q instead of b then we obtain the following inequality:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}a^{pl}b^{q(n-l)}-a^{\frac{p}{p_1}n}b^{\frac{q}{q_1}n}\right]<$$

$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}a^{pl}b^{q(n-l)}-a^n b^n<$$

$$<\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}a^{pl}b^{q(n-l)}-a^{\frac{p}{p_1}n}b^{\frac{q}{q_1}n}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$.

Next inequalities, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6 and Theorem 7 from below are new variants of Theorem 1, Theorem 2, and Theorem 3 from [5] when $n\in\mathbf{N}$. Then some applications to special functions are given in Corollary 3 and Corollary 4.

Theorem 2. *Let $f(z)=\sum_{n=0}^{\infty}p_n z^n$ and $g(z)=\sum_{n=0}^{\infty}q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $a,\ b\in\mathbf{C}$, $a,\ b\neq 0$ so that $|a|^{pn},\ |a|^{qn},\ |b|^{pn},\ |b|^{qn}\in D(0,R)$ and if $p,\ q,\ p_1,\ q_1$ are like in previous inequality and in addition $1<p_1<p$ then we have:*

$$\left(\frac{p_1}{p}\right)^n[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{pl}|b|^{q(n-l)})g_A(|a|^{q(n-l)}|b|^{pl})-$$

$$-f_A(|a|^{\frac{p}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n})]<$$

$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}f_A(|a|^{pl}|b|^{q(n-l)})g_A(|a|^{q(n-l)}|b|^{pl})-f_A(|a|^n|b|^n)g_A(|a|^n|b|^n)<$$

$$<\left(\frac{q_1}{q}\right)^n[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{pl}|b|^{q(n-l)})g_A(|a|^{q(n-l)}|b|^{pl})-$$

$$-f_A(|a|^{\frac{p}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n})].$$

Proof. We start the proof taking into account that the hypothesis $|a|^{pn},\ |a|^{qn},\ |b|^{pn},\ |b|^{qn}\in D(0,R)$ implies the following inclusions $|a|^{pl}|b|^{q(n-l)},\ |a|^{q(n-l)}|b|^{pl},\ |a|^{\frac{p}{p_1}n}|b|^{\frac{q}{q_1}n},\ |a|^n|b|^n\in D(0,R)$, $l=\overline{0,n}$ by calculus.

We use the same method as in [5]. Thus we choose $a=|a|^j|b|^k,\ b=|a|^k|b|^j,\ j,\ k\in\{0,1,2,...,n\}$ in previous inequality and we have:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}|a|^{jpl}|b|^{kpl}|a|^{kq(n-l)}|b|^{jq(n-l)}-|a|^{j\frac{p}{p_1}n}|b|^{k\frac{p}{p_1}n}|a|^{k\frac{q}{q_1}n}|b|^{j\frac{q}{q_1}n}\right]<$$

$$< \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p^l q^{n-l}} |a|^{jpl} |b|^{kpl} |b|^{jq(n-l)} |a|^{kq(n-l)} - |a|^{jn} |b|^{kn} |a|^{kn} |b|^{jn} <$$

$$< \left(\frac{q_1}{q}\right)^n \left[\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p_1^l q_1^{n-l}} |a|^{jpl} |b|^{kpl} |a|^{kq(n-l)} |b|^{jq(n-l)} - |a|^{j\frac{p}{p_1}n} |b|^{k\frac{p}{p_1}n} |a|^{k\frac{q}{q_1}n} |b|^{j\frac{q}{q_1}n} \right].$$

