NEW WEIGHTED INEQUALITIES FOR HIGHER ORDER DERIVATIVES AND APPLICATIONS

SAMET ERDEN, MEHMET ZEKI SARIKAYA, AND HUSEYIN BUDAK

ABSTRACT. We establish a new Ostrowski type inequality for $n + 1$-times differentiable mappings which are bounded. Then, some new inequalities of Hermite-Hadamard type are obtained for functions whose $n + 1.th$ derivatives in absolute value are convex. Spacial cases of these inequalities reduce some well known inequalities. With the help of obtained inequalities, we give applications for the $k-$moment of random variables.

1. INTRODUCTION

In 1938, Ostrowski established the integral inequality which is one of the fundemental inequalities of mathematic as follows (see, [20]):

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup\limits_{t \in (a,b)} |f'(t)| < \infty$. Then, the inequality holds:

$$(1.1) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [7]):

$$(1.2) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

Inequalities (1.1) and (1.2) have wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence, inequality (1.1) and (1.2) have attracted considerable attention and interest from mathematicans and researchers. Now, we give some inequalities related to (1.1) and (1.2) which were proved in recent years (see, [7], [8], [11], [21], [24], [26]).

In [8], Cerone et.al. proved the following inequalities of Ostrowski type and Hadamard type, respectively.

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Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a, b) *and* $f'' : (a, b) \to \mathbb{R}$ *is bounded, i.e.,* $\|f''\|_\infty = \sup\limits_{t \in (a,b)} |f''(t)| < \infty$. *Then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \left(x - \frac{a+b}{2} \right) f'(x) \right|$$

$$\leq \left[\frac{1}{24}(b-a)^2 + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \|f''\|_\infty$$

$$\leq \frac{(b-a)^2}{6} \|f''\|_\infty$$

for all $x \in [a, b]$.

Corollary 1. *Under the above assumptions, we have the mid-point inequality:*

$$(1.3) \qquad \left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{24} \|f''\|_\infty$$

In [11], Kırmacı proved the following results connected with the left part of (1.2).

Theorem 2. *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|$ *is convex on* $[a, b]$, *then we have*

$$(1.4) \qquad \left| \frac{1}{b-a} \int_a^b f(t)\,dt - f\left(\frac{a+b}{2} \right) \right| \leq \frac{b-a}{8} \left(|f'(a)| + |f'(b)| \right).$$

Theorem 3. *Let* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$, *and let* $p > 1$. *If the mapping* $|f'|^{\frac{p}{p-1}}$ *is convex on* $[a, b]$, *then we have*

$$(1.5) \qquad \left| \frac{1}{b-a} \int_a^b f(t)\,dt - f\left(\frac{a+b}{2} \right) \right|$$

$$\leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{\frac{1}{p}} \left[\left(3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right.$$

$$\left. + \left(|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \right].$$

Sarikaya et. al. pointed out the some inequalities in [24], as follows:

Theorem 4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *with* $f'' \in L_1[a, b]$. *If* $|f''|$ *is convex on* $[a, b]$, *then*

$$(1.6) \qquad \left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{24} \left(\frac{|f''(a)| + |f''(b)|}{2} \right).$$

Theorem 5. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I° *such that* $f'' \in L_1[a, b]$ *where* $a, b \in I$, $a < b$, *If* $|f''|^q$ *is convex on* $[a, b]$, $q > 1$, *then*

$$(1.7) \qquad \left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}.$$

In a recent paper, in [4], Barnett and Dragomir obtained a variety of bounds for the variance and expectation of a continuous random variable whose p.d.f. is defined over a finite interval base on the identity:

$$\int\limits_{a}^{b} (t - m_1)^2 \, f(t) \, dt + (m_1 - b)(m_1 - a) = \int\limits_{a}^{b} (t - a)(t - b) \, f(t) \, dt$$

where $m_1 = \int\limits_{a}^{b} u f(u) \, du$.

In recent years, researchers have studied some integral inequalities by using n-time differentiable functions. For example, Authors gave some Ostrowski type inequalities for mappings whose $n-$th derivatives are bounded in [6] and [29]. Sofo established integral inequalities on L_p norm in [27]. In [22] and [23], the authors deduced midpoint and trapezoidal formula for n-times differentiable mappings, respectively. In [1], [2], [9], [18] and [28], researchers obtained some integral inequalities for functions whose obsolute value of $n-$th derivatives are convex, $s-$convex, $m-$convex and $(\alpha, m)-$convex. Kechriniotis and Theodorou proved some integral inequalities via n-times differentiable functions and gave some applications for probability density function in [10]. In [16], [17] and [19], Latif and Dragomir established Hermite-Hadamard type inequalities for n-times differentiable.

