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SOME INEQUALITIES FOR DOUBLE INTEGRALS AND APPLICATIONS FOR CUBATURE FORMULA

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Abstract. We establish an Ostrowski type inequality for double integrals of second order partial derivative functions which are bounded. Then, we deduce some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives in absolute value are convex on the co- ordinates on rectangle from the plane. Finally, some applications in Numerical Analysis in connection with cubature formula are given.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$ [13]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In a recent paper [2], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{x,y}\right\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

Then, we have the inequality:

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right.\\ &\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right| \qquad (1.2)\\ \leq&\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned}$$

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for all $(x,y)\in[a,b]\times[c,d]$.

In [2], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [14], Pachpatte obtained a new inequality in the view (1.2) by using elementary analysis. Latif et al. proved some Ostrowski type inequalities for functions which are co-ordinated convex in [10]. Sarikaya gave inetgral inequalities for bounded functions in [18]. Authors deduced weighted version of Ostrowski type inequalities for double integrals involving functions of two independent variables by using fairly elementary analysis in [1], [16], [17] and [22].

Let us now consider a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta\rightarrow\mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f\left(tx+(1-t)z,ty+(1-t)w\right)\leq tf\left(x,y\right)+(1-t)f\left(z,w\right)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$. A function $f:\Delta\rightarrow\mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\rightarrow\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\rightarrow\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b]$ and $y\in[c,d]$ (see, [4]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function* $f:\Delta\rightarrow\mathbb{R}$ *will be called co-ordinated canvex on* Δ*, for all* $t,s\in[0,1]$ *and* $(x,y),(u,v)\in\Delta$*, if the following inequality holds:*

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)v)\\ \leq\ & tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,v)+(1-t)(1-s)f(y,v).\end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [4]).

Also, in [4], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Suppose that* $f:\Delta\rightarrow\mathbb{R}$ *is co-ordinated convex on* Δ*. Then one has the inequalities:*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \leq\ &\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ \leq\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\ \leq\ &\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\ \leq\ &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\tag{1.3}$$

The above inequalities are sharp.

In recent years, researchers have studied some integral inequalities by using some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. For example, authors gave some Hadamard's type inequalities involving Riemann-Liouville fractional integrals for convex and s-convex functions on the co-ordinates in [3] and [19]. in [5], Dragomir et al. worked Ostrowski type inequality for two dimensional integtals in term of L_p-norms. Erden and Sarikaya deduced weighted version of Hermite-Hadamard type inequalities for functions whose partial derivatives in absolute value are convex on the co- ordinates on rectangle from the plane in [6] and [7]. In [8], [10]-[12], [20] and [21], some integral inequalities are obtained for differentiable co-ordinated convex mappings. In [19], Sarikaya et al. proved some new inequalities that give estimate of the deference between the middle and the right most terms in (1.3) for differentiable co-ordinated convex functions. In [6], [9] and [15], some Hermite-Hadamard type inequalities are developed for veriaty co-ordinated convex functions.

In this study, first of all, we establish an identity for second order partial derivative functions. Then, an inequality of Ostrowski type for double integrals is gotten by using this identity. Also, Hermite-Hadamard type inequalities for convex mappings on the co-ordinates on the rectangle from the plane are obtained. Finally, some applications of the Ostrowski type inequality developed in this work for cubature formula are given.

2. Main Results

In order to prove our main results we need the following lemma:

Lemma 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in[a,b]\times[c,d]$. Then, for all $(x,y)\in[a,b]\times[c,d]$, we have the equality*

$$\begin{aligned}(2.1)\quad &\int_a^b\int_c^d P_h(x,t)\,Q_h(y,s)\,f_{ts}(t,s)\,dsdt\\ =\;&\int_a^b\int_c^d f(t,s)\,dsdt+m_h(x)\int_c^d[f(b,s)-f(a,s)]\,ds\\ &+m_h(y)\int_a^b[f(t,d)-f(t,c)]\,dt-(d-c)\int_a^b f(t,y)\,dt-(b-a)\int_c^d f(x,s)\,ds\\ &+(b-a)(d-c)\,f(x,y)+m_h(x)m_h(y)\,[f(a,c)-f(a,d)-f(b,c)+f(b,d)]\\ &-(d-c)\,m_h(x)\,[f(b,y)-f(a,y)]-(b-a)\,m_h(y)\,[f(x,d)-f(x,c)]\\ =\;&S_h(x,y,s,t)\end{aligned}$$

for

$$P_h(x,t):=\begin{cases}(t-a-m_h(x)) & ,a\le t<x\\ (t-b-m_h(x)) & ,x\le t\le b\end{cases}$$

$$Q_h(y,s):=\begin{cases}(s-c-m_h(y)) & ,c\leq s<y\\ \\ (s-d-m_h(y)) & ,y\leq s\leq d\end{cases}$$

where $m_h(x)=h\left(x-\frac{a+b}{2}\right)$ *and* $m_h(y)=h\left(y-\frac{c+d}{2}\right)$, $h\in[0,2]$.

