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SOME PERTURBED INEQUALITIES OF OSTROWSKI TYPE FOR TWICE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We establish new perturbed Ostrowski type inequalities for functions whose second derivatives are of bounded variation. In addition, we obtain some integral inequalities for absolutely continuous mappings. Finally, some inequalities related to Lipschitzian derivatives are given.

1. INTRODUCTION

In 1938, Ostrowski [28] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum\Delta f(P)$ denotes the sum $\sum_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b(f) := \sup\left\{\sum\Delta f(P) : P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [16], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [13], authors obtained the following Ostroski type inequalities for functions whose second derivatives are bounded:

Theorem 3. *Let$f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'':(a,b)\to\mathbb{R}$ is bounded on $(a;b)$. Then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2\left\|f''\right\|_\infty\\ \leq\ &\frac{\left\|f''\right\|_\infty}{6}(b-a)^2\end{aligned}$$

for all $x\in[a,b]$.

Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many authors have worked on Ostrowski type inequalities. For example, authors gave some Ostrowski type inequalities for function of bounded variation in ([1]-[10],[14],[6]-[18],[26],[27]). The researchers established Ostrowski type integral inequalities for mappings whose second derivatives are bounded in ([13],[15],[29],[30]). Moreover, Dragomir proved some perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to [19]-[25]. In ([11],[12],[31]), some researchers established new perturbed Ostrowski type inequalities for twice differentiable functions.

In this study, some new perturbed Ostrowski type integral inequalities for functions whose second derivatives are of bounded variation, absolutely continuous and Lipschitzian are given.

In [11], Budak et al. deduced the following integral identity:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b). Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex number the following identity holds:*

$$\begin{aligned}&\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\\ &-\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3+\lambda_2(x)(b-x)^3}{3}\right]\\ =\ &\frac{1}{2}\left[\frac{1}{b-a}\int_a^x(t-a)^2\left[f''(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b(t-b)^2\left[f''(t)-\lambda_2(x)\right]dt\right],\end{aligned} \tag{1.3}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

2. Inequalities for Functions Whose Second Derivatives are of Bounded Variation

Now, we establish the following identity by choosing $\lambda_1(x) = \lambda_2(x) = f''(x)$ in (1.3)

$$\begin{aligned}
(2.1)\qquad & \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \\
& -\frac{(b-a)^2}{6}\left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right] f''(x) \\
= {} & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left[f''(t) - f''(x)\right]dt + \int_x^b (t-b)^2\left[f''(t) - f''(x)\right]dt\right],
\end{aligned}$$

for any $x \in [a,b]$.

We start with the following inequality:

Theorem 4. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the second derivative f'' is of bounded variation on $[a,b]$, then*

$$\begin{aligned}
(2.2)\qquad & \left|\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt\right. \\
& \left. -\frac{(b-a)^2}{6}\left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right] f''(x)\right| \\
\leq {} & \frac{1}{6(b-a)}\left[(x-a)^3 \bigvee_a^x (f'') + (b-x)^3 \bigvee_x^b (f'')\right] \\
\leq {} & \frac{(b-a)^2}{6}\begin{cases}
\left[\frac{1}{12} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\left[\frac{1}{2}\bigvee_a^b (f'') + \frac{1}{2}\left|\bigvee_a^x (f'') - \bigvee_x^b (f'')\right|\right], \\[2ex]
\left[\left(\frac{x-a}{b-a}\right)^{3p} + \left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x (f'')\right)^q + \left(\bigvee_x^b (f'')\right)^q\right]^{\frac{1}{q}} \\
p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\[2ex]
\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3 \bigvee_a^b (f'')
\end{cases}
\end{aligned}$$

for any $x \in [a,b]$.

