Received 26/04/16

SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY (s,m)-CONVEX FUNCTIONS IN SECOND SENSE

IMRAN ABBAS BALOCH, İMDAT İŞCAN

Abstract. Authors introduce the concept of harmonically (s,m)-convex functions in second sense in [1].In this article, we establish some Hermite-Hadamard type inequalities of this class of functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex function. Note that some of the classical inequalities for mean can be derived from (1.1) for the appropriate particular selection of mapping f.Both inequalities in (1.1) hold in the reverse direction if f is concave.

In [4], Imdat Iscan introduced the concept of harmonically convex function, and established a variant od Hermite-Hadamard type inequalities which holds for these classes of functions as follows:

Definition 1.1. Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (1.2) is reversed, then f is said to be harmonically concave.

Theorem 1.2. *(see [4]) Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$f(\frac{2ab}{a+b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

Theorem 1.3. *(see [4]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I°,$a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left(\lambda_2|f'(a)|^q + \lambda_3|f'(b)|^q\right)^{\frac{1}{q}}, \tag{1.4}$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_3 = \frac{1}{a(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right).$$

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A51.

Key words and phrases. Harmonically (s,m)-convex function, Hermite-Hadamard type inequalities.

RGMIA Res. Rep. Coll. 19 (2016), Art. 67

Theorem 1.4. *(see [4]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$ and $f' \in [a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1|f'(a)|^q + \mu_2|f'(b)|^q\right)^{\frac{1}{q}}, \tag{1.5}$$

where

$$\mu_1 = \frac{[a^{2-2q} + b^{1-2q}[(b-a)(1-2q)-a]]}{2(b-a)^2(1-q)(1-2q)},$$

$$\mu_2 = \frac{[b^{2-2q} - a^{1-2q}[(b-a)(1-2q)+b]]}{2(b-a)^2(1-q)(1-2q)}.$$

In [5], Imdat Iscan introduced the concept of harmonically s-convex function in second sense as follow:

Definition 1.5. A function $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically s-convex in second sense , if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y) + (1-t)^s f(x) \tag{1.6}$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality in (1.6) is reversed, then f is said to be harmonically s-concave.

Remark 1.6. Note that for $s = 1$, harmonically s-convexity reduces to ordinary harmonically convexity.

In [3], Feixiang Chen and Shanhe Wu generalized Hermite-Hadamard type inequalities given in [4] which hold for harmonically s-convex functions in second sense.

Theorem 1.7. *(see [3]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically s- convex in second sense on $[a,b]$ for some fixed $s \in (0,1]$, $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}C_1^{1-\frac{1}{q}}(a,b)[C_2(s;a,b)|f'(a)|^q + C_3(s;a,b)|f'(b)|^q]^{\frac{1}{q}},$$

where

$$C_1(a,b) = b^{-2}\left({}_2F_1(2,2;3,1-\frac{a}{b}) - {}_2F_1(2,1;2,1-\frac{a}{b}) + \frac{1}{2}{}_2F_1(2,1;3,\frac{1}{2}(1-\frac{a}{b})\right)$$

$$C_2(s,a,b) = b^{-2}\left(\frac{2}{s+2}{}_2F_1(2,s+2;s+3,1-\frac{a}{b}) - \frac{1}{s+1}{}_2F_1(2,s+1;s+2,1-\frac{a}{b})\right.$$
$$\left.+\frac{1}{2^s(s+1)(s+2)}{}_2F_1(2,s+1;s+3,\frac{1}{2}(1-\frac{a}{b}))\right)$$

$$C_3(s,a,b) = b^{-2}\left(\frac{2}{(s+1)(s+2)}{}_2F_1(2,2;s+3,1-\frac{a}{b}) - \frac{1}{s+1}{}_2F_1(2,1;s+2,1-\frac{a}{b})\right.$$
$$\left.+\frac{1}{2}{}_2F_1(2,1;3,\frac{1}{2}(1-\frac{a}{b}))\right)$$

Remark 1.8. Note that for $s = 1$, $C_1(a,b) = \lambda_1$, $C_2(1,a,b) = \lambda_2$ and $C_3(1,a,b) = \lambda_3$. Hence, Theorem 1.3 is particular case of theorem 1.7 for $s = 1$.