In this study, first of all, we derive an identity for $n + 1$ times differentiable functions. Then, some weighted integral inequalities are obtained by using this identity. Some results presented in earlier works related to these inequalities are given. Finally, some applications for random variable whose probability density functions are bounded and their derivatives absolute are convex on the interval of real numbers.

2. Some inequalities for the moments

In order to prove weighted integral inequalities, we need the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $n + 1$ times differentiable function on I°, $a, b \in I^\circ$ with $a < b$, $f^{(n+1)}$ is absolutely continuous on $[a,b]$ and let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$. Then the following equality holds:*

$$(2.1) \qquad \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int\limits_{a}^{b} w(t) f(t) \, dt = \int\limits_{a}^{b} P_w(x,t) f^{(n+1)}(t) \, dt$$

where $n \in \mathbb{N}$, $M_k(x)$ is defined by

$$M_k(x) = \int\limits_{a}^{b} (u - x)^k f(u) \, du, \quad k = 0, 1, 2, \ldots$$

and

$$(2.2) \qquad P_w(x,t) := \begin{cases} \dfrac{1}{n!} \int\limits_{a}^{t} (u - t)^n \, w(u) \, du, & a \leq t < x \\[2em] \dfrac{1}{n!} \int\limits_{b}^{t} (u - t)^n \, w(u) \, du, & x \leq t \leq b. \end{cases}$$

Proof. By integration by parts, we have

$$\int_a^b P_w(x,t) f^{(n+1)}(t) dt$$

$$= \frac{1}{n!} \int_a^x \left(\int_a^t (u-t)^n w(u) du \right) f^{(n+1)}(t) dt + \frac{1}{n!} \int_x^b \left(\int_b^t (u-t)^n w(u) du \right) f^{(n+1)}(t) dt$$

$$= \frac{1}{n!} \left(\int_a^b (u-x)^n w(u) du \right) f^{(n)}(x) + \frac{1}{(n-1)!} \int_a^x \left(\int_a^t (u-t)^{n-1} w(u) du \right) f^{(n)}(t) dt$$

$$+ \frac{1}{(n-1)!} \int_x^b \left(\int_b^t (u-t)^{n-1} w(u) du \right) f^{(n)}(t) dt.$$

By integration by parts $n-$times, we get

$$\int_a^b P_w(x,t) f^{(n+1)}(t) dt = \frac{M_n(x)}{n!} f^{(n)}(x) + \frac{M_{n-1}(x)}{(n-1)!} f^{(n-1)}(x) + ... + \frac{M_2(x)}{2!} f''(x)$$

$$+ M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t) dt$$

which is the required identity in (2.1). Hence, the proof is completed. $\square$

We establish a new inequality for functions whose $n+1-$th derivatives are bounded

Theorem 6. *Suppose that all the assumptions of Lemma 1 hold. Additionally, we assume that $f^{(n+1)} : (a,b) \to \mathbb{R}$ is bounded, i.e., $\left\| f^{(n+1)} \right\|_\infty = \sup_{t \in (a,b)} \left| f^{(n+1)}(t) \right| < \infty$, then we have the inequality*

$$(2.3) \left| \sum_{k=0}^n \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) dt \right|$$

$$\leq \frac{\left\| f^{(n+1)} \right\|_{[a,b],\infty}}{(n+1)!}$$

$$\times \begin{cases} M_{n+1}(x) & , \text{if } n \text{ is an odd number} \\ \\ \left[\int_x^b (u-x)^{n+1} w(u) du - \int_a^x (u-x)^{n+1} w(u) du \right] & , \text{if } n \text{ is an even number} \end{cases}$$

for all $x \in [a,b]$.

Proof. If we take absolute value of both sides of the equality (2.1), because $f^{(n+1)}$ is a bounded function, we can write

$$\left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) dt \right|$$

$$\leq \frac{1}{n!} \int_a^x \left(\int_a^t (t-u)^n w(u) du \right) \left| f^{(n+1)}(t) \right| dt + \frac{1}{n!} \int_x^b \left(\int_t^b (u-t)^n w(u) du \right) \left| f^{(n+1)}(t) \right| dt$$

$$\leq \frac{\left\| f^{(n+1)} \right\|_{[a,x],\infty}}{n!} \int_a^x \left(\int_a^t (t-u)^n w(u) du \right) dt + \frac{\left\| f^{(n+1)} \right\|_{[x,b],\infty}}{n!} \int_x^b \left(\int_t^b (u-t)^n w(u) du \right) dt.$$