Proof. Taking modulus (2.1), we get

$$\begin{aligned}(2.3)\qquad & \left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right. \\ & \left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right| \\ \leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t)-f''(x)\right|dt+\int_x^b (t-b)^2\left|f''(t)-f''(x)\right|dt\right].\end{aligned}$$

Since f'' is of bounded variation on $[a,b]$, we get

$$|f''(t)-f''(x)|\leq \bigvee_t^x (f'')$$

for $t\in[a,x]$ and

$$|f''(t)-f''(x)|\leq \bigvee_x^t (f'')$$

for $t\in[x,b]$.
Herewith,

$$\begin{aligned}(2.4)\qquad \int_a^x (t-a)^2\left|f''(t)-f''(x)\right|dt &\leq \int_a^x (t-a)^2\bigvee_t^x (f'')dt \\ &\leq \frac{(x-a)^3}{3}\bigvee_a^x (f'').\end{aligned}$$

Similarly, we have

$$(2.5)\qquad \int_x^b (t-b)^2\left|f''(t)-f''(x)\right|dt\leq \frac{(b-x)^3}{3}\bigvee_x^b (f'').$$

Substituting the inequalities (2.4) and (2.5) in (2.3), we obtain

$$\begin{aligned} & \left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right. \\ & \left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right| \\ \leq\ & \frac{1}{6(b-a)}\left[(x-a)^3\bigvee_a^x (f'')+(b-x)^3\bigvee_x^b (f'')\right]\end{aligned}$$

whşch completes the proof of first inequality in (2.2).

The second inequality follows by Hölder's inequality

$$mn+pq\leq (m^\alpha+p^\alpha)^{\frac{1}{\alpha}}\left(n^\beta+q^\beta\right)^{\frac{1}{\beta}},\ m,n,p,q\geq 0 \text{ and } \alpha>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1.$$

Thus the theorem is now completely proved. □

Corollary 1. *Under assumptions of Theorem 4 with $x = \frac{a+b}{2}$, we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{72} f''\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{48} \bigvee_a^b (f'').$$

Corollary 2. *If $p \in (a,b)$ is a median point in bounded variation for the second derivative, i.e. $\bigvee_a^p (f'') = \bigvee_p^b (f'')$, then under the assumptions of Theorem 4, we have*

$$\begin{aligned} & \left| \left(p - \frac{a+b}{2}\right) f'(p) - f(p) + \frac{1}{b-a} \int_a^b f(t)dt \right. \\ & \left. - \frac{(b-a)^2}{6} \left[\frac{1}{12} + \frac{(p - \frac{a+b}{2})^2}{(b-a)^2} \right] f''(p) \right| \\ \leq \quad & \frac{(b-a)^2}{12} \left[\frac{1}{12} + \frac{\left(p - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \bigvee_a^b (f''). \end{aligned}$$

3. Inequalities for Absolutely Continious Derivatives

In this section, a perturbed Ostrowski type inequality by utilizing absolutely continuous of f'' are obtained.

Theorem 5. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the second derivative f'' is absolutely continuous on $[a,b]$, then we have*

$$\begin{aligned} (3.1) \qquad & \left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt \right. \\ & \left. - \frac{(b-a)^2}{6} \left[\frac{1}{12} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] f''(x) \right| \\ \leq \quad & \frac{1}{2(b-a)} \begin{cases} \frac{(b-x)^4 + (x-a)^4}{12} \left\| f''' \right\|_{[a,b],\infty} \\ \\ \frac{2q^3 \left[(x-a)^{3q+1} + (b-x)^{3q+1} \right]^{\frac{1}{q}}}{(3q+1)(2q+1)(q+1)} \left\| f''' \right\|_{[a,b],p} \\ \\ \frac{1}{3} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^3 \left\| f''' \right\|_{[a,b],1} \end{cases} \end{aligned}$$

for all $x \in [a,b]$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If we take absolute value of (2.1), we find that

$$\begin{aligned}
&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right.\\
&\left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right|\\
\leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t)-f''(x)\right|dt+\int_x^b (t-b)^2\left|f''(t)-f''(x)\right|dt\right]\\
=\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|\int_t^x f'''(s)ds\right|dt+\int_x^b (t-b)^2\left|\int_x^t f'''(s)ds\right|dt\right]\\
\leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\int_t^x \left|f'''(s)\right|dsdt+\int_x^b (t-b)^2\int_x^t \left|f'''(s)\right|dsdt\right].
\end{aligned}$$