Theorem 1.9. *(see [3]) Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically s- convex in second sense on $[a,b]$ for some fixed $s \in (0,1]$, $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\le \frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{s+1}[{}_2F_1(2q,s+1;s+2,1-\frac{a}{b})|f'(b)|^q + {}_2F_1(2q,1;s+2,1-\frac{a}{b})|f'(a)|^q]\right]^{\frac{1}{q}}$$

Remark 1.10. Note that for $s=1$

$$\mu_1=\frac{1}{2b^{2q}}\cdot {}_2F_1(2q,2,3,1-\frac{a}{b}),$$

and

$$\mu_2=\frac{1}{2b^{2q}}\cdot {}_2F_1(2q,1,3,1-\frac{a}{b}).$$

Hence, Theorem 1.4 is particular case of theorem 1.9 for $s=1$.

In ([7]), Jaekeun Park considered the class of (s,m)-convex functions in second sense. This class of function is defined as follow

Definition 1.11. For some fixed $s\in(0,1]$ and $m\in[0,1]$ a mapping $f:I\subset[0,\infty)\to\mathbb{R}$ is said to be (s,m)-convex in the second sense on I if

$$f(tx+m(1-t)y)\le t^s f(x)+m(1-t)^s f(y)$$

holds, for all $x,y\in I$ and $t\in[0,1]$

In ([6]), Imdat Iscan introduced the concept of harmonically (α,m)-convex functions and established some Hermite-Hadamard type inequalities for this class of function. This class of functions is defined as follow

Definition 1.12. The function $f:(0,\infty)\to\mathbb{R}$ is said to be harmonically (α,m)-convex, where $\alpha\in[0,1]$ and $m\in(0,1]$, if

$$(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\le t^\alpha f(x)+m(1-t^\alpha)f(y) \tag{1.7}$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$. If the inequality in (1.7) is reversed, then f is said to be harmonically (α,m)-concave.

In [1] , authors introduce the concept of Harmonically (s,m)-convex functions in second sense which generalize the notion of Harmonically convex and Harmonically s-convex functions in second sense introduced by Imdat Iscan in [4],[5].
In this paper, we establish some results connected with the right side of new inequality similar to (1.1) for this class of functions such that results given by Imdat Iscan [4] , Feixiang Chen and Shanhe Wu [3] are obtained for the particular values of s,m.

Definition 1.13. The function $f:I\subset(0,\infty)\to\mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s\in(0,1]$ and $m\in(0,1]$ if

$$f(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\le t^s f(x)+m(1-t)^s f(y)$$

$\forall x,y\in I$ and $t\in[0,1]$.

Remark 1.14. Note that for $s=1$, (s,m)-convexity reduces to harmonically m-convexity and for $m=1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense (see [5]) and for $s,m=1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity (see [4]).

Proposition 1.15. *Let $f:(0,\infty)\to\mathbb{R}$ be a function*
a) if f is (s,m)-convex function in second sense and non-decreasing, then$f$ is harmonically (s,m)-convex function in second sense.
b) if f is harmonically (s,m)-convex function in second sense and non-increasing, then f is (s,m)-convex function in second sense.

Proof. For all $t\in[0,1]$, $m\in(0,1)$ and $x,y\in I$, we have

$$t(1-t)(x-my)^2\ge 0$$

Hence, the following inequality holds

$$\frac{mxy}{mty+(1-t)x} \le tx + m(1-t)y \tag{1.8}$$

By the inequality (1.8), the proof is completed. □

Remark 1.16. According to proposition 1.15, every non-decreasing (s,m)-convex function in second sense is also harmonically (s,m)-convex function in second sense.

Example 1.17. (see[2]) Let $0 < s < 1$ and $a,b,c \in \mathbb{R}$, then function $f:(0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & x = 0 \\ bx^s + c, & x > 0 \end{cases}$$

is non-decreasing s-convex function in second sense for $b \ge 0$ and $0 \le c \le a$. Hence, by proposition 1.15, f is harmonically $(s,1)$-convex function.

Proposition 1.18. *Let $s \in [0,1]$, $m \in (0,1]$, $f:[a,mb] \subset (0,\infty) \to \mathbb{R}$, be an increasing function and $g:[a,mb] \to [a,mb]$, $g(x) = \frac{mab}{a+mb-x}$, $a < mb$. Then f is harmonically (s,m)-convex in second sense on $[a,mb]$ if and only if fog is (s,m)-convex in second sense on $[a,mb]$.*

Proof. Since

$$(fog)(ta+m(1-t)b) = f\big(\frac{mab}{mbt+(1-t)a}\big) \tag{1.9}$$

for all $t \in [0,1]$, $m \in (0,1]$. The proof is obvious from equality (1.9). □

The following result of the Hermite-Hadamard type holds.