By using the change of order of integration, because n is a odd number, we get

$$\left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) dt \right|$$

$$\leq \frac{\left\| f^{(n+1)} \right\|_{[a,x],\infty}}{(n+1)!} \int_a^x (x-u)^{n+1} w(u) du + \frac{\left\| f^{(n+1)} \right\|_{[x,b],\infty}}{(n+1)!} \int_x^b (u-x)^{n+1} w(u) du$$

$$\leq \frac{\left\| f^{(n+1)} \right\|_{[a,b],\infty}}{(n+1)!} M_{n+1}(x).$$

Hence, the proof is completed.   □

Remark 1. *If we choose $n = 1$ in Theorem 6, then we obtain*

$$\left| M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t) dt \right| \leq \frac{\left\| f'' \right\|_{[a,b],\infty}}{2} M_2(x).$$

which was given by Sarikaya and Yaldiz in $[25]$.

Remark 2. *If we choose $w(u) = 1$ in Theorem 6, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t) dt \right|$$

$$\leq \frac{\left\| f^{(n+1)} \right\|_{[a,b],\infty}}{(n+2)!} \left[(b-x)^{n+2} + (x-a)^{n+2} \right].$$

for all $n \geq 0$. This inequality was proved by Cerone et al. in $[6]$.

Remark 3. *If we take $w(u) = 1$ and $n = 0$ in Theorem 6, then we get the clasical Ostrowski inequality.*

Remark 4. *If we take $w(u) = 1$ and $n = 1$ in Theorem 6, then the Theorem 6 reduces to the Theorem 1 which is proved by Cerone et.al. in* $[8]$.

Remark 5. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 6, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-a)^{k+1} \left[1 + (-1)^k \right]}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \int_a^b f(t) \, dt \right|$$

$$\leq \quad \frac{\left\| f^{(n+1)} \right\|_{\infty} (b-a)^{n+2}}{2^{n+1} (n+2)!}.$$

for all $n \geq 0$. This inequality was proved by Cerone et al. in [6].

Remark 6. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 6, then the inequality (2.3) becomes the inequality (1.3) which was given by Cerone et.al. in [8].*

Now, we give an inequality for mappings whose absolute value of $n + 1-$th derivatives are convex.

Theorem 7. *Suppose that all the assumptions of Lemma 1 hold. If $\left| f^{(n+1)} \right|$ is convex on $[a, b]$, then, for all $x \in [a, b]$, the following inequality holds:*

$$(2.4) \left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{(n+1)! (b-a)} \left[\left((b-a) \frac{(x-a)^{n+2}}{n+2} + \frac{(b-x)^{n+3} - (x-a)^{n+3}}{n+3} \right) \left| f^{(n+1)}(a) \right| \right.$$

$$\left. + \left(\frac{(x-a)^{n+3} - (b-x)^{n+3}}{n+3} + (b-a) \frac{(b-x)^{n+2}}{n+2} \right) \left| f^{(n+1)}(b) \right| \right]$$

where $\|w\|_{\infty} = \sup\limits_{t \in [a,b]} |w(t)|$.

Proof. By taking absolute value of (2.1) and using the bounded of mapping w, we find that

$$\left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,x],\infty}}{(n+1)!} \int_a^x (t-a)^{n+1} \left| f^{(n+1)}(t) \right| dt + \frac{\|w\|_{[x,b],\infty}}{(n+1)!} \int_x^b (b-t)^{n+1} \left| f^{(n+1)}(t) \right| dt.$$

Since $\left| f^{(n+1)}(t) \right|$ is convex on $[a, b] = [a, x] \cup [x, b]$

$$(2.5) \qquad \left| f^{(n+1)} \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| \leq \frac{b-t}{b-a} \left| f^{(n+1)}(a) \right| + \frac{t-a}{b-a} \left| f^{(n+1)}(b) \right|.$$

Utilising the inequality (2.5), we write

$$
(2.6) \quad \left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t)\, dt \right|
$$

$$
\leq \quad \frac{\|w\|_{[a,x],\infty}}{(n+1)!\,(b-a)} \left(\left| f^{(n+1)}(a) \right| \int_a^x (t-a)^{n+1} (b-t)\, dt + \left| f^{(n+1)}(b) \right| \int_a^x (t-a)^{n+2}\, dt \right)
$$

$$
+ \frac{\|w\|_{[x,b],\infty}}{(n+1)!\,(b-a)} \left(\left| f^{(n+1)}(a) \right| \int_x^b (b-t)^{n+2}\, dt + \left| f^{(n+1)}(b) \right| \int_x^b (b-t)^{n+1} (t-a)\, dt \right).
$$

