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PERTURBED COMPANION OF OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE OF BOUNDED VARIATION

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Abstract. In this paper, some perturbed companions of Ostrowski type integral inequalities for functions whose first derivatives are of bounded variation are established.

1. Introduction

In 1938, Ostrowski [29] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum\Delta f(P)$ denotes the sum $\sum\limits_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b(f):=\sup\left\{\sum\Delta f(P): P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [16], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

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Theorem 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

For a function of bounded variation $v:[a,b]\rightarrow\mathbb{C}$. We define the C*umulative Variation Function* (CVF) $V:[a;b]\rightarrow[0,\infty)$ by

$$V(t):=\bigvee_a^t(v),$$

the total variation of v on the interval $[a,t]$ with $t\in[a,b]$.

It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c\in[a,b]$ if and only if the generating function v is continuing in that point. If v is *Lipschitzian* with the constant $L>0$, i.e.

$$|v(t)-v(s)|\leq L|t-s|\,,\text{ for any } t,s\in[a,b]\,,$$

then V is also Lipschitzian with the same constant.

A simple proof of the following Lemma was given in [17].

Lemma 1. *Let $f,u:[a,b]\rightarrow\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist and*

$$\left|\int_a^b f(t)du(t)\right|\leq\int_a^b|f(t)|\,d\left(\bigvee_a^t(u)\right)\leq\max_{t\in[a,b]}|f(t)|\bigvee_a^b(u). \tag{1.3}$$

In [8], authors obtained the following companion of Ostrowski type inequalities for functions whose first derivatives are of bounded variation:

Theorem 3. *Let $f:[a,b]\rightarrow\mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then we have the inequality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)[f'(x)-f'(a+b-x)]\right|\\ \leq\;&\frac{1}{16}\left[\frac{5(x-a)^2-2(x-a)(b-x)+(b-x)^2}{b-a}+4\left|x-\frac{3a+b}{4}\right|\right]\bigvee_a^b(f')\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

In the past, many authors have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example ([1]-[10], [13]-[19], [27],[28],[30],[32]-[38]). Furthermore, several works were devoted to study of perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to ([11],[12], [20]-[26],[31],[35]). In this study, we establish some perturbed companion of Ostrowski type inequalities for twice differentiable functions whose second derivatives are either bounded or of bounded variation.

2. Some Identities

Before we start our main results, we state and prove the following lemma:

Lemma 2. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b). Then for any $\lambda_i(x)$, $i=1,2,3$ complex number and all $x\in\left[a,\frac{a+b}{2}\right]$ the following identity holds*

$$\begin{aligned}(2.1)\quad&\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt\\&-\frac{1}{6(b-a)}\left[(x-a)^3\left(\lambda_1(x)+\lambda_3(x)\right)+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right]\\=&\ \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\,d\left[f'(t)-\lambda_1(x)t\right]+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)-\lambda_2(x)t\right]\right.\\&\left.+\int_{a+b-x}^b (t-b)^2\,d\left[f'(t)-\lambda_3(x)t\right]\right].\end{aligned}$$

Proof. Using the integration by parts, we have

$$\begin{aligned}(2.2)\qquad&\int_a^x (t-a)^2\,d\left[f'(t)-\lambda_1(x)t\right]\\&=\int_a^x (t-a)^2\,df'(t)-\lambda_1(x)\int_a^x (t-a)^2\,dt\\&=(x-a)^2 f'(x)-2(x-a)f(x)+2\int_a^x f(t)dt-\frac{\lambda_1(x)}{3}(x-a)^3,\end{aligned}$$

$$\begin{aligned}(2.3)\qquad&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)-\lambda_2(x)t\right]\\&=\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 df'(t)-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\\&=\left(\frac{a+b}{2}-x\right)^2\left[f'(a+b-x)-f'(x)\right]\\&\quad-2\left(\frac{a+b}{2}-x\right)\left[f(a+b-x)+f(x)\right]+2\int_x^{a+b-x}f(t)dt\\&\quad-\frac{2}{3}\lambda_2(x)\left(\frac{a+b}{2}-x\right)^3\end{aligned}$$