We multiply last inequality by positive quantities $|p_j||q_k|$ and then summing over j and k from 0 to m we obtain:

$$\left(\frac{p_1}{p}\right)^n [\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p_1^l q_1^{n-l}} \sum_{j=0}^{m} |p_j||a|^{jpl} |b|^{jq(n-l)} \sum_{k=0}^{m} |q_k||b|^{kpl} |a|^{kq(n-l)} -$$

$$- \sum_{j=0}^{m} |p_j||a|^{j\frac{p}{p_1}n} |b|^{j\frac{q}{q_1}n} \sum_{k=0}^{m} |q_k||b|^{k\frac{p}{p_1}n} |a|^{k\frac{q}{q_1}n}] <$$

$$< \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p^l q^{n-l}} \sum_{j=0}^{m} |p_j||a|^{jpl} |b|^{jq(n-l)} \sum_{k=0}^{m} |q_k||b|^{kpl} |a|^{kq(n-l)} -$$

$$- \sum_{j=0}^{m} |p_j||a|^{jn} |b|^{jn} \sum_{k=0}^{m} |q_k||a|^{kn} |b|^{kn} <$$

$$< \left(\frac{q_1}{q}\right)^n [\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p_1^l q_1^{n-l}} \sum_{j=0}^{m} |p_j||a|^{jpl} |b|^{jq(n-l)} \sum_{k=0}^{m} |q_k||b|^{kpl} |a|^{kq(n-l)} -$$

$$- \sum_{j=0}^{m} |p_j||a|^{j\frac{p}{p_1}n} |b|^{j\frac{q}{q_1}n} \sum_{k=0}^{m} |q_k||b|^{k\frac{p}{p_1}n} |a|^{k\frac{q}{q_1}n}].$$

Taking above the limit when $m \to \infty$ we get the desired inequality, because all the series whose partial sums are involved are convergenton the disk $D(0, R)$.

∎

Theorem 3. *Let* $f(z) = \sum_{n=0}^{\infty} p_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} q_n z^n$ *be two power series with real coefficients and convergent on the open disk* $D(0, R)$, $R > 0$. *If* $a, b \in \mathbf{C}$, $a, b \neq 0$ *so that* $|a|^{pn}, |a|^{qn}, |b|^{pn}, |b|^{qn} \in D(0, R)$. *If* p, q, p_1, q_1 *be like in previous inequality and in addition* $1 < p_1 < p$. *Let also* $f(z) = \sum_{n=0}^{\infty} p_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} q_n z^n$ *be two power series with real coefficients and convergent on the open disk* $D(0, R)$, $R > 0$. *If* $a, b \in \mathbf{C}$, $a, b \neq 0$ *so that* $\in D(0, R)$. *Then the following inequality takes place:*

$$\left(\frac{p_1}{p}\right)^n [\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{pl} |b|^{p(n-l)}) g_A(|a|^{q(n-l)} |b|^{ql}) -$$

$$- f_A(|a|^{\frac{p}{p_1}n} |b|^{\frac{p}{q_1}n}) g_A(|a|^{\frac{q}{q_1}n} |b|^{\frac{q}{p_1}n})] <$$

$$< \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p^l q^{n-l}} f_A(|a|^{pl} |b|^{p(n-l)}) g_A(|a|^{q(n-l)} |b|^{ql}) - f_A(|a|^n |b|^{n(p-1)}) g_A(|a|^n |b|^{n(q-1)}) <$$

$$< \left(\frac{q_1}{q}\right)^n [\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{pl} |b|^{p(n-l)}) g_A(|a|^{q(n-l)} |b|^{ql}) -$$

$$- f_A(|a|^{\frac{p}{p_1}n} |b|^{\frac{p}{q_1}n}) g_A(|a|^{\frac{q}{q_1}n} |b|^{\frac{q}{p_1}n})].$$

Proof. We use the hypothesis $|a|^{pn},\ |a|^{qn},\ |b|^{pn},\ |b|^{qn} \in D(0,R)$ which implies the following inclusions $|a|^{pl}|b|^{p(n-l)},\ |a|^{q(n-l)}|b|^{ql},\ |a|^{\frac{p}{p_1}n}|b|^{\frac{p}{q_1}n},\ |a|^{\frac{q}{q_1}n}|b|^{\frac{q}{p_1}n},\ |a|^{n}|b|^{n(p-1)}$, $|a|^{n}|b|^{n(q-1)} \in D(0,R),\ l=\overline{0,n}$ by calculus.