If we calculate the above four inetgrals and also substitute the results in (2.6), because of $\|w\|_{[a,x],\infty}, \|w\|_{[x,b],\infty} \leq \|w\|_{[a,b],\infty}$, we obtain desired inequality (2.4) which completes the proof. $\qquad\square$

Remark 7. *Under the same assumptions of Theorem 7 with $n = 0$, then the following inequality holds:*

$$
\left| f(x) \int_a^b w(t)\, dt - \int_a^b w(t) f(t)\, dt \right|
$$

$$
\leq \quad \frac{\|w\|_{[a,b],\infty}}{(b-a)} \left[\left((b-a) \frac{(x-a)^2}{2} + \frac{(b-x)^3 - (x-a)^3}{3} \right) |f'(a)| \right.
$$

$$
\left. + \left(\frac{(x-a)^3 - (b-x)^3}{3} + (b-a) \frac{(b-x)^2}{2} \right) |f'(b)| \right]
$$

*which is "**weighted Ostrowski**" inequality provided that $|f'|$ is convex on $[a, b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 8. *Under the same assumptions of Theorem 7 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$
\left| f\left(\frac{a+b}{2} \right) \int_a^b w(t)\, dt - \int_a^b w(t) f(t)\, dt \right|
$$

$$
\leq \quad \frac{\|w\|_{[a,b],\infty} (b-a)^2}{4} \left[\frac{|f'(a)| + |f'(b)|}{2} \right]
$$

*which is "**weighted mid-point**" inequality provided that $|f'|$ is convex on $[a, b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 9. *If we choose $n = 1$ in Theorem 7, then we obtain*

$$\left| M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{2(b-a)} \left[\left((b-a) \frac{(x-a)^3}{3} + \frac{(b-x)^4 - (x-a)^4}{4} \right) \left| f''(a) \right| \right.$$

$$\left. + \left(\frac{(x-a)^4 - (b-x)^4}{4} + (b-a) \frac{(b-x)^3}{3} \right) \left| f''(b) \right| \right]$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 2. *Under the same assumptions of Theorem 7 with $w(u) = 1$, then we have the inequality*

$$\left| \sum_{k=0}^n \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t) \, dt \right|$$

$$\leq \left[\left(\frac{(x-a)^{n+2}}{(n+2)!} + \frac{(b-x)^{n+3} - (x-a)^{n+3}}{(b-a)(n+1)!(n+3)} \right) \left| f^{(n+1)}(a) \right| \right.$$

$$\left. + \left(\frac{(x-a)^{n+3} - (b-x)^{n+3}}{(b-a)(n+1)!(n+3)} + \frac{(b-x)^{n+2}}{(n+2)!} \right) \left| f^{(n+1)}(b) \right| \right].$$

Corollary 3. *If we take $w(u) = 1$ and $n = 0$ in Theorem 7, then we have*

$$\left| (b-a) f(x) - \int_a^b f(t) \, dt \right|$$

$$\leq \left[\left(\frac{(x-a)^2}{2} + \frac{(b-x)^3 - (x-a)^3}{3(b-a)} \right) \left| f'(a) \right| \right.$$

$$\left. + \left(\frac{(x-a)^3 - (b-x)^3}{3(b-a)} + \frac{(b-x)^2}{2} \right) \left| f'(b) \right| \right].$$

Remark 10. *If we take $w(u) = 1$ and $n = 1$ in Theorem 7, then we get*

$$\left| (b-a) f(x) + (b-a) \left(\frac{a+b}{2} - x \right) f'(x) - \int_a^b f(t) \, dt \right|$$

$$\leq \left[\left(\frac{(x-a)^3}{6} + \frac{(b-x)^4 - (x-a)^4}{8(b-a)} \right) \left| f''(a) \right| \right.$$

$$\left. + \left(\frac{(x-a)^4 - (b-x)^4}{8(b-a)} + \frac{(b-x)^3}{6} \right) \left| f''(b) \right| \right]$$

which was given by Sarikaya and Yaldiz in [25].

Remark 11. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 7, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-a)^{k+1} \left[1 + (-1)^k\right]}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2}\right) - \int_{a}^{b} f(t)\, dt \right|$$

$$\leq \quad \frac{(b-a)^{n+2}}{2^{n+1} (n+2)!} \left[\frac{\left|f^{(n+1)}(a)\right| + \left|f^{(n+1)}(b)\right|}{2} \right]$$

which was derived by Ozdemir and Yildiz in [22].