and

$$\begin{aligned}
&\int_{a+b-x}^{b} (t-b)^2 \, d\left[f'(t)-\lambda_3(x)t\right] \qquad (2.4)\\
&= \int_{a+b-x}^{b} (t-b)^2 \, df'(t) - \lambda_3(x) \int_{a+b-x}^{b} (t-b)^2 \, dt \\
&= -(x-a)^2 f'(a+b-x) - 2(x-a) f(a+b-x) \\
&\quad +2\int_{a+b-x}^{b} f(t)dt - \frac{\lambda_3(x)}{3}(x-a)^3 .
\end{aligned}$$

If we add the equality (2.2)-(2.4) and divide by $2(b-a)$, we obtain required identity. □

Corollary 1. *Under assumption of Lemma 2 with* $\lambda_i(x)=\lambda_i, i=1,2,3$

i) if we choose $x=a$*, we have*

$$\begin{aligned}
&\frac{b-a}{8}\left[f'(b)-f'(a)\right] - \frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)^2}{24}\lambda_2 \qquad (2.5)\\
&= \frac{1}{2(b-a)}\int_a^b \left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)-\lambda_2 t\right],
\end{aligned}$$

ii) if we choose $x=\frac{a+b}{2}$*, we have*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}(\lambda_1+\lambda_3) \qquad (2.6)\\
&= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (t-a)^2 \, d\left[f'(t)-\lambda_1 t\right] + \int_{\frac{a+b}{2}}^{b} (t-b)^2 \, d\left[f'(t)-\lambda_3 t\right]\right],
\end{aligned}$$

iii) if we choose $x=\frac{3a+b}{4}$*, we have*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] - \frac{(b-a)^2}{384}(\lambda_1+2\lambda_2+\lambda_3) \qquad (2.7)\\
&= \frac{1}{2(b-a)}\left[\int_a^{\frac{3a+b}{4}} (t-a)^2 \, d\left[f'(t)-\lambda_1 t\right] + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)-\lambda_2 t\right]\right.\\
&\quad \left. + \int_{\frac{a+3b}{4}}^{b} (t-b)^2 \, d\left[f'(t)-\lambda_3 t\right]\right].
\end{aligned}$$

Corollary 2. *If we take $\lambda_1 = -\lambda_3$ in (2.6), then we get*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$
$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (t-a)^2\, d\left[f'(t)-\lambda_1 t\right] + \int_{\frac{a+b}{2}}^b (t-b)^2\, d\left[f'(t)+\lambda_1 t\right]\right],$$

and choosing $\lambda_1 = \lambda_3 = -\lambda_2$ in (2.7), we have the inequality

$$\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$
$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{3a+b}{4}} (t-a)^2\, d\left[f'(t)-\lambda_1 t\right] + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)+\lambda_1 t\right] dt\right.$$
$$\left. + \int_{\frac{a+3b}{4}}^b (t-b)^2\, d\left[f'(t)-\lambda_1 t\right]\right].$$

3. Inequalities for Functions Whose First Derivatives are of Bounded Variation

We denote by $\ell : [a,b] \to [a,b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a,b]$.

Theorem 4. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the first derivative f' is of bounded variation on $[a,b]$, then*

(3.1)
$$\left|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2} - \frac{f(x)+f(a+b-x)}{2} + \frac{1}{b-a}\int_a^b f(t)dt\right.$$
$$\left. -\frac{1}{6(b-a)}\left[(x-a)^3\left(\lambda_1(x)+\lambda_3(x)\right)+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right]\right|$$
$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\left(\bigvee_t^x (f'-\lambda_1(x)\ell)\right)dt + \int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\bigvee_x^t (f'-\lambda_2(x)\ell)\right)dt\right.$$
$$\left. + \int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left(\bigvee_t^{a+b-x}(f'-\lambda_2(x)\ell)\right)dt + \int_{a+b-x}^b (b-t)\left(\bigvee_{a+b-x}^t (f'-\lambda_3(x)\ell)\right)dt\right]$$