Proof. By definitions of $P_h(x,t)$ and $Q_h(y,s)$, we have

$$\int_a^b\int_c^d P_h(x,t)\,Q_h(y,s)\,f_{ts}(t,s)\,dsdt \tag{2.2}$$
$$\begin{aligned}
&= \int_a^x\int_c^y [t-a-m_h(x)]\,[s-c-m_h(y)]\,f_{ts}(t,s)\,dsdt\\
&\quad+\int_a^x\int_y^d [t-a-m_h(x)]\,[s-d-m_h(y)]\,f_{ts}(t,s)\,dsdt\\
&\quad+\int_x^b\int_c^y [t-b-m_h(x)]\,[s-c-m_h(y)]\,f_{ts}(t,s)\,dsdt\\
&\quad+\int_x^b\int_y^d [t-b-m_h(x)]\,[s-d-m_h(y)]\,f_{ts}(t,s)\,dsdt.
\end{aligned}$$

Now, we examine the above integrals. By integration by parts twice, we observe that

$$\begin{aligned}
&\int_a^x\int_c^y [t-a-m_h(x)]\,[s-c-m_h(y)]\,f_{ts}(t,s)\,dsdt\\
&= \int_a^x [t-a-m_h(x)]\left\{[y-c-m_h(y)]\,f_t(t,y)\right.\\
&\qquad \left.+m_h(y)f_t(t,c)-\int_c^y f_t(t,s)\,ds\right\}dt\\
&= [x-a-m_h(x)]\,[y-c-m_h(y)]\,f(x,y)+[y-c-m_h(y)]\,m_h(x)f(a,y)\\
&\qquad -[y-c-m_h(y)]\int_a^x f(t,y)\,dt+m_h(y)\,[x-a-m_h(x)]\,f(x,c)\\
&\qquad +m_h(x)m_h(y)f(a,c)-m_h(y)\int_a^x f(t,c)\,dt-[x-a-m_h(x)]\int_c^y f(x,s)\,ds\\
&\qquad -m_h(x)\int_c^y f(a,s)\,ds+\int_a^x\int_c^y f(t,s)\,dsdt.
\end{aligned}$$

If we calculate the other integrals in a similar way and then we substitute the results in (2.2), we obtain desired equality (2.1). The proof is completed. □

Now, we establish a new integral inequality for double integrals and also give some results related to this theorem.

Theorem 3. *Suppose that all the assumptions of Lemma 1 hold. If* $f_{ts} = \frac{\partial^2 f}{\partial t \partial s}$ *exists on* $(a,b) \times (c,d)$ *and is bounded, i.e.,*

$$\|f_{ts}\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| < \infty.$$

Then, we have the inequality:

$$\begin{aligned}(2.3) \qquad & |S_h(x,y,s,t)| \\ \leq & \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 + (h-2)\left(x - \frac{a+b}{2}\right) m_h(x) \right] \\ & \times \left[\left(\frac{d-c}{2}\right)^2 + \left(y - \frac{c+d}{2}\right)^2 + (h-2)\left(y - \frac{c+d}{2}\right) m_h(y) \right] \|f_{ts}\|_\infty\end{aligned}$$

for all $(x,y) \in [a,b] \times [c,d]$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ *and* $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0,2]$.

Proof. We take absolute value of (2.1). Using bounded of the mapping f_{ts}, we find that

$$\begin{aligned}(2.4) |S_h(x,y,s,t)| \leq & \|f_{ts}\|_\infty \int_a^b \int_c^d |P_h(x,t)| \, |Q_h(y,s)| \, ds dt \\ = & \|f_{ts}\|_\infty \left[\int_a^x |t - a - m_h(x)| \, dt + \int_x^b |t - b - m_h(x)| \, dt \right] \\ & \times \left[\int_c^y |s - c - m_h(y)| \, dt + \int_y^d |s - d - m_h(y)| \, ds \right].\end{aligned}$$

We observe the above integrals for the cases $a \leq x \leq \frac{a+b}{2}$ and $\frac{a+b}{2} \leq x \leq b$;

For all $a \leq x \leq \frac{a+b}{2}$, we have

$$\int_a^x |t - a - m_h(x)| \, dt = \frac{(x-a)^2}{2} - (x-a)\, m_h(x)$$

and

$$\int_x^b |t - b - m_h(x)| \, dt = \frac{(b-x)^2}{2} + (b-x)\, m_h(x) + [m_h(x)]^2.$$

For all $\frac{a+b}{2} \leq x \leq b$, we write

$$\int_a^x |t - a - m_h(x)| \, dt = \frac{(x-a)^2}{2} - (x-a)\, m_h(x) + [m_h(x)]^2$$

and

$$\int_{x}^{b} |t-b-m_h(x)|\,dt = \frac{(b-x)^2}{2} + (b-x)\,m_h(x).$$

Then, we get

$$\begin{aligned}(2.5)\qquad & \int_{a}^{x} |t-a-m_h(x)|\,dt + \int_{x}^{b} |t-b-m_h(x)|\,dt \\ = \; & \frac{(b-x)^2+(x-a)^2}{2} + 2\left(\frac{a+b}{2}-x\right) m_h(x) + [m_h(x)]^2\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}(2.6)\qquad & \int_{c}^{y} |s-c-m_h(y)|\,dt + \int_{y}^{d} |s-d-m_h(y)|\,ds \\ = \; & \frac{(d-y)^2+(y-c)^2}{2} + 2\left(\frac{c+d}{2}-y\right) m_h(y) + [m_h(y)]^2 .\end{aligned}$$