Remark 12. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 7, then the inequality (2.4) reduce to the inequality (1.4).*

Remark 13. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 7, then the inequality (2.4) becomes the inequality (1.6).*

We prove some inequalities by using convexity of $\left|f^{(n+1)}\right|^q$.

Theorem 8. *Suppose that all the assumptions of Lemma 1 hold. If $\left|f^{(n+1)}\right|^q$ is convex on $[a,b]$, $q > 1$, then, for all $x \in [a,b]$, we have the inequality*

(2.7)
$$\left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_{a}^{b} w(t) f(t)\, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{(n+1)!} (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{(n+1)p+1} + (x-a)^{(n+1)p+1}}{(n+1)p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{\left|f^{(n+1)}(a)\right|^q + \left|f^{(n+1)}(b)\right|^q}{2} \right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t \in [a,b]} |w(t)|$.

Proof. By similar methods in the proof of Theorem 7 and from Hölder's inequality, we find that

(2.8)
$$\left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_{a}^{b} w(t) f(t)\, dt \right|$$

$$\leq \quad \left(\int_{a}^{b} |P_w(x,t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_{a}^{b} \left|f^{(n+1)}(t)\right|^q\, dt \right)^{\frac{1}{q}}.$$

By simple calculations, we obtain

$$(2.9) \quad \left(\int_a^b |P_w(x,t)|^p \, dt \right)^{\frac{1}{p}}$$

$$= \frac{1}{n!} \left[\int_a^x \left(\int_a^t (t-u)^n \, w(u) du \right)^p dt + \int_x^b \left(\int_t^b (u-t)^n \, w(u) du \right)^p dt \right]^{\frac{1}{p}}$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{(n+1)!} \left[\int_a^x (t-a)^{(n+1)p} \, dt + \int_x^b (b-t)^{(n+1)p} \, dt \right]^{\frac{1}{p}}$$

$$= \frac{\|w\|_{[a,b],\infty}}{(n+1)!} \left[\frac{(b-x)^{(n+1)p+1} + (x-a)^{(n+1)p+1}}{(n+1)p+1} \right]^{\frac{1}{p}}.$$

Since $\left| f^{(n+1)}(t) \right|^q$ is convex on $[a,b] = [a,x] \cup [x,b]$, we have

$$(2.10) \quad \left| f^{(n+1)} \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right|^q \leq \frac{b-t}{b-a} \left| f^{(n+1)}(a) \right|^q + \frac{t-a}{b-a} \left| f^{(n+1)}(b) \right|^q.$$

Using the inequality (2.10), it follows that

$$(2.11) \quad \left(\int_a^b \left| f^{(n+1)}(t) \right|^q dt \right)^{\frac{1}{q}} \leq (b-a)^{\frac{1}{q}} \left[\frac{\left| f^{(n+1)}(a) \right|^q + \left| f^{(n+1)}(b) \right|^q}{2} \right]^{\frac{1}{q}}.$$

Hence, the proof of theorem is completed. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Corollary 4. *Under the same assumptions of Theorem 8 with $n = 0$, then the following inequality hols:*

$$\left| f(x) \int_a^b w(t) \, dt - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \|w\|_{[a,b],\infty} (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{p+1} + (x-a)^{p+1}}{p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

*which is "**weighted Ostrowski**" inequality provided that $|f'|^q$ is convex on $[a,b]$.*

Corollary 5. *Under the same assumptions of Theorem 8 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$\left| f\left(\frac{a+b}{2} \right) \int_a^b w(t) \, dt - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \frac{\|w\|_{[a,b],\infty} (b-a)^2}{2 \cdot (p+1)^{\frac{1}{p}}} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

which is "**weighted mid-point**" inequality provided that $|f'|$ is convex on $[a, b]$.

Remark 14. *If we choose $n = 1$ in Theorem 8, then we obtain*

$$\left| M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t) dt \right|$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{2} (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{2p+1} + (x-a)^{2p+1}}{2p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 6. *Under the same assumptions of Theorem 8 with $w(u) = 1$, then we have the inequality*

$$\left| \sum_{k=0}^n \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t) dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}}}{(n+1)!} \left[\frac{(b-x)^{(n+1)p+1} + (x-a)^{(n+1)p+1}}{(n+1)p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{\left| f^{(n+1)}(a) \right|^q + \left| f^{(n+1)}(b) \right|^q}{2} \right]^{\frac{1}{q}}.$$

Remark 15. *If we take $w(u) = 1$ and $n = 0$ in Theorem 8, then we get*

$$\left| (b-a) f(x) - \int_a^b f(t) dt \right|$$

$$\leq (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{p+1} + (x-a)^{p+1}}{p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}.$$

Remark 16. *If we take $w(u) = 1$ and $n = 1$ in Theorem 8, then we have*

$$\left| (b-a) f(x) + (b-a) \left(\frac{a+b}{2} - x \right) f'(x) - \int_a^b f(t) dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}}}{2} \left[\frac{(b-x)^{2p+1} + (x-a)^{2p+1}}{2p+1} \right]^{\frac{1}{p}}$$

$$\times \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}}$$

which was given by Sarikaya and Yaldiz in [25].