We observe that

$$\begin{aligned}
\int_a^x (t-a)^2\int_x^t \left|f'''(s)\right|dsdt &\leq \int_a^x (t-a)^2(x-t)\left\|f'''\right\|_{[t,x],\infty}dt\\
&\leq \left\|f'''\right\|_{[a,x],\infty}\int_a^x (t-a)^2(x-t)dt\\
&= \frac{(x-a)^4}{12}\left\|f'''\right\|_{[a,x],\infty}.
\end{aligned}$$

Using Hölder's integral inequality, we have

$$\begin{aligned}
\int_a^x (t-a)^2\int_x^t \left|f'''(s)\right|dsdt &\leq \int_a^x (t-a)^2(x-t)^{\frac{1}{q}}\left\|f'''\right\|_{[t,x],p}dt\\
&\leq \left\|f'''\right\|_{[a,x],p}\int_a^x (t-a)^2(x-t)^{\frac{1}{q}}dt\\
&= \frac{2q^3(x-a)^{3+\frac{1}{q}}}{(3q+1)(2q+1)(q+1)}\left\|f'''\right\|_{[a,x],p}
\end{aligned}$$

for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Further,

$$\begin{aligned}
\int_a^x (t-a)^2\int_x^t \left|f'''(s)\right|dsdt &\leq \int_a^x (t-a)^2\left\|f'''\right\|_{[t,x],1}dt\\
&\leq \frac{(x-a)^3}{3}\left\|f'''\right\|_{[a,x],1}.
\end{aligned}$$

Thus, we obtain the inequalities

$$\int_a^x (t-a)^2 \int_x^t |f'''(s)|\, dsdt \leq \begin{cases} \frac{(x-a)^4}{12} \|f'''\|_{[a,x],\infty} \\ \\ \frac{2q^3(x-a)^{3+\frac{1}{q}}}{(3q+1)(2q+1)(q+1)} \|f'''\|_{[a,x],p} \\ \\ \frac{(x-a)^3}{3} \|f'''\|_{[a,x],1} . \end{cases}$$

Similarly, we have

$$\int_x^b (t-b)^2 \int_x^t |f'''(s)|\, dsdt \leq \begin{cases} \frac{(b-x)^4}{12} \|f'''\|_{[x,b],\infty} \\ \\ \frac{2q^3(b-x)^{3+\frac{1}{q}}}{(3q+1)(2q+1)(q+1)} \|f'''\|_{[x,b],p} \\ \\ \frac{(b-x)^3}{3} \|f'''\|_{[x,b],1} . \end{cases}$$

Because of $\|f'''\|_{[a,x],\infty} \leq \|f'''\|_{[a,b],\infty}$ and $\|f'''\|_{[a,x],\infty} \leq \|f'''\|_{[a,b],\infty}$, we obtain

$$\begin{aligned} & (x-a)^4 \|f'''\|_{[a,x],\infty} + (b-x)^4 \|f'''\|_{[x,b],\infty} \\ \leq \quad & \left[(b-x)^4 + (x-a)^4\right] \|f'''\|_{[a,x],\infty} \end{aligned}$$

which completes the proof of the first branch in (3.1).

By Holder's inequality we get

$$\begin{aligned} & (x-a)^{3+\frac{1}{q}} \|f'''\|_{[a,x],p} + (b-x)^{3+\frac{1}{q}} \|f'''\|_{[x,b],p} \\ \leq \quad & \left[(x-a)^{3q+1} + (b-x)^{3q+1}\right]^{\frac{1}{q}} \left[\|f'''\|_{[a,x],p}^p + \|f'''\|_{[x,b],p}^p\right]^{\frac{1}{p}} \\ = \quad & \left[(x-a)^{3q+1} + (b-x)^{3q+1}\right]^{\frac{1}{q}} \|f'''\|_{[a,b],p} \end{aligned}$$

producing the second branch in (3.1).

Finally,

$$\begin{aligned} & (x-a)^3 \|f'''\|_{[a,x],1} + (b-x)^3 \|f'''\|_{[x,b],1} \\ \leq \quad & \max\left\{(x-a)^3, (b-x)^3\right\} \left[\|f'''\|_{[a,x],1} + \|f'''\|_{[x,b],1}\right] \\ = \quad & \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^3 \|f'''\|_{[a,b],1} \end{aligned}$$

gives the final branch in (3.1) where we have used the fact that $\max\{a^n, b^n\} = [\max\{a, b\}]^n$ for $a, b \geq 0$ and $n > 0$.