If we substitute the equality (2.5) and (2.6) in (2.4), we easily deduce required inequality (2.3) which completes the proof. □

Remark 1. *If we take* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$ *in Theorem 3, then we have the mid-point inequality*

$$\begin{aligned}&\left|\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt + (b-a)(d-c)\, f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\ &\left. -(d-c)\int_{a}^{b} f\left(t,\frac{c+d}{2}\right)dt - (b-a)\int_{c}^{d} f\left(\frac{a+b}{2},s\right)ds\right| \\ \leq\; & \frac{1}{16}(b-a)^2(d-c)^2\left\|f_{ts}\right\|_{\infty}\end{aligned}$$

which was given by Barnett and Dragomir in [2].

Remark 2. *Under the same assumptions of Theorem 3 with* $h=1$ *and* $(x,y)=(a,c)$, *then the following inequality hols:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d} f(t,s)\,dsdt\right. \\ &\left. -\frac{1}{2}\left[\frac{1}{d-c}\int_{c}^{d}[f(b,s)+f(a,s)]\,ds + \frac{1}{b-a}\int_{a}^{b}[f(t,d)+f(t,c)]\,dt\right]\right| \\ \leq\; & \frac{(b-a)(d-c)}{16}\left\|f_{ts}\right\|_{\infty} .\end{aligned}$$

Similarly, if we choose $(x,y)=(a,d)$ or $(x,y)=(b,c)$ or $(x,y)=(b,d)$ for $h=1$ in Theorem 3, then we deduce inequalities which are the same of the above result.

Remark 3. *If we choose $h = 0$ in Theorem 3, then the inequality (2.3) reduce to (1.2).*

Theorem 4. *Suppose that all the assumptions of Lemma 1 hold. If $f_{ts} \in L_p(\Delta)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$, then we have the inequality*

$$\begin{aligned} &|S_h(x, y, s, t)| \\ \leq\; & \left[\frac{[b - x + m_h(x)]^{q+1} + [x - a - m_h(x)]^{q+1}}{q+1}\right]^{\frac{1}{q}} \\ & \left[\frac{[d - y + m_h(y)]^{q+1} + [y - c - m_h(y)]^{q+1}}{q+1}\right]^{\frac{1}{q}} \|f_{ts}\|_p \end{aligned} \tag{2.7}$$

for all $(x, y) \in [a, b] \times [c, d]$, where $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ and $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0, 2]$. Also, $\|f_{ts}\|_p$ is defined by

$$\|f_{ts}\|_p = \left(\int_a^b \int_c^d \left|\frac{\partial^2 f(t, s)}{\partial t \partial s}\right|^p dsdt\right)^{\frac{1}{p}}.$$

Proof. Taking absolute value of (2.1) and using Hölder's inequality, we find that

$$\begin{aligned} |S_h(x, y, s, t)| &\leq \left(\int_a^b \int_c^d |P_h(x, t)|^q |Q_h(y, s)|^q dsdt\right)^{\frac{1}{q}} \left(\int_a^b \int_c^d \left|\frac{\partial^2 f(t, s)}{\partial t \partial s}\right|^p dsdt\right)^{\frac{1}{p}} \\ &= \left[\int_a^x |t - a - m_h(x)|^q dt + \int_x^b |t - b - m_h(x)|^q dt\right]^{\frac{1}{q}} \\ &\quad \times \left[\int_c^y |s - c - m_h(y)| dt + \int_y^d |s - d - m_h(y)| ds\right]^{\frac{1}{q}} \|f_{ts}\|_p \end{aligned}$$

We observe the above integrals for the cases $a \leq x \leq \frac{a+b}{2}$ and $\frac{a+b}{2} \leq x \leq b$;

For the case of $a \leq x \leq \frac{a+b}{2}$, we get

$$\int_a^x |t - a - m_h(x)|^q dt = \frac{[x - a - m_h(x)]^{q+1} - [-m_h(x)]^{q+1}}{q+1}$$

and

$$\int_x^b |t - b - m_h(x)|^q dt = \frac{[b - x + m_h(x)]^{q+1} + [-m_h(x)]^{q+1}}{q+1}$$

For the case of $\frac{a+b}{2} \leq x \leq b$, we obtain

$$\int_a^x |t - a - m_h(x)|^q dt = \frac{[x - a - m_h(x)]^{q+1} + [m_h(x)]^{q+1}}{q+1}$$

and

$$\int_x^b |t-b-m_h(x)|^q \, dt = \frac{[b-x+m_h(x)]^{q+1} - [m_h(x)]^{q+1}}{q+1}.$$