Now we choose $a = \frac{|a|^j}{|b|^j}$ and $b = \frac{|a|^k}{|b|^k}$ and use the same method like in Theorem 2.

∎

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty} p_n z^n$ and $g(z)=\sum_{n=0}^{\infty} q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0,R),\ R>0$. If $a,\ b \in \mathbf{C}$, $a,\ b \neq 0$ so that $|a|^{pn},\ |a|^{qn},\ |b|^{pn},\ |b|^{qn} \in D(0,R)$. Let also $p,\ q,\ p_1,\ q_1$ be like in previous inequality and in addition $1<p_1<p$. Then the following inequality takes place:*

$$\left(\frac{p_1}{p}\right)^n \Big[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{ql}|b|^{p(n-l)}) g_A(|a|^{q(n-l)}|b|^{pl}) -$$

$$- f_A(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}) g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n})\Big] <$$

$$< \sum_{l=0}^{n} \binom{n}{l} \frac{1}{p^l q^{n-l}} f_A(|a|^{ql}|b|^{p(n-l)}) g_A(|a|^{q(n-l)}|b|^{pl}) - f_A(|a|^{n(q-1)}|b|^{n(p-1)}) g_A(|a|^{n}|b|^{n}) <$$

$$< \left(\frac{q_1}{q}\right)^n \Big[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{ql}|b|^{p(n-l)}) g_A(|a|^{q(n-l)}|b|^{pl}) -$$

$$- f_A(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}) g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n})\Big].$$

Proof. We use the hypothesis $|a|^{pn},\ |a|^{qn},\ |b|^{pn},\ |b|^{qn} \in D(0,R)$ which implies the following inclusions $|a|^{ql}|b|^{p(n-l)},\ |a|^{q(n-l)}|b|^{pl},\ |a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n},\ |a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n},\ |a|^{n(q-1)}|b|^{n(p-1)}$, $|a|^{n}|b|^{n} \in D(0,R),\ l=\overline{0,n}$ by calculus.

In this case we take $a = \frac{|b|^k}{|b|^j}$ and $b = \frac{|a|^k}{|a|^j}$ and use the same method as in Theorem 2. ∎

Theorem 5. *Let $f(z), g(z)$ and $p,\ q,\ p_1,\ q_1$ be as in Theorem 2, $a, b \in \mathbf{C}$ with $a \neq b,\ |a|^{qn},\ |b|^{pn},\ |a|^{\frac{2}{q}pn},\ |b|^{\frac{2}{p}qn}$ and in addition $1<p_1<p$. Then one has the inequality:*

$$\left(\frac{p_1}{p}\right)^n \Big[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{q(n-l)}|b|^{pl}) g_A(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}) -$$

$$- f_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}) g_A(|a|^{\frac{2}{q}\frac{p}{p_1}n}|b|^{\frac{2}{p}\frac{q}{q_1}n})\Big] <$$

$$< \sum_{l=0}^{n} \binom{n}{l} \frac{1}{p^l q^{n-l}} f_A(|a|^{q(n-l)}|b|^{pl}) g_A(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}) - f_A(|a|^{n}|b|^{n}) g_A(|a|^{\frac{2}{q}n}|b|^{\frac{2}{p}n}) <$$

$$< \left(\frac{q_1}{q}\right)^n \Big[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A(|a|^{q(n-l)}|b|^{pl}) g_A(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}) -$$

$$- f_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}) g_A(|a|^{\frac{2}{q}\frac{p}{p_1}n}|b|^{\frac{2}{p}\frac{q}{q_1}n})\Big].$$

