

**NEW HERMITE HADAMARD TYPE INEQUALITIES FOR
TWICE DIFFERENTIABLE CONVEX MAPPINGS VIA GREEN
FUNCTION AND APPLICATIONS**

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ABSTRACT. We derive some Hermite Hamamard type integral inequalities for functions whose second derivatives absolute value are convex. Some error estimates for the trapezoidal formula are obtained. Finally, some natural applications to special means of real numbers are given.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [2], [5]):

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The inequalities (1.1) have grown into a significant pillar for mathematical analysis and optimization, besides, by looking into a variety of settings, these inequalities are found to have a number of uses. What is more, for a specific choice of the function f , many inequalities with special means are obtainable. Hermite Hadamard's inequality (1.1), for example, is significant in its rich geometry and hence there are many studies on it to demonstrate its new proofs, refinements, extensions and generalizations. You can check ([1], [2], [5], [4] and [10]-[15]) and the references included there.

In [1], Dragomir and Agarwal proved the following inequality connected with the right part of (1.1).

Theorem 1. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$(1.2) \quad \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|).$$

In [6], Pearce and Pécarić proved the following inequality.

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Theorem 2. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, for some $q \geq 1$, then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

In [8], Sarikaya and Aktan established some inequalities related to Hermite-Hadamard's inequality for functions whose second derivatives are convex.

Theorem 3. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a, b]$ and $0 \leq \lambda \leq 1$ where $a, b \in I^\circ$ with $a < b$. If $|f''|$ is a convex on $[a, b]$, then the following inequalities hold:

$$\begin{aligned} & \left| (\lambda - 1) f \left(\frac{a+b}{2} \right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \begin{cases} \frac{(b-a)^2}{12} \left[\left(\lambda^4 + (1+\lambda)(1-\lambda)^3 + \frac{5\lambda-3}{4} \right) |f'(a)| \right. \\ \quad \left. + \left(\lambda^4 + (2-\lambda)\lambda^3 + \frac{1-3\lambda}{4} \right) |f'(b)| \right], & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(b-a)^2(3\lambda-1)}{48} [|f'(a)| + |f'(b)|], & \text{for } \frac{1}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

Remark 1. Under the assumptions Theorem 3 with $\lambda = 1$ in Theorem 3, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{12} \left(\frac{|f''(a)| + |f''(b)|}{2} \right).$$

In [3], Husain et al. proved some inequalities related to Hermite-Hadamard's inequality for s -convex functions:

Theorem 4. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:

$$(1.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{2 \times 6^{\frac{1}{p}}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{(s+2)(s+3)} \right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. If we take $s = 1$ in (1.3), then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}.$$

The interested reader is refer to [3],[7]-[9] for inequalities involving functions whose second derivatives are convex.

In this study, using functions whose second derivatives absolute values are convex, we establish new inequalities that are connected with the right-hand side of Hermite-Hadamard inequality. Then we give some error estimates for trapezoidal quadrature formula by using these inequalities. Finally, we obtain some applications of these inequalities for special means are provided.

2. MAIN RESULTS

In order to prove our main results we need the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a, b]$, the interior of the interval I , where $a, b \in I^\circ$ with $a < b$. Then the following identity holds:*

$$(2.1) \quad \frac{1}{b-a} \int_a^b \int_a^b G(x, y) f''(y) dy dx = \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2}$$

where

$$G(x, y) := \begin{cases} \frac{(x-a)(y-b)}{(b-a)}, & a \leq x \leq y \leq b \\ \frac{(x-b)(y-a)}{(b-a)}, & a \leq y \leq x \leq b. \end{cases}$$

Proof. By integration by parts, we have the following identity:

$$\int_a^b G(x, y) f''(y) dy = G(x, y) f'(y) \Big|_{y=a}^b dx - \int_a^b \frac{\partial G(x, y)}{\partial y} f'(y) dy dx.$$

Because of $G(x, b) = G(x, a) = 0$, we get

$$(2.2) \quad \begin{aligned} \int_a^b G(x, y) f''(y) dy &= -\frac{x-b}{b-a} \int_a^x f'(y) dy - \frac{x-a}{b-a} \int_x^b f'(y) dy \\ &= f(x) - \frac{b-x}{b-a} f(a) - \frac{x-a}{b-a} f(b). \end{aligned}$$

Integrating both sides of (2.2) with respect to x over $[a, b]$ and rewritten, we have

$$\int_a^b \int_a^b G(x, y) f''(y) dy dx = \int_a^b f(x) dx - (b-a) \frac{f(a) + f(b)}{2}.$$

Hence, the proof is completed. \square

We obtain trapezoid inequality by using convexity of $|f''|$.