Corollary 7. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 8, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-a)^{k+1} \left[1 + (-1)^k \right]}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \int_a^b f(t) \, dt \right|$$

$$\leq \frac{(b-a)^{n+2}}{2^{n+1} (n+1)! \left[(n+1)p+1 \right]^{\frac{1}{p}}} \left[\frac{\left| f^{(n+1)}(a) \right|^q + \left| f^{(n+1)}(b) \right|^q}{2} \right]^{\frac{1}{q}}.$$

Remark 17. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 8, then we have*

$$\left| (b-a) f \left(\frac{a+b}{2} \right) - \int_a^b f(t) \, dt \right|$$

$$\leq \frac{(b-a)^2}{2 (p+1)^{\frac{1}{p}}} \left[\frac{\left| f'(a) \right|^q + \left| f'(b) \right|^q}{2} \right]^{\frac{1}{q}}.$$

Remark 18. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 1$ in Theorem 8, then the inequality (2.7) becomes the inequality (1.7).*

Theorem 9. *Suppose that all the assumptions of Lemma 1 hold. If $\left| f^{(n+1)} \right|^q$ is convex on $[a,b]$, $q > 1$, then for all $x \in [a,b]$, we have the inequality*

$$(2.12) \left| \sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) \, dt \right|$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}} (n+1)! \left[(n+1)p+1 \right]^{\frac{1}{p}}}$$

$$\times \left\{ (x-a)^{n+1+\frac{1}{p}} \left[\frac{(b-a)^2 - (b-x)^2}{2} \left| f^{(n+1)}(a) \right|^q + \frac{(x-a)^2}{2} \left| f^{(n+1)}(b) \right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{n+1+\frac{1}{p}} \left[\frac{(b-x)^2}{2} \left| f^{(n+1)}(a) \right|^q + \frac{(b-a)^2 - (x-a)^2}{2} \left| f^{(n+1)}(b) \right|^q \right]^{\frac{1}{q}} \right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup\limits_{t \in [a,b]} |w(t)|$.

Proof. Using similar methods in the proof of Theorem 8 and from Hölder's inequality, we obtain the inequality (2.12). Hence, the proof is completed. □

Remark 19. *Under the same assumptions of Theorem 9 with $n = 0$, then the following inequality holds:*

$$\left| f(x) \int_a^b w(t)\, dt - \int_a^b w(t) f(t)\, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}}$$

$$\times \left\{ (x-a)^{1+\frac{1}{p}} \left[\frac{(b-a)^2 - (b-x)^2}{2} |f'(a)|^q + \frac{(x-a)^2}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{1+\frac{1}{p}} \left[\frac{(b-x)^2}{2} |f'(a)|^q + \frac{(b-a)^2 - (x-a)^2}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right\}$$

*which is "**weighted Ostrowski**" inequality provided that $|f'|^q$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 20. *Under the same assumptions of Theorem 9 with $n = 0$ and $x = \frac{a+b}{2}$, then the following inequality hols:*

$$\left| f\left(\frac{a+b}{2}\right) \int_a^b w(t)\, dt - \int_a^b w(t) f(t)\, dt \right|$$

$$\leq \quad \frac{(b-a)^2}{4 [p+1]^{\frac{1}{p}}} \left\{ \left[\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right]^{\frac{1}{q}} \right\} \|w\|_{[a,b],\infty}$$

*which is "**weighted mid-point**" inequality provided that $|f'|$ is convex on $[a,b]$. This inequality was given by Sarikaya and Erden in [26].*

Remark 21. *If we choose $n = 1$ in Theorem 9, then we obtain*

$$\left| M_1(x) f'(x) + M_0(x) f(x) - \int_a^b w(t) f(t)\, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{2 (b-a)^{\frac{1}{q}} [2p+1]^{\frac{1}{p}}}$$

$$\times \left\{ (x-a)^{2+\frac{1}{p}} \left[\frac{(b-a)^2 - (b-x)^2}{2} |f''(a)|^q + \frac{(x-a)^2}{2} |f''(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{2+\frac{1}{p}} \left[\frac{(b-x)^2}{2} |f''(a)|^q + \frac{(b-a)^2 - (x-a)^2}{2} |f''(b)|^q \right]^{\frac{1}{q}} \right\}$$

which was established by Sarikaya and Yaldiz in [25].