The proof is thus completed. $\square$

Corollary 3. *Under assumptions of Theorem 5 with $x=\frac{a+b}{2}$, we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{72}f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{1}{2(b-a)}\begin{cases}\frac{(b-a)^4}{96}\left\|f'''\right\|_{[a,b],\infty}\\[2ex] \frac{q^3(b-a)^{3+\frac{1}{q}}}{4(3q+1)(2q+1)(q+1)}\left\|f'''\right\|_{[a,b],p}\\[2ex] \frac{(b-a)^3}{24}\left\|f'''\right\|_{[a,b],1}.\end{cases}$$

4. Inequalities for Lipschitzian Derivatives

In this section, we establish a integral inequality for Lipschitzian mappings. In addition, we give some results related to this inequality.

Theorem 6. *Let : $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on I° and $[a,b]\subset I^\circ$. If $\alpha,\beta>-1$ and $L_\alpha,L_\beta>0$ are such that*

$$|f''(t)-f''(x)|\leq L_\alpha(x-t)^\alpha \quad \text{for any } t\in[a,x) \tag{4.1}$$

and

$$|f''(t)-f''(x)|\leq L_\beta(t-x)^\beta \quad \text{for any } t\in(x,b], \tag{4.2}$$

then we have the inequality

$$\begin{aligned}&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right.\\ &\left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right|\\ \leq\ &\frac{1}{(b-a)}\left[\frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}L_\alpha+\frac{(b-x)^{\beta+3}}{(\beta+1)(\beta+2)(\beta+3)}L_\beta\right]\end{aligned} \tag{4.3}$$

for all $x\in(a,b)$.

Proof. Taking absolute value both sides of the equality (2.1), we find that

$$\begin{aligned}&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right.\\ &\left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right|\\ \leq\ &\frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t)-f''(x)\right|dt+\int_x^b (t-b)^2\left|f''(t)-f''(x)\right|dt\right].\end{aligned}$$

Using the properties (4.1) and (4.2), we have

$$\begin{aligned}
\int_a^x (t-a)^2 \left|f''(t)-f''(x)\right| dt &\leq L_\alpha \int_a^x (t-a)^2 (x-t)^\alpha \, dt \\
&= \frac{2(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} L_\alpha
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (t-b)^2 \left|f''(t)-f''(x)\right| dt &\leq \int_x^b (t-b)^2 L_\beta (t-x)^\beta \, dt \\
&= \frac{2(b-x)^{\beta+3}}{(\beta+1)(\beta+2)(\beta+3)} L_\beta .
\end{aligned}$$

From which we get the inequality (4.3) which completes the proof. □

Corollary 4. *Let : $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on I° and $[a,b]\subset I^\circ$. If the second derivative f'' is of $r-H-$Hölder type on $[a,b]$, i.e. we have the condition*

$$|f''(t)-f''(s)| \leq H\,|t-s|^r \quad \text{for any } t,s\in[a,b],$$

where $r\in(0.1]$ and $H>0$ are given, then

$$\begin{aligned}
&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right. \\
&\left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right| \\
\leq\ & \frac{H}{(r+1)(r+2)(r+3)}\left[\left(\frac{x-a}{b-a}\right)^{r+3}+\left(\frac{b-x}{b-a}\right)^{r+3}\right](b-a)^{r+2}
\end{aligned}$$

for all $x\in[a,b]$.
In particular, if f'' Lipschitzian with the constant $L>0$, then we have

$$\begin{aligned}
&\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\right. \\
&\left.-\frac{(b-a)^2}{6}\left[\frac{1}{12}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]f''(x)\right| \\
\leq\ & \frac{L}{24}\left[\left(\frac{x-a}{b-a}\right)^{4}+\left(\frac{b-x}{b-a}\right)^{4}\right](b-a)^{3}
\end{aligned}$$

for all $x\in[a,b]$.

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