Proof. First we check that the corresponding products of $|a|$ and $|b|$ are in $D(0,R)$ using hypothesis and then by choosing $a=|a|^{\frac{2}{q}k}|b|^{j}$ and $b=|a|^{j}|b|^{\frac{2}{p}k}$ and using the same method as in [5] we will obtain the desired inequality.
∎

Theorem 6. *Let $f(z), g(z)$ and $p,\ q,\ p_1,\ q_1$ be as in Theorem 2, $a,\ b\in\mathbf{C}$ with $a\neq b$, $|a|^{qn},\ |a|^{2\frac{q}{p}n},\ |b|^{pn},\ |b|^{2\frac{p}{q}n}\in D(0,R)$ and in addition $1<p_1<p$. Then one has the inequality:*

$$\left(\frac{p_1}{p}\right)^n[\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{ql}|b|^{p(n-l)})g_A(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl})-$$

$$-f_A(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n})g_A(|a|^{\frac{2}{q}\frac{p}{p_1}n}|b|^{\frac{2}{p}\frac{q}{q_1}n})]<$$

$$<\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p^l q^{n-l}}f_A(|a|^{ql}|b|^{p(n-l)})g_A(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl})-f_A(|a|^{n(q-1)}|b|^{n(p-1)})g_A(|a|^{\frac{2}{p}n}|b|^{\frac{2}{q}n})<$$

$$<\left(\frac{q_1}{q}\right)^n[\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{ql}|b|^{p(n-l)})g_A(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl})-$$

$$-f_A(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n})g_A(|a|^{\frac{2}{q}\frac{p}{p_1}n}|b|^{\frac{2}{p}\frac{q}{q_1}n})].$$

Proof. We also check the corresponding products of $|a|$ and $|b|$ are in $D(0,R)$ using hypothesis. This time we replace a by $\frac{|b|^{\frac{2}{q}k}}{|b|^{j}}$ and b by $\frac{|a|^{\frac{2}{p}k}}{|a|^{j}}$ in order to obtain the inequality of the theorem. ∎

Theorem 7. *Let $f(z), g(z)$ and $p,\ q,\ p_1,\ q_1$ be as in Theorem 2, $a,\ b\in\mathbf{C}$, $a\neq b$ and in addition $1<p_1<p$. Then the following inequality takes place:*

$$\left(\frac{p_1}{p}\right)^n[\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{2l}|b|^{q(n-l)})g_A(|a|^{2(n-l)}|b|^{pl})-$$

$$-f_A(|a|^{\frac{2}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{2}{q_1}n}|b|^{\frac{p}{p_1}n})]<$$

$$<\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p^l q^{n-l}}f_A(|a|^{2l}|b|^{q(n-l)})g_A(|a|^{2(n-l)}|b|^{pl})-f_A(|a|^{\frac{2}{p}n}|b|^{n})g_A(|a|^{\frac{2}{q}n}|b|^{n})<$$

$$<\left(\frac{q_1}{q}\right)^n[\sum_{l=0}^{n}\left(\begin{array}{c} n\\ l\end{array}\right)\frac{1}{p_1^l q_1^{n-l}}f_A(|a|^{2l}|b|^{q(n-l)})g_A(|a|^{2(n-l)}|b|^{pl})-$$

$$-f_A(|a|^{\frac{2}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{2}{q_1}n}|b|^{\frac{p}{p_1}n})],$$

if $|a|^{2n},\ |b|^{qn},\ |b|^{pn}\in D(0,R)$.