Corollary 8. *Under the same assumptions of Theorem 9 with $w(u) = 1$, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) - \int_a^b f(t)\, dt \right|$$

$$\leq \quad \frac{1}{(b-a)^{\frac{1}{q}} (n+1)! \left[(n+1)p+1\right]^{\frac{1}{p}}}$$

$$\times \left\{ (x-a)^{n+1+\frac{1}{p}} \left[\frac{(b-a)^2 - (b-x)^2}{2} \left| f^{(n+1)}(a) \right|^q + \frac{(x-a)^2}{2} \left| f^{(n+1)}(b) \right|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{n+1+\frac{1}{p}} \left[\frac{(b-x)^2}{2} \left| f^{(n+1)}(a) \right|^q + \frac{(b-a)^2 - (x-a)^2}{2} \left| f^{(n+1)}(b) \right|^q \right]^{\frac{1}{q}} \right\}.$$

Remark 22. *If we take $w(u) = 1$ and $n = 1$ in Theorem 9, then we get*

$$\left| (b-a) f(x) + (b-a) \left(\frac{a+b}{2} - x \right) f'(x) - \int_a^b f(t)\, dt \right|$$

$$\leq \quad \frac{1}{2(b-a)^{\frac{1}{q}} [2p+1]^{\frac{1}{p}}}$$

$$\times \left\{ (x-a)^{2+\frac{1}{p}} \left[\frac{(b-a)^2 - (b-x)^2}{2} |f''(a)|^q + \frac{(x-a)^2}{2} |f''(b)|^q \right]^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{2+\frac{1}{p}} \left[\frac{(b-x)^2}{2} |f''(a)|^q + \frac{(b-a)^2 - (x-a)^2}{2} |f''(b)|^q \right]^{\frac{1}{q}} \right\}$$

which was given by Sarikaya and Yaldiz in [25].

Remark 23. *If we choose $w(u) = 1$ and $x = \frac{a+b}{2}$ in Theorem 9, then we have the inequality*

$$\left| \sum_{k=0}^{n} \frac{(b-a)^{k+1} \left[1 + (-1)^k\right]}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \int_a^b f(t)\, dt \right|$$

$$\leq \quad \frac{(b-a)^{n+2}}{2^{n+2} (n+1)! \left[(n+1)p+1\right]^{\frac{1}{p}}} \left\{ \left[\frac{3 \left| f^{(n+1)}(a) \right|^q + \left| f^{(n+1)}(b) \right|^q}{4} \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{\left| f^{(n+1)}(a) \right|^q + 3 \left| f^{(n+1)}(b) \right|^q}{4} \right]^{\frac{1}{q}} \right\}$$

which was proved by Ozdemir and Yildiz in [22].

Remark 24. *If we take $w(u) = 1$, $x = \frac{a+b}{2}$ and $n = 0$ in Theorem 9, then the inequality (2.12) reduce to the inequality (1.5).*

Remark 25. *If we take* $w(u) = 1$, $x = \frac{a+b}{2}$ *and* $n = 1$ *in Theorem 9, then we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right|$$

$$\leq \quad \frac{(b-a)^2}{16\,[2p+1]^{\frac{1}{p}}} \left\{ \left[\frac{3\,|f''(a)|^q + |f''(b)|^q}{4} \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{|f''(a)|^q + 3\,|f''(b)|^q}{4} \right]^{\frac{1}{q}} \right\}$$

which was given by Sarikaya and Yaldiz in [25].

Theorem 10. *Suppose that all the assumptions of Lemma 1 hold. If* $\left|f^{(n+1)}\right|^q$ *is convex on* $[a,b]$, $q \geq 1$, *then, for all* $x \in [a,b]$, *the following inequality holds:*

(2.13)
$$\left| \sum_{k=0}^n \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t)\, f(t)\, dt \right|$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{(n+1)!\,(b-a)^{\frac{1}{q}}} \left(\frac{(b-x)^{n+2} + (x-a)^{n+2}}{(n+2)} \right)^{\frac{1}{p}}$$

$$\times \left[\left((b-a)\frac{(x-a)^{n+2}}{n+2} + \frac{(b-x)^{n+3} - (x-a)^{n+3}}{n+3} \right) \left|f^{(n+1)}(a)\right|^q \right.$$

$$\left. + \left(\frac{(x-a)^{n+3} - (b-x)^{n+3}}{n+3} + (b-a)\frac{(b-x)^{n+2}}{n+2} \right) \left|f^{(n+1)}(b)\right|^q \right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|w\|_\infty = \sup\limits_{t \in [a,b]} |w(t)|$.