Theorem 1.19. *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a harmonically (s,m)-convex function in second sense with $s \in [0,1]$ and $m \in (0,1]$. If $0 < a < b < \infty$ and $f \in L[a,b]$, then one has following inequality*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left[\frac{f(a)+mf(\frac{b}{m})}{s+1}, \frac{f(b)+mf(\frac{a}{m})}{s+1}\right]$$

Proof. Since, $f: I \subset (0,\infty) \to \mathbb{R}$ is a harmonically (s,m)-convex function in second sense. We have, for $x,y \in I \subset (0,\infty)$

$$f\big(\frac{xy}{ty+(1-t)x}\big) = f\big(\frac{mx\frac{y}{m}}{mt\frac{y}{m}+(1-t)x}\big) \le t^s f(x) + m(1-t)^s f(\frac{y}{m})$$

which gives

$$f\big(\frac{ab}{tb+(1-t)a}\big) \le t^s f(a) + m(1-t)^s f(\frac{b}{m})$$

and

$$f\big(\frac{ab}{ta+(1-t)b}\big) \le t^s f(b) + m(1-t)^s f(\frac{a}{m})$$

for all $t \in [0,1]$. Integrating on $[0,1]$ w.r.t 't', we obtain

$$\int_0^1 f\big(\frac{ab}{tb+(1-t)a}\big)dt \le \frac{f(a)+mf(\frac{b}{m})}{s+1}$$

and

$$\int_0^1 f\big(\frac{ab}{ta+(1-t)b}\big)dt \le \frac{f(b)+mf(\frac{a}{m})}{s+1}$$

However,

$$\int_0^1 f\big(\frac{ab}{tb+(1-t)a}\big)dt = \int_0^1 f\big(\frac{ab}{tb+(1-t)a}\big)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

Hence, required inequality is established. □

Corollary 1.20. *If we take $m=1$ in theorem 1.19, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{s+1}$$

Corollary 1.21. *If we take $s=1$ in theorem 1.19, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \min\left[\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf(\frac{a}{m})}{2}\right]$$

2. Main Results

For finding some new inequalities of Hermite-Hadamard type for the functions whose derivatives are harmonically (s,m)-convex in second sense, we need the following lemma

Lemma 2.1. *Let $f: I\subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on I° with $a<b$. If $f\in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)}f'\big(\frac{ab}{(tb+(1-t)a)}\big)dt$$

Theorem 2.2. *Let $f: I\subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, \frac{b}{m} \in I^\circ$ with $a<b$, $m\in(0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2^{2-\frac{1}{q}}}\left[\rho_1(s,q;a,b)|f'(a)|^q + m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

where

$$\begin{aligned}\rho_1(s,q;a,b) &= \frac{\beta(1,s+2)}{b^{2q}}\cdot{}_2F_1\big(2q,1;s+3;1-\frac{a}{b}\big) - \frac{\beta(2,s+1)}{b^{2q}}\cdot{}_2F_1\big(2q,2;s+3;1-\frac{a}{b}\big)\\ &+ \frac{2^{2q-s}\beta(2,s+1)}{(a+b)^{2q}}\cdot{}_2F_1\big(2q,2;s+3;1-\frac{2a}{a+b}\big)\end{aligned}$$

$$\begin{aligned}\rho_2(s,q;a,b) &= \frac{\beta(s+1,2)}{2^s b^{2q}}\cdot{}_2F_1\big(2q,s+1;s+3,\frac{1}{2}(1-\frac{a}{b})\big) - \frac{\beta(s+1,2)}{b^{2q}}\cdot{}_2F_1\big(2q,s+1;s+3,1-\frac{a}{b}\big)\\ &+ \frac{\beta(s+2,1)}{b^{2q}}\cdot{}_2F_1\big(2q,s+2;s+3,1-\frac{a}{b}\big)\end{aligned}$$

β is Euler Beta function defined by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b;c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1$$

Proof. From above Lemma and using power mean inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| &\leq \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\big(\frac{ab}{tb+(1-t)a}\big)\right|dt\\ &\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\\ &\times \left(\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^{2q}}\left|f'\big(\frac{ab}{tb+(1-t)a}\big)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