$$\leq \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x(f'-\lambda_1(x)\ell)+\left(\frac{a+b}{2}-x\right)^2\bigvee_x^{a+b-x}(f'-\lambda_2(x)\ell)\right.$$
$$\left.+(x-a)^2\bigvee_{a+b-x}^{b}(f'-\lambda_3(x)\ell)\right]$$
$$\leq \frac{1}{2(b-a)}\begin{cases}(b-a)^2\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]^2 \\ \times\left[\bigvee_a^x(f'-\lambda_1(x)\ell)+\bigvee_x^{a+b-x}(f'-\lambda_2(x)\ell)+\bigvee_{a+b-x}^{b}(f'-\lambda_3(x)\ell)\right] \\ \left[2(x-a)^2+\left(\frac{a+b}{2}-x\right)^2\right] \\ \times\max\left\{\bigvee_a^x(f'-\lambda_1(x)\ell),\bigvee_x^{a+b-x}(f'-\lambda_2(x)\ell),\bigvee_{a+b-x}^{b}(f'-\lambda_3(x)\ell)\right\}\end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Taking the modulus identity (2.1) and using Lemma 1, we have

(3.2)
$$\left|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt\right.$$
$$\left.-\frac{1}{6(b-a)}\left[(x-a)^3(\lambda_1(x)+\lambda_3(x))+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right]\right|$$
$$=\frac{1}{2(b-a)}\left[\left|\int_a^x(t-a)^2d\left[f'(t)-\lambda_1(x)t\right]\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2d\left[f'(t)-\lambda_2(x)t\right]\right|\right.$$
$$\left.+\left|\int_{a+b-x}^{b}(t-b)^2d\left[f'(t)-\lambda_3(x)t\right]\right|\right]$$
$$\leq\frac{1}{2(b-a)}\left[\int_a^x(t-a)^2d\left(\bigvee_a^t(f'-\lambda_1(x)\ell)\right)\right.$$
$$+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2d\left(\bigvee_a^t(f'-\lambda_2(x)\ell)\right)$$
$$\left.+\int_{a+b-x}^{b}(t-b)^2d\left(\bigvee_a^t(f'-\lambda_3(x)\ell)\right)\right].$$

Using the integration by parts in the Riemann-Stieltjes integral, we get

$$
\begin{aligned}
(3.3)\qquad & \int_a^x (t-a)^2\, d\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right) \\
&= (t-a)^2 \bigvee_a^t (f'-\lambda_1(x)\ell)\Big|_a^x - 2\int_a^x (t-a)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right) dt \\
&= (x-a)^2 \bigvee_a^x (f'-\lambda_1(x)\ell) - 2\int_a^x (t-a)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right) dt \\
&= 2\int_a^x (t-a)\left(\bigvee_a^x (f'-\lambda_1(x)\ell)\right) dt - 2\int_a^x (t-a)\left(\bigvee_a^t (f'-\lambda_1(x)\ell)\right) dt \\
&= 2\int_a^x (t-a)\left(\bigvee_t^x (f'-\lambda_1(x)\ell)\right) dt,
\end{aligned}
$$

$$
\begin{aligned}
(3.4)\qquad & \int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)^2 d\left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right) \\
&= \left(t-\frac{a+b}{2}\right)^2 \left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right)\Bigg|_x^{a+b-x} - 2\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)\left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right) dt \\
&= \left(\frac{a+b}{2}-x\right)^2 \left(\bigvee_a^{a+b-x} (f'-\lambda_2(x)\ell)\right) - \left(x-\frac{a+b}{2}\right)^2 \left(\bigvee_a^x (f'-\lambda_2(x)\ell)\right) \\
&\quad -2\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)\left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right) dt \\
&= 2\int_{\frac{a+b}{2}}^{a+b-x} \left(t-\frac{a+b}{2}\right)\left(\bigvee_a^{a+b-x} (f'-\lambda_2(x)\ell)\right) dt - 2\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left(\bigvee_a^x (f'-\lambda_2(x)\ell)\right) dt \\
&\quad +2\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right) dt - 2\int_{\frac{a+b}{2}}^{a+b-x} \left(t-\frac{a+b}{2}\right)\left(\bigvee_a^t (f'-\lambda_2(x)\ell)\right) dt \\
&= 2\int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left(\bigvee_x^t (f'-\lambda_2(x)\ell)\right) dt + 2\int_{\frac{a+b}{2}}^{a+b-x} \left(t-\frac{a+b}{2}\right)\left(\bigvee_t^{a+b-x} (f'-\lambda_2(x)\ell)\right) dt
\end{aligned}
$$