Then, we can write

$$\begin{aligned} &\int_a^x |t-a-m_h(x)|^q \, dt + \int_x^b |t-b-m_h(x)|^q \, dt \\ &= \frac{[b-x+m_h(x)]^{q+1} + [x-a-m_h(x)]^{q+1}}{q+1}. \end{aligned} \tag{2.8}$$

Similarly, we easily deduce the identity

$$\begin{aligned} &\int_c^y |s-c-m_h(y)|^q \, dt + \int_y^d |s-d-m_h(y)|^q \, ds \\ &= \frac{[d-y+m_h(y)]^{q+1} + [y-c-m_h(y)]^{q+1}}{q+1}. \end{aligned} \tag{2.9}$$

Using the equality (2.8) and (2.9), we easily deduce required inequality (2.7) . Hence, the proof is completed. □

Remark 4. *If we take* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in Theorem 4, then we have the mid-point inequality*

$$\begin{aligned} &\left| \int_a^b \int_c^d f(t,s) \, ds dt + (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\ &\left. -(d-c) \int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a) \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right| \\ &\leq \frac{(b-a)^{1+\frac{1}{q}} (d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \|f_{ts}\|_\infty \end{aligned}$$

which was pgiven by Dragomir et al. in [5].

Remark 5. *If we choose* $h = 0$ *in Theorem 4, then we have*

$$\begin{aligned} &\left| (b-a)(d-c) f(x,y) - (d-c) \int_a^b f(t,y) dt \right. \\ &\left. -(b-a) \int_c^d f(x,s) ds + \int_a^b \int_c^d f(t,s) ds dt \right| \\ &\leq \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\ &\times \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \end{aligned}$$

which was proved by Dragomir et al. in [5].

Remark 6. *Under the same assumptions of Theorem 4 with* $h=1$ *and* $(x,y)=(a,c)$, *then the following inequality hols:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right.$$
$$\left.-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d[f(b,s)+f(a,s)]\,ds+\frac{1}{b-a}\int_a^b[f(t,d)+f(t,c)]\,dt\right]\right|$$
$$\leq \frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\|f_{ts}\|_\infty .$$

Similarly, if we choose $(x,y)=(a,d)$ or $(x,y)=(b,c)$ or $(x,y)=(b,d)$ for $h=1$ in Theorem 4, then we deduce inequalities which are the same of the above result.

For convenience, we give the following notations used to simplify the details of the next theorem,

$$A = (b-a)\left[\frac{(x-a)^2}{2}-(x-a)\,m_h(x)\right]+\frac{(b-x)^3-(x-a)^3}{3}$$
$$+\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]m_h(x)-\frac{[m_h(x)]^3}{3},$$

$$B = (b-a)\left[\frac{(b-x)^2}{2}+(b-x)\,m_h(x)\right]-\frac{(b-x)^3-(x-a)^3}{3}$$
$$-\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]m_h(x)+\frac{[m_h(x)]^3}{3},$$

$$C = (d-c)\left[\frac{(y-c)^2}{2}-(y-c)\,m_h(y)\right]+\frac{(d-y)^3-(y-c)^3}{3}$$
$$+\left[\left(\frac{d-c}{2}\right)^2+\left(y-\frac{c+d}{2}\right)^2\right]m_h(y)-\frac{[m_h(y)]^3}{3}$$

and

$$D = (d-c)\left[\frac{(d-y)^2}{2}+(d-y)\,m_h(y)\right]-\frac{(d-y)^3-(y-c)^3}{3}$$
$$-\left[\left(\frac{d-c}{2}\right)^2+\left(y-\frac{c+d}{2}\right)^2\right]m_h(y)+\frac{[m_h(y)]^3}{3}.$$

We give some inequalities by using convexity of $|f_{ts}(t,s)|$ in the following theorem.

Theorem 5. *Suppose that all the assumptions of Lemma 1 hold. If* $|f_{ts}(t,s)|$ *is a convex function on the co-ordinates on* $[a,b]\times[c,d]$, *then the following inequalities* *hold:*

$$
\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \frac{\left|f_{ts}(a,c)\right|}{(b-a)(d-c)}AC+\frac{\left|f_{ts}(a,d)\right|}{(b-a)(d-c)}A\left[D+(d-c)\left[m_h(y)\right]^2\right] \\
&+\frac{\left|f_{ts}(b,c)\right|}{(b-a)(d-c)}\left[B+(b-a)\left[m_h(x)\right]^2\right]C \\
&+\frac{\left|f_{ts}(b,d)\right|}{(b-a)(d-c)}\left[B+(b-a)\left[m_h(x)\right]^2\right]\left[D+(d-c)\left[m_h(y)\right]^2\right]
\end{aligned}
\tag{2.10}
$$