Proof. From (2.3), using the properties of modulus and from Hölder's inequality, we get

(2.14)
$$\left| \sum_{k=0}^n \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t)\, f(t)\, dt \right|$$

$$\leq \quad \left(\int_a^b |P_w(x,t)|\, dt \right)^{\frac{1}{p}} \left(\int_a^b |P_w(x,t)| \left|f^{(n+1)}(t)\right|^q dt \right)^{\frac{1}{q}}.$$

By simple calculations, we obtain

(2.15)
$$\int_a^b |P_w(x,t)|\, dt \leq \|w\|_{[a,b],\infty} \frac{(b-x)^{n+2} + (x-a)^{n+2}}{(n+2)!}.$$

Because of convexity of $\left|f^{(n+1)}\right|^q$ and bounded of w, appliying similar methods in the proof of Theorem 7 and using the inequality (2.10), we find that

$$(2.16) \qquad \int_a^b |P_w(x,t)| \left|f^{(n+1)}(t)\right|^q dt$$

$$\leq \quad \frac{\|w\|_{[a,b],\infty}}{(n+1)!(b-a)}$$

$$\times \left[\left((b-a) \frac{(x-a)^{n+2}}{n+2} + \frac{(b-x)^{n+3}-(x-a)^{n+3}}{n+3} \right) \left|f^{(n+1)}(a)\right|^q \right.$$

$$\left. \left(\frac{(x-a)^{n+3}-(b-x)^{n+3}}{n+3} + (b-a) \frac{(b-x)^{n+2}}{n+2} \right) \left|f^{(n+1)}(b)\right|^q \right].$$

Substituting the inequalities (2.15) and (2.16) in (2.14), we easily deduce required inequality (2.13) which completes the proof. $\qquad\square$

Remark 26. *In case $(p,q)=(\infty,1)$, if we take limit as $p \to \infty$ in Theorem 10, then the inequality (2.13) becomes the inequality (2.4). Thus, we obtain all of the results which are similar to Theorem 7.*

Remark 27. *If we take $w(u)=1$, $x=\frac{a+b}{2}$ and $n=0$ in Theorem 10, then we obtain*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

which was established by Pearce and Pečarić in [21].

3. SOME APPLICATIONS FOR THE MOMENTS

Distribution functions and density functions provide complete descriptions of the distribution of probality for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied (see, [3]-[5], [12]-[15], [25]).

Set X to denote a random variable whose probability function is $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$ is a convex function on the interval of real numbers I and let $a,b \in I$, $(a < b)$. Denote by $M_r(x)$ the rth moment about any arbitrary point x of the random variable $X, r \geq 0$, defined as

$$M_r(x) = \int_a^b (u-x)^r f(u)\,du, \quad r = 0,1,2,...$$

Now, we reconsider the identity (2.1) by changing conditions given in Lemma 1. Herewith, we deduce an identity involving r^{th} moment.

Lemma 2. *Let X be random variable whose p.d.f. $f : I \subset \mathbb{R} \to \mathbb{R}_+$ be $n+1$ times differentiable function on I°, $a,b \in I^\circ$ with $a < b$, $f^{(n+1)}$ is absolutely continuous*

on $[a, b]$ and let $w : [a, b] \to \mathbb{R}$ be nonnegative and continuous on $[a, b]$. Then the following equality holds:

$$\sum_{k=0}^{n} \frac{M_k(x)}{k!} f^{(k)}(x) - \int_a^b w(t) f(t) dt = \int_a^b P_w(x, t) f^{(n+1)}(t) dt$$

where $n \in \mathbb{N}$, $M_k(x)$ is the kth moment and $P_w(x, t)$ is defined as (2.2).

Similarly, using boundedness of $f^{(n+1)}$, convexity of $\left| f^{(n+1)} \right|$ or convexity of $\left| f^{(n+1)} \right|^q$ in addition to conditions of Lemma 2, we obtain same of the inequalities given in previous section for random variable.

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Department of Mathematics, Faculty of Science, Bartin University, Bartin-Turkey
E-mail address: erdem1627@gmail.com

Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce-TURKEY
E-mail address: sarikayamz@gmail.com

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY
E-mail address: hsyn.budak@gmail.com