Received 10/05/16

GENERALIZED WEIGHTED ČEBYSEV AND OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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Abstract. In this paper, we firstly establish generalized weighted Montgomery identity for double integrals. Then, some generalized weighted Čebysev and Ostrowski type inequalities for double integrals are given.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty ,$$

for all $x\in[a,b]$[10]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.* For some results which generalize, improve and extend the inequality (1.1) see [2, 5, 18, 19, 21] and the references therein.

In [4], P. L. Čebysev proved the following important integral inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2\|f'\|_\infty \|g'\|_\infty \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_\infty[a,b]$ and

$$T(f,g)=\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int\limits_a^b f(x)dx\right)\left(\frac{1}{b-a}\int\limits_a^b g(x)dx\right) \tag{1.2}$$

which is called the Čebysev functional, provided the integrals in (1.2) exist. In recent years many researchers have given the generalization of Čebysev type inequalities, we can mention the works [1, 3, 6, 9, 12, 13, 14, 16, 20].

Let $w_1:[a,b]\to[0,\infty)$ a weight function. We define $m_1(a,b)=\int\limits_a^b w_1(s)ds$ and $m_1(a,t)=\int\limits_a^t w_1(s)ds$, so that $m_1(a,t)=0$ for $t<a$.

In [13], the authours proved the following weighted Montgomery's indentity:

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. Čebysev type inequalities, Ostrowski type inequalities, weighted integral inequalities.

Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $\varphi_1:\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function on $\mathbb{R}_+$ with $\varphi_1(0)=0$, $\varphi_1(m_1(a,b))\neq 0$ and φ_1' is integrable on $\mathbb{R}_+$, then

$$f(x) = \frac{1}{\varphi_1(m_1(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t)dt + \frac{1}{\varphi_1(m_1(a,b))}\int_a^b P_{w_1,\varphi_1}(x,t)f'(t)dt \tag{1.3}$$

for all $x\in[a,b]$, where

$$P_{w_1,\varphi_1}(x,t)=\begin{cases}\varphi_1(m_1(a,t)), & a\le t\le x\\ \varphi_1(m_1(a,t))-\varphi_1(m_1(a,b)), & x\le t\le b.\end{cases} \tag{1.4}$$

Recently, many authors have studied on Čebysev inequality for double integrals, please see [7, 8, 11, 15]. In [8], authors established a weighted Čebysev type inequality for double integrals using the probability density functions. In this paper, we obtain a generalized weighted Čebysev type inequality similir to this inequality for double integrals using the weighted funtions which are not necessarily the probability density functions. Moreover, we established an Ostrowski type inequality for double integral which is the generalization of the inequality given in [17].

2. Generalized Weighted Montgomery Identity for Double Integrals

In order to prove our main theorems, we need prove following identities:

Let $w_2:[c,d]\to[0,\infty)$ a weight function. We define $m_2(c,d)=\int_c^d w_2(u)du$ and $m_2(c,s)=\int_c^s w_2(u)du$, so that $m_2(c,s)$ for $s<c$. $\varphi_2:\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function on $\mathbb{R}_+$ with $\varphi_2(0)=0$, $\varphi_2(m_2(c,d))\neq 0$ and φ_2' is integrable on $\mathbb{R}_+$.

Theorem 1. *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a partial differentiable function such that second derivative $\frac{\partial^2 f(t,s)}{\partial s\partial t}$ is integrable on Δ. Then for all $(x,y)\in\Delta$ we have*

$$\begin{aligned} f(x,y) &= \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \\ &\times\left[\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt\right. \\ &+\int_a^b\int_c^d w_1(t)\varphi_1'(m_1(a,t))Q_{w_2,\varphi_2}(y,s)\frac{\partial f(t,s)}{\partial s}dsdt \\ &+\int_a^b\int_c^d w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dsdt \\ &\left.+\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial s\partial t}dsdt\right] \end{aligned} \tag{2.1}$$

where $P_{w_1,\varphi_1}(x,t)$ is defined as in (1.4) and $Q_{w_2,\varphi_2}(y,s)$ defined by

$$Q_{w_2,\varphi_2}(y,s)=\begin{cases}\varphi_2(m_2(c,s)), & c\le s\le y\\ \varphi_2(m_2(c,s))-\varphi_2(m_2(c,d)), & y\le s\le d.\end{cases}\tag{2.2}$$