for all $a\leq x\leq\frac{a+b}{2}$ *and* $c\leq y\leq\frac{c+d}{2}$

$$
\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \frac{\left|f_{ts}(a,c)\right|}{(b-a)(d-c)}A\left[C+(d-c)\left[m_h(y)\right]^2\right]+\frac{\left|f_{ts}(a,d)\right|}{(b-a)(d-c)}AD \\
&+\frac{\left|f_{ts}(b,c)\right|}{(b-a)(d-c)}\left[B+(b-a)\left[m_h(x)\right]^2\right]\left[C+(d-c)\left[m_h(y)\right]^2\right] \\
&+\frac{\left|f_{ts}(b,d)\right|}{(b-a)(d-c)}\left[B+(b-a)\left[m_h(x)\right]^2\right]D
\end{aligned}
\tag{2.11}
$$

for all $a\leq x\leq\frac{a+b}{2}$ *and* $\frac{c+d}{2}\leq y\leq d$

$$
\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \frac{\left|f_{ts}(a,c)\right|}{(b-a)(d-c)}\left[A+(b-a)\left[m_h(x)\right]^2\right]C \\
&+\frac{\left|f_{ts}(a,d)\right|}{(b-a)(d-c)}\left[A+(b-a)\left[m_h(x)\right]^2\right]\left[D+(d-c)\left[m_h(y)\right]^2\right] \\
&+\frac{\left|f_{ts}(b,c)\right|}{(b-a)(d-c)}BC+\frac{\left|f_{ts}(b,d)\right|}{(b-a)(d-c)}B\left[D+(d-c)\left[m_h(y)\right]^2\right]
\end{aligned}
\tag{2.12}
$$

for all $\frac{a+b}{2}\leq x\leq b$ *and* $c\leq y\leq\frac{c+d}{2}$

$$
\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \frac{\left|f_{ts}(a,c)\right|}{(b-a)(d-c)}\left[A+(b-a)\left[m_h(x)\right]^2\right]\left[C+(d-c)\left[m_h(y)\right]^2\right] \\
&+\frac{\left|f_{ts}(a,d)\right|}{(b-a)(d-c)}\left[A+(b-a)\left[m_h(x)\right]^2\right]D \\
&+\frac{\left|f_{ts}(b,c)\right|}{(b-a)(d-c)}B\left[C+(d-c)\left[m_h(y)\right]^2\right]+\frac{\left|f_{ts}(b,d)\right|}{(b-a)(d-c)}BD
\end{aligned}
\tag{2.13}
$$

for all $\frac{a+b}{2}\leq x\leq b$ *and* $\frac{c+d}{2}\leq y\leq d$*, where* $m_h(x)=h\left(x-\frac{a+b}{2}\right)$ *and* $m_h(y)=h\left(y-\frac{c+d}{2}\right)$, $h\in[0,2]$.

Proof. If we take absolute value of (2.1), then we get

$$
\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \int_a^b\int_c^d\left|P_h(x,t)\right|\left|Q_h(y,s)\right|\left|f_{ts}(t,s)\right|dsdt.
\end{aligned}
$$

Since $|f_{ts}(t,s)|$ is a convex function on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right| \leq \frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)| +\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|. \tag{2.14}$$

Utilizing the inequality (2.14), we obtain

$$\begin{aligned} &|S_h(x,y,s,t)| \\ \leq\ & \frac{|f_{ts}(a,c)|}{(b-a)(d-c)}\left[\int_a^b(b-t)|P_h(x,t)|\,dt\right]\left[\int_c^d(d-s)|Q_h(y,s)|\,ds\right] \\ &+\frac{|f_{ts}(a,d)|}{(b-a)(d-c)}\left[\int_a^b(b-t)|P_h(x,t)|\,dt\right]\left[\int_c^d(s-c)|Q_h(y,s)|\,ds\right] \\ &+\frac{|f_{ts}(b,c)|}{(b-a)(d-c)}\left[\int_a^b(t-a)|P_h(x,t)|\,dt\right]\left[\int_c^d(d-s)|Q_h(y,s)|\,ds\right] \\ &+\frac{|f_{ts}(b,d)|}{(b-a)(d-c)}\left[\int_a^b(t-a)|P_h(x,t)|\,dt\right]\left[\int_c^d(s-c)|Q_h(y,s)|\,ds\right] \end{aligned} \tag{2.15}$$

We observe that

$$\int_a^b(b-t)|P_h(x,t)|\,dt=(b-a)\int_a^x|t-a-m_h(x)|\,dt -\int_a^x(t-a)|t-a-m_h(x)|\,dt+\int_x^b(b-t)|t-b-m_h(x)|\,dt. \tag{2.16}$$

Now, let us observe that

$$\begin{aligned}\int_p^r|t-p||t-q|\,dt &= \int_p^q(t-p)(q-t)\,dt+\int_q^r(t-p)(t-q)\,dt \\ &= \frac{(q-p)^3}{3}+\frac{(r-p)^3}{3}-\frac{(q-p)(r-p)^2}{2}\end{aligned} \tag{2.17}$$

for all r,p,q such that $p\leq q\leq r$.