and

$$\begin{aligned}
(3.5)\quad & \int_{a+b-x}^{b} (t-b)^2 \, d\left(\bigvee_a^t (f' - \lambda_3(x)\ell) \right) \\
&= (t-b)^2 \bigvee_a^t (f' - \lambda_3(x)\ell) \Big|_{a+b-x}^{b} - 2\int_{a+b-x}^{b} (t-b) \bigvee_a^t (f' - \lambda_3(x)\ell) dt \\
&= -(x-a)^2 \left(\bigvee_a^{a+b-x} (f' - \lambda_3(x)\ell) \right) - 2\int_{a+b-x}^{b} (t-b) \left(\bigvee_a^t (f' - \lambda_3(x)\ell) \right) dt \\
&= -2\int_{a+b-x}^{b} (b-t) \left(\bigvee_a^{a+b-x} (f' - \lambda_3(x)\ell) \right) dt + 2\int_{a+b-x}^{b} (b-t) \left(\bigvee_a^t (f' - \lambda_3(x)\ell) \right) dt \\
&= 2\int_{a+b-x}^{b} (b-t) \left(\bigvee_{a+b-x}^{t} (f' - \lambda_3(x)\ell) \right) dt.
\end{aligned}$$

If we substitute the equalities (3.3)-(3.5) in (3.2), we have the first inequality in (3.1).

Here, we have

$$\begin{aligned}
(3.6)\quad \int_a^x (t-a) \left(\bigvee_t^x (f' - \lambda_1(x)\ell) \right) dt &\leq \left(\bigvee_a^x (f' - \lambda_1(x)\ell) \right) \int_a^x (t-a)\, dt \\
&= \frac{(x-a)^2}{2} \bigvee_a^x (f' - \lambda_1(x)\ell),
\end{aligned}$$

$$\begin{aligned}
(3.7)\quad & \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left(\bigvee_x^t (f' - \lambda_2(x)\ell) \right) dt \\
&\leq \left(\bigvee_x^{\frac{a+b}{2}} (f' - \lambda_2(x)\ell) \right) \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) dt \\
&= \frac{1}{2} \left(\frac{a+b}{2} - x \right)^2 \bigvee_x^{\frac{a+b}{2}} (f' - \lambda_2(x)\ell),
\end{aligned}$$

$$\begin{aligned}
(3.8)\qquad & \int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left(\bigvee_{t}^{a+b-x}(f'-\lambda_2(x)\ell)\right)dt \\
\leq\ & \left(\bigvee_{\frac{a+b}{2}}^{a+b-x}(f'-\lambda_2(x)\ell)\right)\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\
=\ & \frac{1}{2}\left(\frac{a+b}{2}-x\right)^2\bigvee_{\frac{a+b}{2}}^{a+b-x}(f'-\lambda_2(x)\ell)
\end{aligned}$$

and

$$\begin{aligned}
(3.9)\qquad & \int_{a+b-x}^{b}(b-t)\left(\bigvee_{a+b-x}^{t}(f'-\lambda_3(x)\ell)\right)dt \\
\leq\ & \left(\bigvee_{a+b-x}^{b}(f'-\lambda_3(x)\ell)\right)\int_{a+b-x}^{b}(b-t)\,dt \\
=\ & \frac{(x-a)^2}{2}\bigvee_{a+b-x}^{b}(f'-\lambda_3(x)\ell).
\end{aligned}$$

With the inequalities (3.6)-(3.9), we obtain the second inequality in (3.1).
The last inequality obvious by maximum properties. □