Proof. Aplying the identity (1.3) for the partial derivative $\frac{\partial f(t,y)}{\partial t}$, we have

$$\begin{aligned}f(x,y) &= \frac{1}{\varphi(m(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt\\ &+\frac{1}{\varphi_1(m_1(a,b))}\int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt\end{aligned}\tag{2.3}$$

for all $(x,y)\in\Delta$. Similarly, applying the identity (1.3) for the partial derivative $\frac{\partial f(t,s)}{\partial s}$, we get

$$\begin{aligned}f(t,y) &= \frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(t,s)ds\\ &+\frac{1}{\varphi_2(m_2(c,d))}\int_c^d Q_{w_2,\varphi_2}(y,s)\frac{\partial f(t,s)}{\partial s}ds\end{aligned}\tag{2.4}$$

for all $(t,y)\in\Delta$. For partial derivative of (2.4) according to t, we have

$$\begin{aligned}\frac{\partial f(t,y)}{\partial t} &= \frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))\frac{\partial f(t,s)}{\partial t}ds\\ &+\frac{1}{\varphi_2(m_2(c,d))}\int_c^d Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial s\partial t}ds\end{aligned}\tag{2.5}$$

for all $(t,y)\in\Delta$. If we subsitute the equalities (2.4) and (2.5) in (2.3), then we obtain the required result. □

Remark 1. *If we choose $\varphi_1(u)\equiv\varphi_2(u)\equiv u$ in the Theorem 1, then the Theorem 1 reduces the Lemma 2 in* [17].

Theorem 2. *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable function such that second derivative $\frac{\partial^2 f(t,s)}{\partial s\partial t}$ is integrable on Δ. Then we have the following generalized* *weighted Montgomery's identity,*

$$
\begin{aligned}
(2.6) f(x,y) &- \frac{1}{\varphi_1(m_1(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt \\
&-\frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(x,s)ds \\
&+\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt \\
&= \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt
\end{aligned}
$$

for all $(x,y)\in\Delta$.

Proof. Using the integration by parts we have,

$$
\begin{aligned}
(2.7) &\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt \\
&= \int_a^b P_{w_1,\varphi_1}(x,t)\left[\int_c^d \varphi_2(m_2(c,s))\frac{\partial^2 f(t,s)}{\partial t\partial s}ds - \varphi_2(m_2(c,d))\int_c^d \frac{\partial^2 f(t,s)}{\partial t\partial s}ds\right]dt \\
&= \int_a^b P_{w_1,\varphi_1}(x,t)\left[\varphi_2(m_2(c,d))\frac{\partial f(t,y)}{\partial t} - \int_c^d w_2(s)\varphi_2'(m_2(c,s))\frac{\partial f(t,s)}{\partial t}ds\right]dt \\
&= \varphi_2(m_2(c,d))\int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt \\
&\quad -\int_c^d\int_a^b w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dtds.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(2.8) \qquad &\int_a^b P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,y)}{\partial t}dt \\
&= \int_a^b \varphi_1(m_1(a,t))\frac{\partial f(t,y)}{\partial t}dt - \varphi_1(m_1(a,b))\int_x^b \frac{\partial f(t,y)}{\partial t}dt \\
&= \varphi_1(m_1(a,b))f(x,y) - \int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.9)\int_c^d\int_a^b & w_2(s)\varphi_2'(m_2(c,s))P_{w_1,\varphi_1}(x,t)\frac{\partial f(t,s)}{\partial t}dtds \\
= & \int_c^d w_2(s)\varphi_2'(m_2(c,s))\left[\int_a^b \varphi_1(m_1(a,t))\frac{\partial f(t,s)}{\partial t}dt-\varphi_1(m(a,b))\int_x^b\frac{\partial f(t,s)}{\partial t}dt\right]ds \\
= & \int_c^d w_2(s)\varphi_2'(m_2(c,s))\left[\varphi(m_1(a,b))f(x,s)-\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,s)dt\right]ds \\
= & \varphi_1(m_1(a,b))\int_c^d w_2(s)\varphi_2'(m_2(c,s))f(x,s)ds \\
& -\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dt.
\end{aligned}
$$

If we subsitute the equalities (2.8) and (2.9) in (2.7) and divide the result identity by $\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))$, then we obtain the required identity (2.6). □

Remark 2. *If w_1 and w_2 are probability density functions in (2.6), then the identity (2.6) reduces the identity (6) in* [8].