We investigate integrals given in the equality (2.16) for the cases $a\leq x\leq\frac{a+b}{2}$ and $\frac{a+b}{2}\leq x\leq b$;

For all $a\leq x\leq\frac{a+b}{2}$, we have

$$\int_a^x|t-a-m_h(x)|\,dt=\frac{(x-a)^2}{2}-(x-a)\,m_h(x),$$

$$\int_a^x (t-a)\left|t-a-m_h(x)\right| dt = \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x)$$

and using the equality (2.17) for second integral, we get

$$\int_x^b |b-t|\left|t-b-m_h(x)\right| dt = -\frac{[m_h(x)]^3}{3} + \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x).$$

For all $\frac{a+b}{2} \leq x \leq b$, we have

$$\int_a^x \left|t-a-m_h(x)\right| dt = \frac{(x-a)^2}{2} - (x-a)\, m_h(x) + [m_h(x)]^2,$$

$$\int_x^b |b-t|\left|t-b-m_h(x)\right| dt = \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x)$$

and using the equality (2.17), we obtain

$$\int_a^x |a-t|\left|t-a-m_h(x)\right| dt = \frac{[m_h(x)]^3}{3} + \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x).$$

Then, we write

$$\int_a^b (b-t)\left|P_h(x,t)\right| dt = A$$

for all $a \leq x \leq \frac{a+b}{2}$ and

$$\int_a^b (b-t)\left|P_h(x,t)\right| dt = A + (b-a)\,[m_h(x)]^2$$

for all $\frac{a+b}{2} < x \leq b$.

Similarly, we easily deduce the other integrals given in the inequality (2.15) for cases $a \leq x \leq \frac{a+b}{2}$, $\frac{a+b}{2} < x \leq b$, $c \leq y \leq \frac{c+d}{2}$ and $\frac{c+d}{2} \leq y \leq d$. If we substitute the resulting inequalities for all cases in (2.15), we obtain desired inequalities. The proof is thus completed. □

Remark 7. *If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 5, then we have the mid-point inequality*

$$\left| \int_a^b \int_c^d f(t,s)\, ds dt + (b-a)(d-c)\, f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.$$
$$\left. -(d-c)\int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a)\int_c^d f\left(\frac{a+b}{2}, s\right) ds \right|$$
$$\leq \frac{(b-a)^2 (d-c)^2}{16}\left[\frac{|f_{ts}(a,c)| + |f_{ts}(a,d)| + |f_{ts}(b,c)| + |f_{ts}(b,d)|}{4}\right]$$

which was given by Latif and Dragomir in [10].

Corollary 1. *Under the same assumptions of Theorem 5 with* $h=0$, *we get the inequality*

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)dtds+(d-c)(b-a)f(x,y)\right.\\
&\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right|\\
&\leq \left[(b-a)\frac{(x-a)^2}{2}+\frac{(b-x)^3-(x-a)^3}{3}\right]\\
&\quad\times\left\{\frac{|f_{ts}(a,c)|}{(b-a)(d-c)}\left[(d-c)\frac{(y-c)^2}{2}+\frac{(d-y)^3-(y-c)^3}{3}\right]\right.\\
&\quad\left.+\frac{|f_{ts}(a,d)|}{(b-a)(d-c)}\left[(d-c)\frac{(d-y)^2}{2}-\frac{(d-y)^3-(y-c)^3}{3}\right]\right\}\\
&\quad+\left[(b-a)\frac{(b-x)^2}{2}-\frac{(b-x)^3-(x-a)^3}{3}\right]\\
&\quad\times\left\{\frac{|f_{ts}(b,c)|}{(b-a)(d-c)}\left[(d-c)\frac{(y-c)^2}{2}+\frac{(d-y)^3-(y-c)^3}{3}\right]\right.\\
&\quad\left.+\frac{|f_{ts}(b,d)|}{(b-a)(d-c)}\left[(d-c)\frac{(d-y)^2}{2}-\frac{(d-y)^3-(y-c)^3}{3}\right]\right\}
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$.

Remark 8. *If we take* $(x,y)=(a,c)$ *for* $h=1$ *in the inequality (2.10), then we have the result*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right.\\
&\left.-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d[f(b,s)+f(a,s)]\,ds+\frac{1}{b-a}\int_a^b[f(t,d)+f(t,c)]\,dt\right]\right|\\
&\leq \frac{(b-a)(d-c)}{16}\left[\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4}\right]
\end{aligned}$$

which was proved Sarikaya et al. in [19].

Similarly, if we choose $(x,y)=(a,d)$ in (2.11) or $(x,y)=(b,c)$ in (2.12) or $(x,y)=(b,d)$ in (2.13) for $h=1$, then we obtain inequalities which are the same of the above result.