Corollary 3. *Under assumption of Theorem 4 with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$,*
i) if we choose $x = a$, then we have

$$\begin{aligned}
(3.10)\qquad & \left|\frac{b-a}{8}\left[f'(b)-f'(a)\right]-\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-a)^2}{24}\lambda_2\right| \\
\leq\ & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\bigvee_a^t(f'-\lambda_2\ell)\right)dt\right. \\
& \left.+\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)\left(\bigvee_t^b(f'-\lambda_2\ell)\right)dt\right] \\
\leq\ & \frac{(b-a)}{8}\bigvee_a^b(f'-\lambda_2\ell).
\end{aligned}$$

ii) if we choose $x = \frac{a+b}{2}$, then we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}[\gamma_1 + \gamma_3] \right| \tag{3.11}$$
$$\leq \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} (t-a) \left(\bigvee_t^{\frac{a+b}{2}} (f' - \lambda_1 \ell) \right) dt + \right.$$
$$\left. + \int_{\frac{a+b}{2}}^b (b-t) \left(\bigvee_{\frac{a+b}{2}}^t (f' - \lambda_3 \ell) \right) dt \right]$$
$$\leq \frac{(b-a)}{8} \left[\bigvee_a^{\frac{a+b}{2}} (f' - \lambda_1 \ell) + \bigvee_{\frac{a+b}{2}}^b (f' - \lambda_3 \ell) \right].$$

iii) if we choose $x = \frac{3a+b}{4}$, then we have

(3.12)
$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{(b-a)^2}{384}[\lambda_1 + 2\gamma_2 + \lambda_3] \right|$$
$$\leq \frac{1}{b-a} \left[\int_a^{\frac{3a+b}{4}} (t-a) \left(\bigvee_t^{\frac{3a+b}{4}} (f' - \lambda_1 \ell) \right) dt + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left(\bigvee_{\frac{3a+b}{4}}^t (f' - \lambda_2 \ell) \right) dt \right.$$
$$\left. + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} \left(t - \frac{a+b}{2} \right) \left(\bigvee_t^{\frac{a+3b}{4}} (f' - \lambda_2 \ell) \right) dt + \int_{\frac{a+3b}{4}}^b (b-t) \left(\bigvee_{\frac{a+3b}{4}}^t (f' - \lambda_3 \ell) \right) dt \right]$$
$$\leq \frac{(b-a)}{32} \left[\bigvee_a^{\frac{3a+b}{4}} (f' - \lambda_1 \ell) + \bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} (f' - \lambda_2 \ell) \bigvee_{\frac{a+3b}{4}}^b (f' - \lambda_3 \ell) \right].$$

Corollary 4. *If we choose $\gamma_1 = -\gamma_3$ in (3.11) and $\gamma_1 = \gamma_3 = -\gamma_2$ in (3.12), then we have the following inequality respectively,*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| \tag{3.13}$$
$$\leq \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} (t-a) \left(\bigvee_t^{\frac{a+b}{2}} (f' - \lambda_1 \ell) \right) dt + + \int_{\frac{a+b}{2}}^b (b-t) \left(\bigvee_{\frac{a+b}{2}}^t (f' + \lambda_1 \ell) \right) dt \right]$$
$$\leq \frac{(b-a)}{8} \left[\bigvee_a^{\frac{a+b}{2}} (f' - \lambda_1 \ell) + \bigvee_{\frac{a+b}{2}}^b (f' + \lambda_1 \ell) \right].$$

and

(3.14)

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|$$

$$\leq \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}(t-a)\left(\bigvee_t^{\frac{3a+b}{4}}(f'-\lambda_1\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\bigvee_{\frac{3a+b}{4}}^{t}(f'+\lambda_1\ell)\right)dt\right.$$

$$\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left(\bigvee_t^{\frac{a+3b}{4}}(f'+\lambda_1\ell)\right)dt+\int_{\frac{a+3b}{4}}^{b}(b-t)\left(\bigvee_{\frac{a+3b}{4}}^{t}(f'-\lambda_1\ell)\right)dt\right]$$

$$\leq \frac{(b-a)}{32}\left[\bigvee_a^{\frac{3a+b}{4}}(f'-\lambda_1\ell)+\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f'+\lambda_1\ell)\bigvee_{\frac{a+3b}{4}}^{b}(f'-\lambda_1\ell)\right].$$