3. New Generalized Weighted Ostrowski and Čebysev Inequalities

Theorem 3. *Let $f:\Delta\to\mathbb{R}$ has continuous partial derivatives $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$ and $\frac{\partial^2 f(t,s)}{\partial s\partial t}$ on Δ. Then we have the following weighted Ostrowski inequality*

$$
\begin{aligned}
& \left|f(x,y)-\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))f(t,s)dsdt\right| \\
\leq & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\left[m_1(a,b)\left\|\frac{\partial f(t,s)}{\partial s}\right\|_\infty\|\varphi_1'\|_\infty H_2(y)\right. \\
& \left.+m_2(c,d)\left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty\|\varphi_2'\|_\infty H_1(x)+\left\|\frac{\partial f(t,s)}{\partial t}\right\|_\infty H_1(x)H_2(y)\right]
\end{aligned}
$$

where

$$H_1(x)=\int_a^b\left|P_{w_1,\varphi_1}(x,t)\right|dt,$$

and

$$H_2(y)=\int_c^d\left|Q_{w_2,\varphi_2}(y,s)\right|ds.$$

Proof. Taking modulus in Theorem 1, we have

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \int_a^b \int_c^d w_1(t) w_2(s) \varphi_1'(m_1(a,t)) \varphi_2'(m_2(c,s)) f(t,s) ds dt \right| \\
\leq\ & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \left[\int_a^b \int_c^d w_1(t) \left|\varphi_1'(m_1(a,t))\right| \left|Q_{w_2,\varphi_2}(y,s)\right| \left| \frac{\partial f(t,s)}{\partial s} \right| ds dt \right. \\
&+ \int_a^b \int_c^d w_2(s) \left|\varphi_2'(m_2(c,s))\right| \left|P_{w_1,\varphi_1}(x,t)\right| \left| \frac{\partial f(t,s)}{\partial t} \right| ds dt \\
&\left. + \int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| \left| \frac{\partial^2 f(t,s)}{\partial s \partial t} \right| ds dt \right] \\
\leq\ & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \left[\left\| \frac{\partial f(t,s)}{\partial s} \right\|_\infty \left\|\varphi_1'\right\|_\infty \int_a^b \int_c^d w_1(t) \left|Q_{w_2,\varphi_2}(y,s)\right| ds dt \right. \\
&+ \left\| \frac{\partial f(t,s)}{\partial t} \right\|_\infty \left\|\varphi_2'\right\|_\infty \int_a^b \int_c^d w_2(s) \left|P_{w_1,\varphi_1}(x,t)\right| ds dt \\
&\left. + \left\| \frac{\partial f(t,s)}{\partial t} \right\|_\infty \int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| ds dt \right].
\end{aligned}$$

Here, we have the equalities

$$\int_a^b \int_c^d w_1(t) \left|Q_{w_2,\varphi_2}(y,s)\right| ds dt = \left(\int_a^b w_1(t) dt \right) \left(\int_c^d \left|Q_{w_2,\varphi_2}(y,s)\right| ds \right) = m_1(a,b) H_2(y),$$

$$\int_a^b \int_c^d w_2(s) \left|P_{w_1,\varphi_1}(x,t)\right| ds dt = \left(\int_c^d w_2(s) ds \right) \left(\int_a^b \left|P_{w_1,\varphi_1}(x,t)\right| dt \right) = m_2(c,d) H_1(x)$$

and

$$\begin{aligned}
&\int_a^b \int_c^d \left|P_{w_1,\varphi_1}(x,t)\right| \left|Q_{w_2,\varphi_2}(y,s)\right| ds dt \\
=\ & \left(\int_a^b \left|P_{w_1,\varphi_1}(x,t)\right| dt \right) \left(\int_c^d \left|Q_{w_2,\varphi_2}(y,s)\right| ds \right) = H_1(x) H_2(y)
\end{aligned}$$

which complete the proof. □

Remark 3. *If we choose $\varphi_1(u) \equiv \varphi_2(u) \equiv u$ in the Theorem 3, then the Theorem 3 reduces the Theorem 2 in* [17].