Theorem 6. *Suppose that all the assumptions of Lemma 1 hold. If* $|f_{ts}(t,s)|^q$ *is a convex function on the co-ordinates on* $[a,b]\times[c,d]$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$, *then* *the following inequality holds:*

$$\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}\left[\frac{[b-x+m_h(x)]^{p+1}+[x-a-m_h(x)]^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
&\times\left[\frac{[d-y+m_h(y)]^{p+1}+[y-c-m_h(y)]^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
&\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$*, where* $m_h(x)=h\left(x-\frac{a+b}{2}\right)$ *and* $m_h(y)=h\left(y-\frac{c+d}{2}\right)$*,* $h\in[0,2]$*.*

Proof. Taking absolute value of (2.1) and using Hölder's inequality, we find that

$$\begin{aligned}
&\left|S_h(x,y,s,t)\right| \\
\leq\ & \left(\int_a^b\int_c^d |P_h(x,t)|^p\,|Q_h(y,s)|^p\,dsdt\right)^{\frac{1}{q}}\left(\int_a^b\int_c^d |f_{ts}(t,s)|^q\,dsdt\right)^{\frac{1}{q}}
\end{aligned}$$

By similar methods in the proof of Theorem 4, we obtain

$$\begin{aligned}
&\left[\int_a^b\int_c^d |P_h(x,t)|^p\,|Q_h(y,s)|^p\,dsdt\right]^{\frac{1}{q}} \\
=\ & \left[\frac{[b-x+m_h(x)]^{p+1}+[x-a-m_h(x)]^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
&\times\left[\frac{[d-y+m_h(y)]^{p+1}+[y-c-m_h(y)]^{p+1}}{p+1}\right]^{\frac{1}{p}}.
\end{aligned}$$

Since $|f_{ts}(t,s)|^q$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}
&\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right|^q \\
\leq\ & \frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|^q+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)|^q \\
&+\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|^q+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|^q.
\end{aligned}\tag{2.18}$$

Using the inequality (2.18), it follows that

$$\begin{aligned}
&\left(\int_a^b\int_c^d |f_{ts}(t,s)|^q\,dsdt\right)^{\frac{1}{q}} \\
\leq\ & (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} \\
&\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}
\end{aligned}$$

The proof is thus completed. □

Remark 9. *If we take* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in Theorem 6, then we have the mid-point inequality*

$$\left| \int_a^b \int_c^d f(t,s)\,dsdt + (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.$$
$$\left. -(d-c)\int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a)\int_c^d f\left(\frac{a+b}{2}, s\right) ds \right|$$
$$\leq \frac{(b-a)^2 (d-c)^2}{4(q+1)^{\frac{2}{q}}} \left\{ \frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}}$$

which was deduced by Latif and Dragomir in [10].

Corollary 2. *If we choose* $h = 0$ *in Theorem 6, then we have*

$$\left| (b-a)(d-c) f(x,y) - (d-c)\int_a^b f(t,y)dt \right.$$
$$\left. -(b-a)\int_c^d f(x,s)ds + \int_a^b \int_c^d f(t,s)dsdt \right|$$
$$\leq (b-a)^{\frac{1}{q}} \left[\frac{(b-x)^{p+1} + (x-a)^{p+1}}{p+1} \right]^{\frac{1}{p}}$$
$$\times (d-c)^{\frac{1}{q}} \left[\frac{(d-y)^{p+1} + (y-c)^{p+1}}{p+1} \right]^{\frac{1}{p}}$$
$$\times \left\{ \frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}}$$

which is a Ostrowski type inequality for co-ordinated convex mappings.

Remark 10. *Under the same assumptions of Theorem 6 with* $h = 1$ *and* $(x,y) = (a,c)$, *then the following inequality hols:*

$$\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt \right.$$
$$\left. -\frac{1}{2}\left[\frac{1}{d-c}\int_c^d [f(b,s) + f(a,s)]\,ds + \frac{1}{b-a}\int_a^b [f(t,d) + f(t,c)]\,dt \right] \right|$$
$$\leq \frac{(b-a)^2 (d-c)^2}{4(q+1)^{\frac{2}{q}}} \left\{ \frac{|f_{ts}(a,c)|^q + |f_{ts}(a,d)|^q + |f_{ts}(b,c)|^q + |f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}}$$

which was proved Sarikaya et al. in [19].

Similarly, if we choose $(x,y) = (a,d)$ or $(x,y) = (b,c)$ or $(x,y) = (b,d)$ for $h = 1$ in Theorem 6, then we deduce inequalities which are the same of the above result.

3. Applications to Cubature Formulae

We now consider applications of the integral inequalities developed in the previous section, to obtain estimates of cubature formula which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = d$ be divisions of the intervals $[a, b]$ and $[c, d]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. Consider the sum

$$
\begin{aligned}
(3.1)\quad & T(f, I_n, J_m, \xi, \eta) \\
:= & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(t, \eta_j\right) dt + \\
& \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i \int_{y_j}^{y_{j+1}} f\left(\xi_i, s\right) ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i l_j f\left(\xi_i, \eta_j\right) \\
& - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\xi_i) \int_{y_j}^{y_{j+1}} \left[f\left(x_{i+1}, s\right) - f\left(x_i, s\right)\right] ds \\
& - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\eta_j) \int_{x_i}^{x_{i+1}} \left[f\left(t, y_{j+1}\right) - f\left(t, y_j\right)\right] dt \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j m_h(\xi_i) \left[f\left(x_{i+1}, \eta_j\right) - f\left(x_i, \eta_j\right)\right] \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i m_h(\eta_j) \left[f\left(\xi_i, y_{j+1}\right) - f\left(\xi_i, y_j\right)\right] \\
& - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\xi_i) m_h(\eta_j) \left[f\left(x_i, y_j\right) - f\left(x_i, y_{j+1}\right) - f\left(x_{i+1}, y_j\right) + f\left(x_{i+1}, y_{j+1}\right)\right]
\end{aligned}
$$

where $k_i = x_{i+1} - x_i,\ l_j = y_{j+1} - y_j\ (i = 0, ..., n-1;\ j = 0, ..., m-1)$, $m_h(\xi_i) = h\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)$ and $m_h(\eta_j) = h\left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)$.

Theorem 7. *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t, s) \in [a, b] \times [c, d]$. If $f_{ts} = \frac{\partial^2 f}{\partial t \partial s}$ exists on $(a, b) \times (c, d)$ and is bounded, i.e.,*

$$\left\| f_{ts} \right\|_\infty = \sup_{(t,s) \in (x_i, x_{i+1}) \times (y_j, y_{j+1})} \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| < \infty.$$

Then we have the representation

$$\int_a^b \int_c^d f(t, s) ds dt = T(f, I_n, J_m, \xi, \eta) + R(f, I_n, J_m, \xi, \eta)$$

where $S(f,f',\xi,I_n)$ is as defined in (3.1) and the remainder satisfies the astimations:

$$
\begin{aligned}
(3.2)\qquad & |R(f,I_n,J_m,\xi,\eta)| \\
\leq\ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{k_i^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2+(h-2)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)m_h(\xi_i)\right] \\
& \times\left[\frac{l_i^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2+(h-2)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)m_h(\eta_j)\right]\|f_{ts}\|_\infty
\end{aligned}
$$

for all $(\xi_i,\eta_j)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ with $(i=0,...,n-1;\ j=0,...,m-1)$, where $m_h(\xi_i)=h\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)$ and $m_h(\eta_j)=h\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)$ with $h\in[0,2]$.

Proof. Applying Theorem 3 on the interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, $(i=0,...,n-1;\ j=0,...,m-1)$, we obtain

$$
\begin{aligned}
& \Bigg|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt-l_j\int_{x_i}^{x_{i+1}} f\left(t,\eta_j\right)dt-k_i\int_{y_j}^{y_{j+1}} f\left(\xi_i,s\right)ds+k_il_jf\left(\xi_i,\eta_j\right) \\
& +m_h(\xi_i)\int_{y_j}^{y_{j+1}}\left[f\left(x_{i+1},s\right)-f\left(x_i,s\right)\right]ds+m_h(\eta_j)\int_{x_i}^{x_{i+1}}\left[f\left(t,y_{j+1}\right)-f\left(t,y_j\right)\right]dt \\
& -l_jm_h(\xi_i)\left[f\left(x_{i+1},\eta_j\right)-f\left(x_i,\eta_j\right)\right]-k_im_h(\eta_j)\left[f\left(\xi_i,y_{j+1}\right)-f\left(\xi_i,y_j\right)\right] \\
& +m_h(\xi_i)m_h(\eta_j)\left[f\left(x_i,y_j\right)-f\left(x_i,y_{j+1}\right)-f\left(x_{i+1},y_j\right)+f\left(x_{i+1},y_{j+1}\right)\right]\Bigg| \\
\leq\ & \|f_{ts}\|_\infty\left[\frac{k_i^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2+(h-2)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)m_h(\xi_i)\right] \\
& \times\left[\frac{l_i^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2+(h-2)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)m_h(\eta_j)\right]
\end{aligned}
$$

for all $i=0,...,n-1;\ j=0,...,m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ using the generalized triangle inequality we obtain the estimations (3.2). □

Remark 11. *If we take $h=0$ in Theorem 7, then we recapture the cubature formula*

$$\int_a^b\int_c^d f(t,s)dsdt=T(f,I_n,J_m,\xi,\eta)+R(f,I_n,J_m,\xi,\eta)$$

where the remainder $R(f,I_n,J_m,\xi,\eta)$ satisfies the estimation:

$$
\begin{aligned}
(3.3)\qquad & |R(f,I_n,J_m,\xi,\eta)| \\
\leq\ & \|f_{ts}\|_\infty\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{k_i^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\left[\frac{l_i^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right]
\end{aligned}
$$

which was given by Barnett and Dragomir in [2].

It is clear that inequalities (3.2) and (3.3) are much better than the clasical avereges of the remainders of the Midpoint cubatures.

Remark 12. *If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$ *and* $\eta_j = \frac{y_j + y_{j+1}}{2}$ *in Theorem 7, then we recapture the midpoint cubature formula*

$$\int_a^b \int_c^d f(t,s)dsdt = T_M(f, I_n, J_m) + R_M(f, I_n, J_m)$$

where the remainder $R_M(f, I_n, J_m)$ *satisfies the estimation:*

$$|R_M(f, I_n, J_m)| \leq \frac{\|f_{ts}\|_\infty}{16} \sum_{i=0}^{n-1} k_i^2 \sum_{j=0}^{m-1} l_j^2.$$

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