Proof. It is easily to check that the corresponding products of $|a|$ and $|b|$ are in $D(0,R)$ using hypothesis. Then we choose $a=|a|^{\frac{2}{p}j}|b|^{k}$ and $b=|a|^{\frac{2}{q}k}|b|^{j}$ and repeat the same method as above. ∎

Corollary 1. *Let $f(z)$, $g(z)$, a, b and p, q, p_1, q_1 be as in Theorem 2, $1 < p_1 < p$ and in addition, we take $n = 1$ in the same Theorem 2. Then we obtain:*

$$\frac{p_1}{p}\left[\frac{1}{q_1}f_A(|b|^q)g_A(|a|^q)+\frac{1}{p_1}f_A(|a|^p)g_A(|b|^p)-f_A(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})g_A(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]<$$

$$<\frac{1}{q}f_A(|b|^q)g_A(|a|^q)+\frac{1}{p}f_A(|a|^p)g_A(|b|^p)-f_A(|ab|)g_A(|ab|)<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}f_A(|b|^q)g_A(|a|^q)+\frac{1}{p_1}f_A(|a|^p)g_A(|b|^p)-f_A(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})g_A(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right].$$

Corollary 2. *(a) If we take $f(z) = g(z)$ in previous inequality, in Corollary 1, then we obtain:*

$$\frac{p_1}{p}\left[\frac{1}{q_1}f_A(|b|^q)f_A(|a|^q)+\frac{1}{p_1}f_A(|a|^p)f_A(|b|^p)-f_A(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})f_A(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]<$$

$$<\frac{1}{q}f_A(|b|^q)f_A(|a|^q)+\frac{1}{p}f_A(|a|^p)f_A(|b|^p)-f_A^2(|ab|)<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}f_A(|b|^q)f_A(|a|^q)+\frac{1}{p_1}f_A(|a|^p)f_A(|b|^p)-f_A(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})f_A(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right].$$

(b) We can also state the following form of the left side of the inequality from Theorem 2, a form where appears the functions $f(z)$ and $g(z)$.

$$|f(a^nb^n)g(a^nb^n)|<$$

$$<[1-(\frac{p_1}{p})^n]\sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\left(\frac{1}{p^lq^{n-l}}-\frac{1}{p_1^lq_1^{n-l}}\right)f_A(|a|^{pl}|b|^{q(n-l)})g_A(|a|^{q(n-l)}|b|^{pl})+$$

$$+\left(\frac{p_1}{p}\right)^n f_A(|a|^{\frac{p}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}).$$

(c) The inequality from Theorem 2 can be also rewritten like below:

$$f_A(|a|^n|b|^n)g_A(|a|^n|b|^n)-\left(\frac{p_1}{p}\right)^n f_A(|a|^{\frac{p}{p_1}n}|b|^{\frac{q}{q_1}n})g_A(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n})<$$

$$<[1-(\frac{p_1}{p})^n]\sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\left(\frac{1}{p^lq^{n-l}}-\frac{1}{p_1^lq_1^{n-l}}\right)f_A(|a|^{pl}|b|^{q(n-l)})g_A(|a|^{q(n-l)}|b|^{pl}).$$

Corollary 3. *(a) If we consider the function $f(z) = \sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$, $z \in$ $\mathbf{C}$ then $f_A(z) = \sinh(z)$, $z \in \mathbf{C}$ and under condition from Corollary 2 (a), inequality becomes:*

$$\frac{p_1}{p}\left[\frac{1}{q_1}\sinh(|b|^q)\sinh(|a|^q)+\frac{1}{p_1}\sinh(|a|^p)\sinh(|b|^p)-\sinh(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})\sinh(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]<$$

$$<\frac{1}{q}\sinh(|b|^q)\sinh(|a|^q)+\frac{1}{p}\sinh(|a|^p)\sinh(|b|^p)-\sinh^2(|ab|)<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}\sinh(|b|^q)\sinh(|a|^q)+\frac{1}{p_1}\sinh(|a|^p)\sinh(|b|^p)-\sinh(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})\sinh(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right].$$