4. Inequalities for Functions Whose First Derivatives are Lipschitzian

Theorem 5. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on I° and $[a,b]\subset I^\circ$. If $f'-\lambda_1(x)\ell$ is Lipschitzian with the constant $K_1(x)$ on the interval $[a,x]$, $f'-\lambda_2(x)\ell$ is Lipschitzian with the constant $K_2(x)$ on the interval $[x,a+b-x]$, and $f'-\lambda_1(x)\ell$ is Lipschitzian with the constant $K_3(x)$ on the interval $[a+b-x,b]$ then, for any $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_i(x)$, $i=1,2,3$ complex numbers, we have the inequalities*

(4.1)

$$\left|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt\right.$$

$$\left.-\frac{1}{6(b-a)}\left[(x-a)^3(\lambda_1(x)+\lambda_3(x))+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right]\right|$$

$$\leq \frac{1}{6(b-a)}\left[K_1(x)(x-a)^3+2K_2(x)\left(\frac{a+b}{2}-x\right)^3+K_3(x)(x-a)^3\right]$$

$$\leq \frac{1}{3(b-a)}\left[(x-a)^3+\left(\frac{a+b}{2}-x\right)^3\right]\max\{K_1(x),K_2(x),K_1(x)\}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $K>0$, then the Riemann-Stieltje integral $\int_c^d g(t)du(t)$ exist and

$$\left|\int_c^d g(t)du(t)\right|\leq K\int_c^d|g(t)|\,dt.$$

Taking the modulus (2.1), we get

$$
\begin{aligned}
&\left|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt\right. \\
&\left.-\frac{1}{6(b-a)}\left[(x-a)^3(\lambda_1(x)+\lambda_3(x))+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right]\right| \\
\leq\ &\frac{1}{2(b-a)}\left[\left|\int_a^x (t-a)^2 d\left[f'(t)-\lambda_1(x)t\right]\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 d\left[f'(t)-\lambda_2(x)t\right]\right|\right. \\
&\left.+\left|\int_{a+b-x}^b (t-b)^2 d\left[f'(t)-\lambda_3(x)t\right]\right|\right] \\
\leq\ &\frac{1}{2(b-a)}\left[K_1(x)\int_a^x (t-a)^2 dt+K_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right. \\
&\left.+K_3(x)\int_{a+b-x}^b (t-b)^2 dt\right] \\
=\ &\frac{1}{6(b-a)}\left[K_1(x)(x-a)^3+2K_2(x)\left(\frac{a+b}{2}-x\right)^3+K_3(x)(x-a)^3\right]
\end{aligned}
\tag{4.2}
$$

which completes the proof of the first inequality in (4.1).

For the second inequality, using the property of maximum in the last line in (4.2), we have

$$
\begin{aligned}
&K_1(x)(x-a)^3+2K_2(x)\left(\frac{a+b}{2}-x\right)^3+K_3(x)(x-a)^3 \\
\leq\ &2\left[(x-a)^3+\left(\frac{a+b}{2}-x\right)^3\right]\max\left\{K_1(x),K_2(x),K_1(x)\right\}.
\end{aligned}
$$

This proves the theorem. □

Corollary 5. *Under the assumption of Theorem 5, we have the following inequalities for the spacial cases,*

i) for $x=\frac{a+b}{2}$,

$$
\left|\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{48}\left[\lambda_1+\lambda_3\right]\right|\leq\frac{(b-a)^2}{24}\left[\frac{K_1+K_3}{2}\right]
$$

ii) for $x = \frac{3a+b}{4}$

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[f\left(\frac{3a+b}{4} \right) + f\left(\frac{a+3b}{4} \right) \right] - \frac{(b-a)^2}{384} \left[\lambda_1 + 2\lambda_2 + \lambda_3 \right] \right|$$

$$\leq \frac{(b-a)^2}{96} \left[\frac{K_1 + 2K_2 + K_3}{4} \right].$$

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