Theorem 5. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f''|$ is convex on $[a, b]$, then, for all $y \in [a, b]$, the following inequality holds:*

$$(2.3) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{12} \left(\frac{|f''(a)| + |f''(b)|}{2} \right).$$

Proof. We take absolute value of (2.1), we find that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b |G(x, y)| |f''(y)| dy dx \\ & = \int_a^b \int_a^x \frac{(b-x)(y-a)}{(b-a)^2} |f''(y)| dy dx + \int_a^b \int_x^b \frac{(b-y)(x-a)}{(b-a)^2} |f''(y)| dy dx. \end{aligned}$$

Since $|f''(y)|$ is convex on $[a, b]$, we have

$$(2.4) \quad \left| f'' \left(\frac{b-y}{b-a} a + \frac{y-a}{b-a} b \right) \right| \leq \frac{b-y}{b-a} |f''(a)| + \frac{y-a}{b-a} |f''(b)|.$$

From (2.4), it follows that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \\ & \leq |f''(a)| \int_a^b \int_a^x \frac{(b-x)(y-a)(b-y)}{(b-a)^3} dy dx + |f''(b)| \int_a^b \int_a^x \frac{(b-x)(y-a)^2}{(b-a)^3} dy dx \\ & \quad |f''(a)| \int_a^b \int_x^b \frac{(x-a)(b-y)^2}{(b-a)^3} dy dx + |f''(b)| \int_a^b \int_x^b \frac{(x-a)(b-y)(y-a)}{(b-a)^3} dy dx. \end{aligned}$$

If we calculate the above integrals and also use elementary analysis, then we easily deduce required inequality (2.3) which completes the proof. \square

We derive trapezoid inequality by using convexity of $|f''|^q$.

Theorem 6. Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, for all $y \in [a, b]$, then the following inequality holds:

$$(2.5) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} [B(p+2, p+1) + B(p+1, p+2)]^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $B(p, q)$ is Euler's Beta function.

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$\begin{aligned}
(2.6) \quad & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \\
& \leq \frac{1}{b-a} \int_a^b \int_a^b |G(x, y)| |f''(y)| dy dx \\
& \leq \frac{1}{b-a} \left(\int_a^b \int_a^b |G(x, y)|^p dy dx \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |f''(y)|^q dy dx \right)^{\frac{1}{q}}.
\end{aligned}$$

We need to calculate above integrals to prove the theorem.

Let's calculate first integral

$$\begin{aligned}
& \int_a^b \int_a^b |G(x, y)|^p dy dx \\
& = \int_a^b \int_a^x \frac{(b-x)^p (y-a)^p}{(b-a)^p} dy dx + \int_a^b \int_x^b \frac{(b-y)^p (x-a)^p}{(b-a)^p} dy dx \\
& = \int_a^b \frac{(b-x)^p (x-a)^{p+1}}{(b-a)^p (p+1)} dx + \int_a^b \frac{(x-a)^p (b-x)^{p+1}}{(b-a)^p (p+1)} dx.
\end{aligned}$$

Using the change of the variables $x-a = (b-a)u$ and $b-x = (b-a)u$ for the above integrals, respectively, we write

$$(2.7) \quad \int_a^b \int_a^b |G(x, y)|^p dy dx = \frac{(b-a)^{p+2}}{p+1} [B(p+2, p+1) + B(p+1, p+2)]$$

where $B(p, q) = \int_0^1 u^{p-1} (1-u)^{q-1} du$, ($p, q > 0$).

Since $|f''(t)|^q$ is convex on $[a, b]$, we have

$$(2.8) \quad \left| f'' \left(\frac{b-y}{b-a}a + \frac{y-a}{b-a}b \right) \right|^q \leq \frac{b-y}{b-a} |f''(a)|^q + \frac{y-a}{b-a} |f''(b)|^q.$$

From (2.8), it follows that

$$\begin{aligned}
(2.9) \quad & \int_a^b \int_a^b |f''(y)|^q dy dx \\
& \leq |f''(a)|^q \int_a^b \int_a^b \frac{b-y}{b-a} dy dx + |f''(b)|^q \int_a^b \int_a^b \frac{y-a}{b-a} dy dx \\
& = \frac{|f''(a)|^q + |f''(b)|^q}{2} (b-a)^2.
\end{aligned}$$

If we substitute (2.7) and (2.9) in (2.6), then we easily deduce required inequality (2.5) which completes the proof. \square

Now, we establish trapezoid inequality in a different way by using convexity of $|f''|^q$.