(b) If $f(z)=\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=f_A(z),\ z\in\mathbf{C}$ *then under conditions from Corollary 2 (a) the inequality will be the following:*

$$\frac{p_1}{p}\left[\frac{1}{q_1}\exp(|b|^q+|a|^q)+\frac{1}{p_1}\exp(|a|^p+|b|^p)-\exp(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}}+|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]<$$

$$<\frac{1}{q}\exp(|b|^q+|a|^q)+\frac{1}{p}\exp(|a|^p+|b|^p)-\exp^2(|ab|)<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}\exp(|b|^q+|a|^q)+\frac{1}{p_1}\exp(|a|^p+|b|^p)-\exp(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}}+|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right].$$

(c) If $f(z)=\frac{1}{1-z}=f_A(z),\ z\in D(0,1)$ *and* $a,\ b$ *are complex numbers as in Theorem 2, then we have:*

$$\frac{p_1}{p}\left[\frac{1}{q_1}\frac{1}{1-|b|^q}\frac{1}{1-|a|^q}+\frac{1}{p_1}\frac{1}{1-|b|^p}\frac{1}{1-|a|^p}-\frac{1}{1-|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}}}\frac{1}{1-|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}}\right]<$$

$$<\frac{1}{q}\frac{1}{1-|b|^q}\frac{1}{1-|a|^q}+\frac{1}{p}\frac{1}{1-|b|^p}\frac{1}{1-|a|^p}-\left(\frac{1}{1-|ab|}\right)^2<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}\frac{1}{1-|b|^q}\frac{1}{1-|a|^q}+\frac{1}{p_1}\frac{1}{1-|b|^p}\frac{1}{1-|a|^p}-\frac{1}{1-|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}}}\frac{1}{1-|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}}\right].$$

Corollary 4. *If* $Li_n(z)$ *is the polylogarithm function, that is* $Li_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n}$ *then we have*

$$\frac{p_1}{p}\left[\frac{1}{q_1}Li_n(|b|^q)Li_n(|a|^q)+\frac{1}{p_1}Li_n(|a|^p)Li_n(|b|^p)-Li_n(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})Li_n(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]<$$

$$<\frac{1}{q}Li_n(|b|^q)Li_n(|a|^q)+\frac{1}{p}Li_n(|a|^p)Li_n(|b|^p)-Li_n^2(|ab|)<$$

$$<\frac{q_1}{q}\left[\frac{1}{q_1}Li_n(|b|^q)Li_n(|a|^q)+\frac{1}{p_1}Li_n(|a|^p)Li_n(|b|^p)-Li_n(|a|^{\frac{p}{p_1}}|b|^{\frac{q}{q_1}})Li_n(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}})\right]$$

for any $a,b\in\mathbf{C},\ a,\ b\neq 0$ *under conditions of Corollary 2 (a) when* $D(0,R)$ *is* $D(0,1)$.

References

[1] Alzer, H., Fonseca, C. M., Kovacec, A., Young-type inequalities and their matrix analogues, Linear and Multilinear Algebra, 63, 3, (2015), 622-635.

[2] Ciurdariu L., Craciunescu A., On spectral representation on gramian normal operators on pseudo-Hilbert spaces, Analele Universitatii de Vest din Timisoara, Seria Matematica Informatica, XLV, 1, pp.131-149, (2007).

[3] Ciurdariu L., Inequalities for selfadjoint operators on Hilbert spaces and pseudo-Hilbert spaces, RGMIA, Res. Rep. Coll., 2015.

[4] Dragomir S. S., Cebysev's type inequalities for functions on selfadjoint operators in Hilbert spaces, Res. Rep. Coll., RGMIA, 2009.

[5] Ibrahim A., Dragomir S. S., Darus M., Power series inequalities via Young's inequality with applications, *Journal of Inequalities and Applications*, 2013, 2013:314.

[6] Kittaneh,F., Manasrah, Y., Reverse Young and Heinz inequalities for matrices, *Linear and Multilinear Algebra*, **59**, 9 (2011), 1031–1037.

[7] Kittaneh, F., Manasrah, Y., Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, **361** (2010), 262–269.

Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara