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QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED TO A WEAKEN DAVIS-CHOI-JENSEN'S INEQUALITY FOR POSITIVE MAPS

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Abstract. In this paper we establish some quasilinearity properties of some functionals associated to a weaken Davis-Choi-Jensen's inequality for positive maps and convex (concave) functions. Applications for power function and logarithm are also provided.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [14, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

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The following Jensen's type result is well known [2]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \le \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \le \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \le \Psi(f(A)). \tag{1.2}$$

In the recent paper [9] we established the following *weaken version of Davis-Choi-Jensen's inequality* that holds for the larger class of convex functions:

Theorem 2. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ a normalised positive linear map. Then for any selfadjoint operator A whose spectrum* $\operatorname{Sp}(A)$ *is contained in I we have*

$$f(\langle\Phi(A)y, y\rangle) \le \langle\Phi(f(A))y, y\rangle \tag{1.3}$$

for any $y \in K$, $\|y\| = 1$.

If the normality condition is dropped, then we have:

Corollary 1. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I and $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$. Then for any selfadjoint operator A whose spectrum* $\operatorname{Sp}(A)$ *is contained in I we have*

$$f\left(\frac{\langle\Psi(A)v, v\rangle}{\langle\Psi(1_H)v, v\rangle}\right) \le \frac{\langle\Psi(f(A))v, v\rangle}{\langle\Psi(1_H)v, v\rangle} \tag{1.4}$$

for any $v \in K$ with $v \ne 0$.

For Jensen's type operator inequalities see [3]-[13] and the references therein.

We define by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is positive invertible operator in K and define the functional $\triangle_{f,A,v} : \mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\triangle_{f,A,v}(\Psi) = \langle\Psi(1_H)v, v\rangle f\left(\frac{\langle\Psi(A)v, v\rangle}{\langle\Psi(1_H)v, v\rangle}\right),$$

where $f : I \to \mathbb{R}$ is a convex (concave) function on the interval I, A is a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \ne 0$.

In this paper we establish some quasilinearity properties of some functionals associated to a weaken Davis-Choi-Jensen's inequality (1.4) for positive maps and convex (concave) functions. Applications for power function and logarithm are also provided.

2. The Main Results

The following result holds:

Theorem 3. *Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$. If Ψ_1, $\Psi_2 \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0,1]$, then*

$$\triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \leq (\geq)(1-\lambda)\triangle_{f,A,v}(\Psi_1) + \lambda\triangle_{f,A,v}(\Psi_2), \tag{2.1}$$

namely $\triangle_{f,A,v}$ is convex (concave) on $\mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$.

In particular, we have that

$$\triangle_{f,A,v}(\Psi_1 + \Psi_2) \leq (\geq) \triangle_{f,A,v}(\Psi_1) + \triangle_{f,A,v}(\Psi_2), \tag{2.2}$$

namely $\triangle_{f,A,v}$ is subadditive (superadditive) on $\mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$.

Proof. Assume that $f : I \to \mathbb{R}$ is a convex function on the interval I and $v \in K$, $v \neq 0$.

Let Ψ_1, $\Psi_2 \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$ and $\lambda \in [0,1]$, then

$$\begin{aligned} &\triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &= \langle((1-\lambda)\Psi_1 + \lambda\Psi_2)(1_H)v, v\rangle f\left(\frac{\langle((1-\lambda)\Psi_1 + \lambda\Psi_2)(A)v, v\rangle}{\langle((1-\lambda)\Psi_1 + \lambda\Psi_2)(1_H)v, v\rangle}\right) \\ &= [(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle] \\ &\times f\left(\frac{(1-\lambda)\langle\Psi_1(A)v, v\rangle + \lambda\langle\Psi_2(A)v, v\rangle}{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle}\right). \end{aligned} \tag{2.3}$$

Using the convexity of f we have

$$\begin{aligned} &f\left(\frac{(1-\lambda)\langle\Psi_1(A)v, v\rangle + \lambda\langle\Psi_2(A)v, v\rangle}{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle}\right) \\ &= f\left(\frac{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle \frac{\langle\Psi_1(A)v,v\rangle}{\langle\Psi_1(1_H)v,v\rangle} + \lambda\langle\Psi_2(1_H)v, v\rangle \frac{\langle\Psi_2(A)v,v\rangle}{\langle\Psi_2(1_H)v,v\rangle}}{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle}\right) \\ &\leq \frac{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle f\left(\frac{\langle\Psi_1(A)v,v\rangle}{\langle\Psi_1(1_H)v,v\rangle}\right) + \lambda\langle\Psi_2(1_H)v, v\rangle f\left(\frac{\langle\Psi_2(A)v,v\rangle}{\langle\Psi_2(1_H)v,v\rangle}\right)}{(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle} \end{aligned} \tag{2.4}$$

and by multiplying (2.4) with $(1-\lambda)\langle\Psi_1(1_H)v, v\rangle + \lambda\langle\Psi_2(1_H)v, v\rangle > 0$ and by using (2.3), we get

$$\begin{aligned} &\triangle_{f,A,v}((1-\lambda)\Psi_1 + \lambda\Psi_2) \\ &\leq (1-\lambda)\langle\Psi_1(1_H)v, v\rangle f\left(\frac{\langle\Psi_1(A)v, v\rangle}{\langle\Psi_1(1_H)v, v\rangle}\right) + \lambda\langle\Psi_2(1_H)v, v\rangle f\left(\frac{\langle\Psi_2(A)v, v\rangle}{\langle\Psi_2(1_H)v, v\rangle}\right) \\ &= (1-\lambda)\triangle_{f,A,v}(\Psi_1) + \lambda\triangle_{f,A,v}(\Psi_2), \end{aligned}$$

which proves the convexity of $\triangle_{f,A,v}$.

We have by (2.1) that

$$\begin{aligned} \triangle_{f,A,v}(\Psi_1 + \Psi_2) &= \triangle_{f,A,v}\left(\frac{2\Psi_1 + 2\Psi_2}{2}\right) \leq \frac{\triangle_{f,A,v}(2\Psi_1) + \triangle_{f,A,v}(2\Psi_2)}{2} \\ &= \frac{2\triangle_{f,A,v}(\Psi_1) + 2\triangle_{f,A,v}(\Psi_2)}{2} = \triangle_{f,A,v}(\Psi_1) + \triangle_{f,A,v}(\Psi_2) \end{aligned}$$

for any Ψ_1, $\Psi_2 \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$, which proves (2.2). □

For $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ we denote that $\Psi_2\succ_I\Psi_1$ if $\Psi_2-\Psi_1\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. This means that $\Psi_2-\Psi_1$ is a linear positive functional and $\Psi_2(1_H)-\Psi_1(1_H)\in\mathcal{B}^{++}(K)$.

We have:

Corollary 2. *Let $f:I\to[0,\infty)$ be a concave function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v\in K$, $v\neq 0$.*

(i) If $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2\succ_I\Psi_1$ then

$$\triangle_{f,A,v}(\Psi_2)\geq\triangle_{f,A,v}(\Psi_1), \tag{2.5}$$

namely $\triangle_{f,A,v}$ is operator monotonic in the order "$\succ_I$" of $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

(ii) If $\Psi,\ \Upsilon\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, $t,\ T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$ then

$$T\triangle_{f,A,v}(\Upsilon)\geq\triangle_{f,A,v}(\Psi)\geq t\triangle_{f,A,v}(\Upsilon). \tag{2.6}$$

Proof. (i) Let $\Psi_1,\ \Psi_2\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2\succ_I\Psi_1$, then by (2.2) we have

$$\triangle_{f,A,v}(\Psi_2)=\triangle_{f,A,v}(\Psi_1+\Psi_2-\Psi_1)\geq\triangle_{f,A,v}(\Psi_1)+\triangle_{f,A,v}(\Psi_2-\Psi_1)$$

implying that

$$\triangle_{f,A,v}(\Psi_2)-\triangle_{f,A,v}(\Psi_1)\geq\triangle_{f,A,v}(\Psi_2-\Psi_1).$$

Since f is positive and $\Psi_2-\Psi_1\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi_2(1_H)-\Psi_1(1_H)\in\mathcal{B}^{++}(K)$ it follows that $\triangle_{f,A,v}(\Psi_2-\Psi_1)\geq 0$ and the inequality (2.5) is proved.

(ii) The proof follows by (2.5) on taking first $\Psi_2=T\Upsilon$, $\Psi_1=\Psi$ and then $\Psi_2=\Psi$, $\Psi_1=t\Upsilon$ and by the positive homogeneity of $\triangle_{f,A,v}$. □

We consider now the functional $\triangle_{f,A,v}:\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]\to\mathcal{B}(K)$ defined by

$$\begin{aligned}\square_{f,A,v}(\Psi)&:=\langle\Psi(f(A))v,v\rangle-\triangle_{f,A,v}(\Psi)\\&=\langle\Psi(f(A))v,v\rangle-\langle\Psi(1_H)v,v\rangle f\left(\frac{\langle\Psi(A)v,v\rangle}{\langle\Psi(1_H)v,v\rangle}\right),\end{aligned} \tag{2.7}$$

where $f:I\to\mathbb{R}$ is a convex (concave) function on the interval I, A is a selfadjoint operator whose spectrum is contained in I and $v\in K$, $v\neq 0$.

We can state the following result:

Theorem 4. *Let $f:I\to\mathbb{R}$ be a convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in I and $v\in K$, $v\neq 0$. Then the functional $\square_{f,A,v}$ is positive (negative) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, it is positive homogeneous and concave (convex) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. $\square_{f,A,v}$ is also superadditive (subadditive) on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.*

Proof. We consider only the convex case. The positivity of $\square_{f,A,v}$ on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$ is equivalent to the inequality for general positive linear maps (1.4). The positive homogeneity follows by the same property of $\triangle_{f,A,v}$ and the definition of $\triangle_{f,A,v}$.

If $\Psi_1,\ \Psi_2 \in \mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, $\lambda\in[0,1]$ and $v\in K$, $v\neq 0$, then by Theorem 3 we have

$$\begin{aligned}
&\square_{f,A,v}\left((1-\lambda)\Psi_1+\lambda\Psi_2\right)\\
&=\left\langle\left((1-\lambda)\Psi_1+\lambda\Psi_2\right)(f(A))v,v\right\rangle-\triangle_{f,A,v}\left((1-\lambda)\Psi_1+\lambda\Psi_2\right)\\
&\geq(1-\lambda)\left\langle\Psi_1(f(A))v,v\right\rangle+\lambda\left\langle(\Psi_2 f(A))v,v\right\rangle\\
&-(1-\lambda)\triangle_{f,A,v}(\Psi_1)-\lambda\triangle_{f,A,v}(\Psi_2)\\
&=(1-\lambda)\left[\left\langle\Psi_1(f(A))v,v\right\rangle-\triangle_{f,A,v}(\Psi_1)\right]\\
&+\lambda\left[\left\langle(\Psi_2 f(A))v,v\right\rangle-\triangle_{f,A,v}(\Psi_2)\right]\\
&=(1-\lambda)\square_{f,A,v}(\Psi_1)+\lambda\square_{f,A,v}(\Psi_2)
\end{aligned}$$

that proves the operator concavity of $\square_{f,A,v}$.

The operator superadditivity follows in a similar way and we omit the details. □

Corollary 3. *Let $f:I\to\mathbb{R}$ be a convex function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v\in K$, $v\neq 0$. If Ψ, $\Upsilon\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, t, $T>0$ with $T>t$ and $T\Upsilon\succ_I\Psi\succ_I t\Upsilon$ then*

$$T\square_{f,A,v}(\Upsilon)\geq\square_{f,A,v}(\Psi)\geq t\square_{f,A,v}(\Upsilon) \tag{2.8}$$

or, equivalently,

$$\begin{aligned}
T\left(\left\langle\Upsilon(f(A))v,v\right\rangle-\triangle_{f,A,v}(\Upsilon)\right)&\geq\left\langle\Psi(f(A))v,v\right\rangle-\triangle_{f,A,v}(\Psi)\\
&\geq t\left(\left\langle\Upsilon(f(A))v,v\right\rangle-\triangle_{f,A,v}(\Upsilon)\right)\geq 0.
\end{aligned} \tag{2.9}$$

Now, assume that A a selfadjoint operator whose spectrum is contained in $[m,M]$ for some real constants $M>m$. If f is convex, then for any $t\in[m,M]$ we have

$$f(t)\leq\frac{(M-t)f(m)+(t-m)f(M)}{M-m}. \tag{2.10}$$

If A a selfadjoint operator whose spectrum is contained in $[m,M]$, then $m1_H\leq A\leq M1_H$ and by taking the map Ψ we get $m\Psi(1_H)\leq\Psi(A)\leq M\Psi(1_H)$ for $\Psi\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$. This is equivalent to

$$m\leq\frac{\left\langle\Psi(A)v,v\right\rangle}{\left\langle\Psi(1_H)v,v\right\rangle}\leq M$$

for any $v\in K$, $v\neq 0$.

If we take $t=\frac{\langle\Psi(A)v,v\rangle}{\langle\Psi(1_H)v,v\rangle}$, $v\in K$, $v\neq 0$ in (2.10), then we get

$$f\left(\frac{\left\langle\Psi(A)v,v\right\rangle}{\left\langle\Psi(1_H)v,v\right\rangle}\right)\leq\frac{(M-\frac{\langle\Psi(A)v,v\rangle}{\langle\Psi(1_H)v,v\rangle})f(m)+\left(\frac{\langle\Psi(A)v,v\rangle}{\langle\Psi(1_H)v,v\rangle}-m\right)f(M)}{M-m}$$

that is equivalent to

$$\triangle_{f,A,v}(\Psi)\leq\Diamond_{f,A,v}(\Psi)$$

where

$$\Diamond_{f,A,v}(\Psi):=\frac{\left\langle\left(M\Psi(1_H)-\Psi(A)\right)v,v\right\rangle f(m)+\left\langle\left(\Psi(A)-m\Psi(1_H)\right)v,v\right\rangle f(M)}{M-m}$$

for $\Psi\in\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$, is a trapezoidal type functional. We observe that $\Diamond_{f,A,v}$ is *additive* and *positive homogeneous* on $\mathfrak{P}_I[\mathcal{B}(H),\mathcal{B}(K)]$.

We define the functional $\Diamond_{f,A,v} : \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)] \to \mathcal{B}(K)$ by

$$\begin{aligned}\Diamond_{f,A,v}(\Psi) &:= \Diamond_{f,A,v}(\Psi) - \triangle_{f,A,v}(\Psi) \\ &= \frac{\langle (M\Psi(1_H) - \Psi(A)) v, v\rangle f(m) + \langle (\Psi(A) - m\Psi(1_H)) v, v\rangle f(M)}{M-m} \\ &- \langle \Psi(1_H) v, v\rangle f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right).\end{aligned}$$

We observe that if f is convex (concave) on $[m, M]$ and $m1_H \leq A \leq M1_H$, then

$$\Diamond_{f,A,v}(\Psi) \geq (\leq)\, 0 \text{ for any } \Psi \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]. \tag{2.11}$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I and A a selfadjoint operator whose spectrum is contained in $[m, M]$ and $v \in K$, $v \neq 0$. Then the functional $\Diamond_{f,A,v}$ is positive (negative) on $\mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$, it is positive homogeneous and concave (convex) on $\mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$. $\Diamond_{f,A,v}$ is also superadditive (subadditive) on $\mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$.*

The proof is similar to the one from Theorem 4 and we omit the details.

Corollary 4. *Let $f : I \to \mathbb{R}$ be a convex function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in K$, $v \neq 0$. If Ψ, $\Upsilon \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$, t, $T > 0$ with $T > t$ and $T\Upsilon \succ_I \Psi \succ_I t\Upsilon$ then*

$$T\Diamond_{f,A,v}(\Upsilon) \geq \Diamond_{f,A,v}(\Psi) \geq t\Diamond_{f,A,v}(\Upsilon) \tag{2.12}$$

or, equivalently,

$$\begin{aligned}&T\left[\frac{\langle (M\Upsilon(1_H) - \Upsilon(A)) v, v\rangle f(m) + \langle (\Upsilon(A) - m\Upsilon(1_H)) v, v\rangle f(M)}{M-m}\right. \\ &\left.- \langle \Upsilon(1_H) v, v\rangle f\left(\frac{\langle \Upsilon(A) v, v\rangle}{\langle \Upsilon(1_H) v, v\rangle}\right)\right] \\ &\geq \frac{\langle (M\Psi(1_H) - \Psi(A)) v, v\rangle f(m) + \langle (\Psi(A) - m\Psi(1_H)) v, v\rangle f(M)}{M-m} \\ &- \langle \Psi(1_H) v, v\rangle f\left(\frac{\langle \Psi(A) v, v\rangle}{\langle \Psi(1_H) v, v\rangle}\right) \\ &\geq t\left[\frac{\langle (M\Upsilon(1_H) - \Upsilon(A)) v, v\rangle f(m) + \langle (\Upsilon(A) - m\Upsilon(1_H)) v, v\rangle f(M)}{M-m}\right. \\ &\left.- \langle \Upsilon(1_H) v, v\rangle f\left(\frac{\langle \Upsilon(A) v, v\rangle}{\langle \Upsilon(1_H) v, v\rangle}\right)\right] \\ &\geq 0.\end{aligned} \tag{2.13}$$

3. Some Examples

Let A_i be selfadjoint operators on H with $\operatorname{Sp}(A_i) \subset I$, $i \in \{1, ..., n\}$ and $p = (p_1, ..., p_n)$ an n-tuple of nonnegative weights with $P_n := \sum_{i=1}^n p_i > 0$. We write $p \in \mathbb{R}^n_{++}$. Consider also the n-tuple of normalised positive maps $\Phi = (\phi_1, ..., \phi_n)$ with $\phi_i \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(H)]$ for $i \in \{1, ..., n\}$.

If we put

$$\tilde{A} := \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix},$$

then we have $\operatorname{Sp}\left(\tilde{A}\right) \subset I$. We can define the positive map

$$\Psi_{p,\Phi} : \mathcal{B}(H) \oplus ... \oplus \mathcal{B}(H) \to \mathcal{B}(H)$$

by

$$\Psi_{p,\Phi}(A_1 \oplus ... \oplus A_n) = \sum_{i=1}^{n} p_i \phi_i (A_i).$$

Using the functional calculus for continuous functions f on I we have

$$\Psi_{p,\Phi}\left(f\left(\tilde{A}\right)\right) = \sum_{i=1}^{n} p_i \phi_i (f(A_i)) \text{ and } f\left(\Psi_{p,\Phi}\left(\tilde{A}\right)\right) = f\left(\sum_{i=1}^{n} p_i \phi_i (A_i)\right).$$

Since

$$\Psi_{p,\Phi}(1_H \oplus ... \oplus 1_H) = \sum_{i=1}^{n} p_i \phi_i (1_H) = P_n 1_H$$

and $P_n > 0$ it follows that $\Psi_{p,\Phi} \in \mathfrak{P}_I[\mathcal{B}(H) \oplus ... \oplus \mathcal{B}(H), \mathcal{B}(H)]$.

If $p, q \in \mathbb{R}^n_{++}$ with $p \geq q$, namely $p_i \geq q_i$ for $i \in \{1, ..., n\}$ and $P_n > Q_n$ then

$$\Psi_{p,\Phi} \succ_I \Psi_{q,\Phi}.$$

Assume also that $r = \min_{i \in \{1,\ldots,n\}} \left\{\frac{p_i}{q_i}\right\}$, $R = \max_{i \in \{1,\ldots,n\}} \left\{\frac{p_i}{q_i}\right\}$ and $r < \frac{P_n}{Q_n} < R$. Then

$$\Psi_{p,\Phi}\left(\tilde{A}\right) - r\Psi_{q,\Phi}\left(\tilde{A}\right) = \sum_{i=1}^{n} (p_i - rq_i) \phi_i (A_i) \geq 0$$

for $\tilde{A} \geq \tilde{0}$,

$$\Psi_{p,\Phi}\left(\tilde{1}_H\right) - r\Psi_{q,\Phi}\left(\tilde{1}_H\right) = \sum_{i=1}^{n} (p_i - rq_i) \phi_i (1_H) = (P_n - rQ_n) 1_H$$

and

$$R\Psi_{q,\Phi}\left(\tilde{1}_H\right) - \Psi_{p,\Phi}\left(\tilde{1}_H\right) = \sum_{i=1}^{n} (Rq_i - p_i) \phi_i (1_H) = (RQ_n - P_n) 1_H$$

showing that

$$R\Psi_{q,\Phi} \succ_I \Psi_{p,\Phi} \succ_I r\Psi_{q,\Phi}. \tag{3.1}$$

Now, observe that for $v \in H$, $\|v\| = 1$ we have

$$\triangle_{f,\tilde{A},v}(\Psi_{p,\Phi}) = P_n f\left(\frac{\left\langle \sum_{i=1}^{n} p_i \phi_i (A_i) v, v\right\rangle}{P_n}\right),$$

where $p \in \mathbb{R}^n_{++}$.

Let $f : I \to \mathbb{R}$ be a convex (concave) function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v \in H$, $\|v\| = 1$. If $p, q \in \mathbb{R}^n_{++}$ then we have by Theorem 3 that

$$\triangle_{f,\tilde{A},v}\left(\Psi_{(1-\lambda)p+\lambda q,\Phi}\right) \leq (\geq) (1-\lambda) \triangle_{f,\tilde{A},v}(\Psi_{p,\Phi}) + \lambda \triangle_{f,\tilde{A},v}(\Psi_{q,\Phi}) \tag{3.2}$$

for any $\lambda \in [0, 1]$ and, in particular

$$\triangle_{f,\tilde{A},v}(\Psi_{p+q,\Phi}) \leq (\geq) \triangle_{f,\tilde{A},v}(\Psi_{p,\Phi}) + \triangle_{f,\tilde{A},v}(\Psi_{q,\Phi}). \tag{3.3}$$

By using (2.6) for $p,q\in\mathbb{R}_{++}^{n}$ with $r=\min_{i\in\{1,...,n\}}\left\{\frac{p_i}{q_i}\right\}$, $R=\max_{i\in\{1,...,n\}}\left\{\frac{p_i}{q_i}\right\}$ and $r<\frac{P_n}{Q_n}<R$ we have

$$RQ_n f\left(\frac{\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right)\geq P_n f\left(\frac{\left\langle\sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle}{P_n}\right)\geq rQ_n f\left(\frac{\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right),\tag{3.4}$$

provided $f:I\to[0,\infty)$ is a concave function on the interval I, A a selfadjoint operator whose spectrum is contained in I and $v\in H$, $\|v\|=1$.

If we take $f(t)=t^s$, $s\in(0,1)$ and assume that $A_i\geq 0$, $i\in\{1,...,n\}$ then by (3.4) we have the power inequality

$$R^{1/s}Q_n^{1/s-1}\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle\geq P_n^{1/s-1}\left\langle\sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle\geq r^{1/s}Q_n^{1/s-1}\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle,\tag{3.5}$$

for $v\in H$, $\|v\|=1$.

By taking the supremum in this inequality over $v\in H$, $\|v\|=1$, we get the norm inequality

$$R^{1/s}Q_n^{1/s-1}\left\|\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)\right\|\geq P_n^{1/s-1}\left\|\sum_{i=1}^{n}p_i\phi_i\left(A_i\right)\right\|\geq r^{1/s}Q_n^{1/s-1}\left\|\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)\right\|.\tag{3.6}$$

We also have

$$\square_{f,\tilde{A},v}\left(\Psi_{p,\Phi}\right):=\sum_{i=1}^{n}p_i\left\langle\phi_i\left(f\left(A_i\right)\right)v,v\right\rangle-P_n f\left(\frac{\left\langle\sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle}{P_n}\right),\tag{3.7}$$

where $p\in\mathbb{R}_{++}^{n}$.

By utilising (2.9) we can state that

$$\begin{aligned}&R\left[\sum_{i=1}^{n}q_i\left\langle\phi_i\left(f\left(A_i\right)\right)v,v\right\rangle-Q_n f\left(\frac{\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right)\right]\\&\geq\sum_{i=1}^{n}p_i\left\langle\phi_i\left(f\left(A_i\right)\right)v,v\right\rangle-P_n f\left(\frac{\left\langle\sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle}{P_n}\right)\\&\geq r\left[\sum_{i=1}^{n}q_i\left\langle\phi_i\left(f\left(A_i\right)\right)v,v\right\rangle-Q_n f\left(\frac{\left\langle\sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right)\right]\end{aligned}\tag{3.8}$$

for $p,q\in\mathbb{R}_{++}^{n}$ with $r=\min_{i\in\{1,...,n\}}\left\{\frac{p_i}{q_i}\right\}$, $R=\max_{i\in\{1,...,n\}}\left\{\frac{p_i}{q_i}\right\}$ and $r<\frac{P_n}{Q_n}<R$.

If we take $f(t)=|t|^{\alpha}$, $t\in\mathbb{R}$ with $\alpha\geq 1$ then for any selfadjoint operators A_i, $i\in\{1,...,n\}$ we have

$$R\left[\sum_{i=1}^{n}q_i\left\langle \phi_i\left(|A_i|^{\alpha}\right)v,v\right\rangle - Q_n^{1-\alpha}\left|\left\langle \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle\right|^{\alpha}\right] \tag{3.9}$$
$$\geq \sum_{i=1}^{n}p_i\left\langle \phi_i\left(|A_i|^{\alpha}\right)v,v\right\rangle - P_n^{1-\alpha}\left|\left\langle \sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle\right|^{\alpha}$$
$$\geq r\left[\sum_{i=1}^{n}q_i\left\langle \phi_i\left(|A_i|^{\alpha}\right)v,v\right\rangle - Q_n^{1-\alpha}\left|\left\langle \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle\right|^{\alpha}\right].$$

Finally, since

$$\begin{aligned}\Diamond_{f,\tilde{A},v}\left(\Psi_{p,\Phi}\right) = \frac{1}{M-m}&\left[\left\langle\left(MP_n 1_H - \sum_{i=1}^{n}p_i\phi_i\left(A_i\right)\right)v,v\right\rangle f\left(m\right)\right. \\ &\left. + \left\langle\left(\sum_{i=1}^{n}p_i\phi_i\left(A_i\right) - mP_n 1_H\right)v,v\right\rangle f\left(M\right)\right] \\ &- P_n f\left(\frac{\left\langle \sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle}{P_n}\right),\end{aligned}$$

then by (2.13) we have

$$\begin{aligned}R&\left\{\frac{1}{M-m}\left[\left\langle\left(MQ_n 1_H - \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)\right)v,v\right\rangle f\left(m\right)\right.\right. \\ &\left. + \left\langle\left(\sum_{i=1}^{n}q_i\phi_i\left(A_i\right) - mQ_n 1_H\right)v,v\right\rangle f\left(M\right)\right] \\ &\left. -Q_n f\left(\frac{\left\langle \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right)\right\} \\ &\geq \frac{1}{M-m}\left[\left\langle\left(MP_n 1_H - \sum_{i=1}^{n}p_i\phi_i\left(A_i\right)\right)v,v\right\rangle f\left(m\right)\right. \\ &\left. + \left\langle\left(\sum_{i=1}^{n}p_i\phi_i\left(A_i\right) - mP_n 1_H\right)v,v\right\rangle f\left(M\right)\right] \\ &- P_n f\left(\frac{\left\langle \sum_{i=1}^{n}p_i\phi_i\left(A_i\right)v,v\right\rangle}{P_n}\right) \\ &\geq r\left\{\frac{1}{M-m}\left[\left\langle\left(MQ_n 1_H - \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)\right)v,v\right\rangle f\left(m\right)\right.\right. \\ &\left. + \left\langle\left(\sum_{i=1}^{n}q_i\phi_i\left(A_i\right) - mQ_n 1_H\right)v,v\right\rangle f\left(M\right)\right] \\ &\left. -Q_n f\left(\frac{\left\langle \sum_{i=1}^{n}q_i\phi_i\left(A_i\right)v,v\right\rangle}{Q_n}\right)\right\}\end{aligned}$$

for $p, q \in \mathbb{R}_{++}^n$ with $r = \min_{i \in \{1,\dots,n\}} \left\{ \frac{p_i}{q_i} \right\}$, $R = \max_{i \in \{1,\dots,n\}} \left\{ \frac{p_i}{q_i} \right\}$ and $r < \frac{P_n}{Q_n} < R$.

Several other inequalities may be obtained if one chooses the convex functions $f(t) = -\ln t$, $t \ln t$, t^β where $t > 0$ and $\beta \in (-\infty, 0) \cup [1, \infty)$ or $f(t) = \exp(\gamma t)$, $t, \gamma \in \mathbb{R}$ and $\gamma \neq 0$. The details are omitted.

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