Theorem 7. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, for all $y \in [a, b]$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b-a)^{2+\frac{1}{q}}}{(p+1)^{\frac{1}{p}} 2^{\frac{1}{q}}} \left\{ [B(p+2, p+1)]^{\frac{1}{p}} \left(\frac{2|f''(a)|^q + |f''(b)|^q}{3} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + [B(p+1, p+2)]^{\frac{1}{p}} \left(\frac{|f''(a)|^q + 2|f''(b)|^q}{3} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $B(p, q)$ is Euler's Beta function.

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b |G(x, y)| |f''(y)| dy dx \\ & = \int_a^b \int_a^x \frac{(b-x)(y-a)}{(b-a)^2} |f''(y)| dy dx + \int_a^b \int_x^b \frac{(b-y)(x-a)}{(b-a)^2} |f''(y)| dy dx \\ & \leq \left(\int_a^b \int_a^x \frac{(b-x)^p (y-a)^p}{(b-a)^{2p}} dy dx \right)^{\frac{1}{p}} \left(\int_a^b \int_a^x |f''(y)|^q dy dx \right)^{\frac{1}{q}} \\ & \quad + \left(\int_a^b \int_x^b \frac{(b-y)^p (x-a)^p}{(b-a)^{2p}} dy dx \right)^{\frac{1}{p}} \left(\int_a^b \int_x^b |f''(y)|^q dy dx \right)^{\frac{1}{q}}. \end{aligned}$$

If we calculate the above interals utilizing the convexity of $|f''(y)|^q$, from (2.8), then we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \\ & \leq \left(\frac{(b-a)^2}{p+1} B(p+2, p+1) \right)^{\frac{1}{p}} \left(\frac{(b-a)^3}{2} \times \frac{2|f''(a)|^q + |f''(b)|^q}{3} \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{(b-a)^2}{p+1} B(p+1, p+2) \right)^{\frac{1}{p}} \left(\frac{(b-a)^3}{2} \times \frac{|f''(a)|^q + 2|f''(b)|^q}{3} \right)^{\frac{1}{q}}. \end{aligned}$$

The proof is thus completed. □

Theorem 8. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q \geq 1$, for all $y \in [a, b]$, then the following inequality holds:*

$$(2.10) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^2}{12 \cdot 2^{\frac{1}{p}}} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p} + \frac{1}{q}$ can be written instead of 1. Using Holder's inequality, we find that

$$\begin{aligned} (2.11) \quad & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \\ & \leq \frac{1}{b-a} \int_a^b \int_a^b |G(x, y)|^{\frac{1}{p} + \frac{1}{q}} |f''(y)| dy dx \\ & \leq \frac{1}{b-a} \left(\int_a^b \int_a^b |G(x, y)| dy dx \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |G(x, y)| |f''(y)|^q dy dx \right)^{\frac{1}{q}}. \end{aligned}$$

We calculate the above integrals, respectively,

$$(2.12) \quad \int_a^b \int_a^b |G(x, y)| dy dx = \frac{(b-a)^3}{24}.$$

Since $|f''(t)|^q$ is convex on $[a, b]$, we get

$$(2.13) \quad \int_a^b \int_a^b |G(x, y)| |f''(y)|^q dy dx = \frac{(b-a)^3}{12} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2} \right).$$

Substituting the equalities (2.12) and (2.13) in (2.11), we obtain the inequality (2.10) which completes the proof. □

3. APPLICATIONS TO NUMERICAL INTEGRATION

We now consider applications of the integral inequalities developed in the previous section, to obtain estimates of composite quadrature rules which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) and consider the trapezoidal rule

$$(3.1) \quad A_T(f, I_n) = \sum_{i=0}^{n-1} \left[\frac{f(x_i) + f(x_{i+1})}{2} \right] h_i$$

where $h_i = (x_{i+1} - x_i)$, $i = 0, \dots, n-1$.

Theorem 9. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f''|$ is convex on $[a, b]$, for all $y \in [a, b]$, then we have the representation*

$$\int_a^b f(x) dx = A_T(f, I_n) + R_T(f, I_n)$$

where $A_T(f, I_n)$ is as defined in (3.1) and the remainder satisfies the estimations:

$$(3.2) \quad |R_T(f, I_n)| \leq \frac{1}{12} \sum_{i=0}^{n-1} \left[\frac{|f''(x_i)| + |f''(x_{i+1})|}{2} \right] h_i^3$$

for $i = 0, \dots, n-1$.

Proof. Applying Theorem 5 on the interval $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$, we obtain

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(x) dx - \left[\frac{f(x_i) + f(x_{i+1})}{2} \right] h_i \right| \\ & \leq \frac{1}{12} \left(\frac{|f''(x_i)| + |f''(x_{i+1})|}{2} \right) h_i^3 \end{aligned}$$

for $i = 0, \dots, n-1$.

Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (3.2). \square

Remark 3. *If we consider the inequality (2.5), then we get the quadrature formula*

$$\int_a^b f(x) dx = A_T(f, I_n) + R_T(f, I_n)$$

where $A_T(f, I_n)$ is trapezoidal formula and the remainder term satisfies the estimate

$$\begin{aligned} |R_T(f, I_n)| & \leq \frac{[B(p+2, p+1) + B(p+1, p+2)]^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}} \\ & \times \sum_{i=0}^{n-1} \left[\frac{|f''(x_i)|^q + |f''(x_{i+1})|^q}{2} \right]^{\frac{1}{q}} h_i^3. \end{aligned}$$

Remark 4. If we consider the inequality (2.10), then we get the quadrature formula

$$\int_a^b f(x)dx = A_T(f, I_n) + R_T(f, I_n)$$

where $A_T(f, I_n)$ is trapezoidal formula and the remainder term satisfies the estimate

$$|R_T(f, I_n)| \leq \frac{1}{12 \cdot 2^{\frac{1}{p}}} \sum_{i=0}^{n-1} \left[\frac{|f''(x_i)|^q + |f''(x_{i+1})|^q}{2} \right]^{\frac{1}{q}} h_i^3.$$

4. APPLICATIONS TO SOME SPECIAL MEANS

Let us recall the following means:

(a) The *arithmetic mean*:

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0$$

(b) The *Geometric mean*:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0$$

(c) The *Harmonic mean*:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0$$

(d) The *Logarithmic mean*:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

(e) The *Identric mean*:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

(f) The *p-logarithmic mean*:

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$.

The following simple relationships are known in literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

(1) Consider the mapping $f : (0, \infty) \rightarrow \mathbb{R}$, $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $|n(n-1)| \geq 3$.

Then, we have, for $0 < a < b$,

$$\frac{1}{b-a} \int_a^b f(t) dt = L_n^n$$

and

$$\frac{f(a) + f(b)}{2} = A(a^n, b^n).$$

Using the inequality (2.3), we have the result:

$$|L_n^n - A(a^n, b^n)| \leq |n(n-1)| \frac{(b-a)^2}{12} A(a^{n-2}, b^{n-2}).$$

Also, from (2.5), we get the inequality

$$\begin{aligned} |L_n^n - A(a^n, b^n)| &\leq |n(n-1)| [B(p+2, p+1) + B(p+1, p+2)]^{\frac{1}{p}} \\ &\quad \times \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}} [A(a^{q(n-2)}, b^{q(n-2)})]^{\frac{1}{q}}. \end{aligned}$$

(2) Consider the mapping $f(x) = \frac{1}{x}$, $x \in [a, b] \subset (0, \infty)$.

Then, we have

$$\frac{1}{b-a} \int_a^b f(t) dt = L_{-1}^{-1} = \frac{1}{L}$$

and

$$\frac{f(a) + f(b)}{2} = H^{-1}.$$

Using the inequality (2.3), we have the result:

$$|L_{-1}^{-1} - H^{-1}| \leq \frac{(b-a)^2}{12} H^{-1}(a^3, b^3).$$

Also, from (2.10), we get the inequality

$$|L_{-1}^{-1} - H^{-1}| \leq \frac{(b-a)^2}{12 \cdot 2^{\frac{1}{p}}} [H^{-1}(a^{3q}, b^{3q})]^{\frac{1}{q}}.$$

(3) Consider the mapping $f(x) = -\ln x$, $x \in [a, b] \subset (0, \infty)$.

Then, we have

$$\frac{1}{b-a} \int_a^b f(t) dt = -\ln I(a, b)$$

and

$$\frac{f(a) + f(b)}{2} = -A(\ln a, \ln b).$$

Using the inequality (2.3), we have the result:

$$|A(\ln a, \ln b) - \ln I(a, b)| \leq \frac{(b-a)^2}{12} H^{-1}(a^2, b^2).$$

Also, from (2.10), we get the inequality

$$|A(\ln a, \ln b) - \ln I(a, b)| \leq \frac{(b-a)^2}{12 \cdot 2^{\frac{1}{p}}} [H^{-1}(a^{2q}, b^{2q})]^{\frac{1}{q}}.$$

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