Since, $|f'|^q$ is harmonically (s,m)-convex function in second sense, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{ab(b-a)}{2}(\frac{1}{2})^{1-\frac{1}{q}}\bigg(\int_0^1\frac{|1-2t|[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{(tb+(1-t)a)^{2q}}dt\bigg)^{\frac{1}{q}}$$
$$=\frac{ab(b-a)}{2}(\frac{1}{2})^{1-\frac{1}{q}}\left[|f'(a)|^q\int_0^1\frac{|1-2t|t^s}{(tb+(1-t)a)^{2q}}dt+m|f'(\frac{b}{m})|^q\int_0^1\frac{|1-2t|(1-t)^s}{(tb+(1-t)a)^{2q}}dt\right]^{\frac{1}{q}}$$
$$=\frac{ab(b-a)}{2}(\frac{1}{2})^{1-\frac{1}{q}}\left[\rho_1(s,q;a,b)|f'(a)|^q+m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

It is easy to check that

$$\begin{aligned}\int_0^1\frac{|1-2t|t^s}{(tb+(1-t)a)^{2q}}dt &= \int_0^{\frac{1}{2}}\frac{|1-2t|t^s}{(tb+(1-t)a)^{2q}}dt+\int_{\frac{1}{2}}^1\frac{|1-2t|t^s}{(tb+(1-t)a)^{2q}}dt\\ &= \frac{2^{2q-s}\beta(2,s+1)}{(a+b)^{2q}}.{}_2F_1(2q,2;s+3,1-\frac{2a}{a+b})-\frac{\beta(2,s+1)}{b^{2q}}.{}_2F_1(2q,2;s+3,1-\frac{a}{b})\\ &+ \frac{\beta(1,s+2)}{b^{2q}}.{}_2F_1(2q,1;s+3,1-\frac{a}{b})\\ &:= \rho_1(s,q;a,b)\end{aligned}$$

and

$$\begin{aligned}\int_0^1\frac{|1-2t|(1-t)^s}{(tb+(1-t)a)^{2q}}dt &= \frac{\beta(s+1,2)}{2^sb^{2q}}.{}_2F_1(2q,s+1;s+3,\frac{1}{2}(1-\frac{a}{b}))-\frac{\beta(s+2,1)}{b^{2q}}.{}_2F_1(2q,s+2;s+3,(1-\frac{a}{b}))\\ &+ \frac{\beta(s+2,1)}{b^{2q}}.{}_2F_1(2q,s+2;s+3,(1-\frac{a}{b})):=\rho_2(s,q;a,b)\end{aligned}$$

This completes the proof. □

If we take $s=m=1$ in Theorem 2.2, we get the following

Corollary 2.3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be differentiable function on $I\circ$, $a,b\in I\circ$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is $(1,1)$-harmonically convex in second sense or harmonically convex function on $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2^{2-\frac{1}{q}}}\left[\rho_1(1,q;a,b)|f'(a)|^q+\rho_2(1,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.4. *If we take $m=1$ in Theorem 2.2, then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2^{2-\frac{1}{q}}}\left[\rho_1(s,q;a,b)|f'(a)|^q+\rho_2(s,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.5. *If we take $s=1$ in Theorem 2.2, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2^{2-\frac{1}{q}}}\left[\rho_1(1,q;a,b)|f'(a)|^q+m\rho_2(1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Theorem 2.6. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I, $ma,b\in I^\circ$ with $a<b$, $m\in(0,1]$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(s,q;a,b)|f'(a)|^q+m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Proof. From Lemma, Power mean inequality and harmonically (s,m)-convexity in second sense of $|f'|^q$ on $[a,\frac{b}{m}]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{|1-2t|[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{(tb+(1-t)a)^2}dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(s,q;a,b)|f'(a)|^q+m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}
\end{aligned}$$

□

Corollary 2.7. *If we take $m=1$ in Theorem 2.6, then we get*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(s,q;a,b)|f'(a)|^q+\rho_2(s,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}
\end{aligned}$$

This is Theorem 1.7 proved by Feixiang Chen and Shanhe Wu in [3].

Corollary 2.8. *If we take $s=1$ in Theorem 2.6, we get*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(1,q;a,b)|f'(a)|^q+m\rho_2(1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}
\end{aligned}$$

Corollary 2.9. *If we take $s=m=1$ in Theorem 2.6, we get*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(1,q;a,b)|f'(a)|^q+\rho_2(1,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}
\end{aligned}$$

which is Theorem 1.3 proved by Imdat Iscan in [4].