Theorem 4. *Let $f, g : \Delta \to \mathbb{R}$ be be a partial differentiable functions such that their second derivatives $\frac{\partial^2 f(t,s)}{\partial s \partial t}$ and $\frac{\partial^2 g(t,s)}{\partial s \partial t}$ are integrable on Δ. Then we have the* *weighted Čebysev inequality*

$$
\begin{aligned}
(3.1) \qquad & |T(w_1,\varphi_1,w_2,\varphi_2,f,g)| \\
\leq \quad & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\|\varphi_1'\|_\infty\|\varphi_2'\|_\infty \\
& \times\int_a^b\int_c^d w_1(x)w_2(y)H_1^2(x)H_2^2(y)dydx
\end{aligned}
$$

where $H_1(x)$ and $H_2(y)$ are defined as in (3) and

$$
\begin{aligned}
& T(w_1,\varphi_1,w_2,\varphi_2,f,g) \\
= \quad & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \\
& \times\left[\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y)g(x,y)dydx\right. \\
& -\frac{1}{\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y) \\
& \times\left(\int_c^d w_2(s)\varphi_2'(m_2(c,s))g(x,s)ds\right)dydx \\
& -\frac{1}{\varphi_1(m_1(a,b))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))g(x,y) \\
& \times\left(\int_a^b w_1(t)\varphi_1'(m_1(a,t))f(t,y)dt\right)dydx \\
& +\frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))} \\
& \times\left(\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))f(x,y)dydx\right) \\
& \left.\times\left(\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))g(x,y)dydx\right)\right].
\end{aligned}
$$

Proof. From Theorem 2, writting again the identity (2.6) for the function $g(x,y)$, we have

$$
\begin{aligned}
(3.2) g(x,y) &- \frac{1}{\varphi_1(m_1(a,b))}\int_a^b w_1(t)\varphi_1'(m_1(a,t))g(t,y)dt \\
&- \frac{1}{\varphi_2(m_2(c,d))}\int_c^d w_2(s)\varphi_2'(m_2(c,s))g(x,s)ds \\
&+ \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d w_1(t)w_2(s)\varphi_1'(m_1(a,t))\varphi_2'(m_2(c,s))g(t,s)dsdt \\
= & \frac{1}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt.
\end{aligned}
$$

After multiplying the identities (2.6) and (3.2), multiplying both sides result by $\frac{w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y))}{\varphi_1(m_1(a,b))\varphi_2(m_2(c,d))}$ and integrating over Δ, we have

$$
\begin{aligned}
(3.3)\quad & T(w_1,\varphi_1,w_2,\varphi_2,f,g) \\
= & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\int_a^b\int_c^d \left[\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 f(t,s)}{\partial t\partial s}dsdt\right] \\
& \times\left[\int_a^b\int_c^d P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)\frac{\partial^2 g(t,s)}{\partial t\partial s}dsdt\right]dydx.
\end{aligned}
$$

Taking the modulus in (3.3), we obtain

$$
\begin{aligned}
& |T(w_1,\varphi_1,w_2,\varphi_2,f,g)| \\
\leq & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\int_a^b\int_c^d w_1(x)w_2(y)\varphi_1'(m_1(a,x))\varphi_2'(m_2(c,y)) \\
& \times\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|dsdt\right] \\
& \times\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\left|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right|dsdt\right]dydx \\
\leq & \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\left\|\frac{\partial^2 g(t,s)}{\partial t\partial s}\right\|_\infty\|\varphi_1'\|_\infty\|\varphi_2'\|_\infty \\
& \times\int_a^b\int_c^d w_1(x)w_2(y)\left[\int_a^b\int_c^d |P_{w_1,\varphi_1}(x,t)Q_{w_2,\varphi_2}(y,s)|\,dsdt\right]^2 dydx
\end{aligned}
$$

$$= \frac{1}{\varphi_1^3(m_1(a,b))\varphi_2^3(m_2(c,d))} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t,s)}{\partial t \partial s} \right\|_\infty \|\varphi_1'\|_\infty \|\varphi_2'\|_\infty$$

$$\times \int_a^b \int_c^d w_1(x) w_2(y) H_1^2(x) H_2^2(y) dy dx.$$

This completes the proof. □

Remark 4. *If w_1 and w_2 are probability density functions in (3.1), then the identity (3.1) reduces the identity (14) in* [8].

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