Theorem 2.10. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $ma,b\in I^\circ$ with $a<b$, $m\in(0,1]$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a,\frac{b}{m}]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ with $s\in[0,1]$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\nu_1(s,q;a,b)|f'(a)|^q+m\nu_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}
\end{aligned}$$

where

$$\nu_1(s,q;a,b)=\frac{\beta(1,s+1)}{b^{2q}}.{}_2F_1(2q,1;s+2,1-\frac{a}{b})$$

and

$$\nu_2(s,q;a,b)=\frac{\beta(s+1,1)}{b^{2q}}.{}_2F_1(2q,s+1;s+2,1-\frac{a}{b})$$

Proof. From Lemma, Hölder's inequality and harmonically (s,m)-convexity of $|f'|^q$ on $[a, \frac{b}{m}]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt\right.$$

$$\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|^q\int_0^1 \frac{t^s}{(tb+(1-t)a)^{2q}}dt + m|f'(\frac{b}{m})|^q\int_0^1 \frac{(1-t)^s}{(tb+(1-t)a)^{2q}}dt\right]^{\frac{1}{q}}$$

$$= \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|^q\nu_1(s,q;a,b) + m|f'(\frac{b}{m})|^q\nu_2(s,q;a,b)\right]^{\frac{1}{q}}$$

where an easy calculation gives

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1}$$

$$\int_0^1 \frac{t^s}{(tb+(1-t)a)^{2q}}dt = \frac{\beta(1,s+1)}{b^{2q}}.{}_2F_1(2q,1;s+2,1-\frac{a}{b}) := \nu_1(s,q;a,b)$$

and

$$\int_0^1 \frac{(1-t)^s}{(tb+(1-t)a)^{2q}}dt = \frac{\beta(s+1,1)}{b^{2q}}.{}_2F_1(2q,s=1;s+2,1-\frac{a}{b}) := \nu_2(s,q;a,b)$$

This completes the proof. □

Corollary 2.11. *If we take $m=1$ in Theorem 2.10, then we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2}(\frac{1}{p+1})^{\frac{1}{p}}\left[\nu_1(s,q;a,b)|f'(a)|^q + \nu_2(s,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

this is Theorem 1.9 proved by Feixiang Chen and Shanhe Wu in [3].

Corollary 2.12. *If we take $s=1$ in above Theorem, then we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2}(\frac{1}{p+1})^{\frac{1}{p}}\left[\nu_1(1,q;a,b)|f'(a)|^q + m\nu_2(1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Corollary 2.13. *If we take $s=m=1$ in above Theorem, then we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\leq \frac{ab(b-a)}{2}(\frac{1}{p+1})^{\frac{1}{p}}\left[\nu_1(1,q;a,b)|f'(a)|^q + \nu_2(1,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

This is Theorem 1.4 proved by Imdat Iscan in [4].

References

[1] I. A. Baloch, I.Işcan, Some Ostrowski Type Inequalities For Harmonically (s,m)-convex functoins in Second Sense, International Journal of Analysis,Volume 2015, Article ID 672675, 9 pages. http://dx.doi.org/10.1155/2015/672675

[2] W.W. Breckner,"*Stetigkeitsaussagenfür eine klasse verallgemeinerter konvexer funktonen in topologischen linearen Räumen*,Publ.Inst.Math.(Beograd),23,(1978),13-20.

[3] F.Chen and S.Wu,"Some Hermite-Hadamard type inequalities for harmonically s-convex functions,"The scientific World Journal,vol2014,Article ID 279158,7pages,2014.

[4] *İ.İ*scan,"Hermite-Hadamard type inequalities for harmonically convex functions,"Hacettepe Journal of Mathematics and statistics, vol 43 (6) (2014),935-942.

[5] *İ.İ*scan,"Ostrowski type inequalities for harmonically s-convex functions,"Konuralp journal of Mathematics, 3(1) (2015), 63-74 .

[6] *İ.İ*scan,"Hermite-Hadamard type inequalities for harmonically (α,m) convex functions,"Hacettepe Journal of Mathematics and statistics. Accepted for publication "arxiv:1307.5402v2[math.CA]".

[7] J.Park,"New Ostrowski-Like type inequalities for differentiable (s,m)-convex mappings,International journal of pure and applied mathematics,vol.78 No.8 2012,1077-1089.

[8] E.Set, *İ.İ*scan, F.Zehir, On some new inequalities of Hermite-Hadamard type involving harmonically convex function via fractional integrals,"Konuralp journal of Mathematics, 3(1) (2015), 42-55 .

Imran Abbas Baloch, Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
E-mail address: iabbasbaloch@gmail.com, iabbasbaloch@sms.edu.pk

İmdat İşcan, Department of Mathematics, Faculty of Arts and Sciences,, Giresun University, 28200, Giresun, TURKEY
E-mail address: imdat.iscan@giresun.edu.tr