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WEIGHTED INEQUALITIES FOR RIEMANN-STIELTJES INTEGRALS

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Abstract. In this paper, we define firstly a new functional which is weighted version of the functional defined by Dragomir and Fedotov. Then, some inequalities involving this functional are obtained. Finally, we apply this results to establish new bounds for weighted Chebysev functional.

1. Introduction

The following definitions will be frequently used to prove our results.

Definition 1. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number*

$$\bigvee_a^b (f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [17] Dragomir and Fedotov have established the following functional

$$D(f, u) = \int_a^b f(t)du(t) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t)dt. \tag{1.1}$$

In the same paper, authors proved the following inequality:

Theorem 1. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is of bounded variation on $[a, b]$ and f is Lipschitzian with the constant $L > 0$. Then we have*

$$|D(f, u)| \leq \frac{1}{2} L (b - a) \bigvee_a^b (u) \tag{1.2}$$

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The constant $\frac{1}{2}$ is sharp in the sense that it can not be replaced by a smaller one.

In [1], Alomari gave the following inequality:

Theorem 2. *Let $x\in[a,b]$. Let $f,u:[a,b]\to\mathbb{R}$ be a continuous mappings on $[a,b]$. Assume that u is monotonic nondecreasing mapping on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on both intervals $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|D(f,u)|\leq 2u(b)\left[f(b)-\int_a^b f(t)dt\right].$$

Dragomir gave some new bounds for the function $D(f,u)$ in [22]. One of them is following inequality,

Theorem 3. *Assume that $f,u:[a,b]\to\mathbb{R}$ are of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist. Then,*

$$\begin{aligned}(1.3)\qquad & |D(f,u)| \\ \leq\ & \frac{1}{b-a}\left[\int_a^b [2t-a-b]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)\right. \\ & \left.-\int_a^b\left(\bigvee_a^t(f)\right)dt\bigvee_a^b(u)+2\int_a^b\bigvee_a^t(f)\bigvee_a^t(u)dt\right] \\ \leq\ & \frac{1}{b-a}\int_a^b[2t-a-b]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)+\frac{1}{b-a}\int_a^b\left(\bigvee_a^t(f)\bigvee_a^t(u)\right)dt \\ \leq\ & \frac{1}{b-a}\int_a^b[2t-a-b]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)+\bigvee_a^b(f)\bigvee_a^b(u).\end{aligned}$$

Lemma 1. *Let $f,u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist and*

$$(1.4)\qquad \left|\int_a^b f(t)du(t)\right|\leq\int_a^b|f(t)|\,d\left(\bigvee_a^t(u)\right)\leq\max_{t\in[a,b]}|f(t)|\bigvee_a^b(u).$$

A great many of authors worked on inequalities for Riemann-Stieltjes integral via functions of bounded variation (or derivatives of bounded variation), for some of them please see in [1]-[26]

The main purpose of this paper is to obtain some weighted inequalities for Riemann-Stieltjes integral. First of all, we define a weighted version the functional $D(f,u)$. Then , we establish some bounds for this functional acording to cases of the functions f and u. Finally, some new bounds for the weighted Chebysev funtional are also given.

This paper is divided into the following six sections. In Section 2, we establish some identitities will be used to prove our results. In Section 3 and Section 4, some weighted integral inequalities for the case when the function u is bounded variation and when the function f is bounded variation are given, respectively. In the next section, we give a inequality for the case when u is $(l, L)-$Lipschitzan. Finally, in Section 6, we presented some applications for weighted Chebysev functional using the results given previous sections.

2. Some Identities

Let $w : [a, b] \to \mathbb{R}$ be nonnegative and continuous on $[a, b]$. We define $m(a, b) = \int_a^b w(s)ds$ and $m_1(a, t) = \int_a^t w(s)ds$, so that $m(a, t) = 0$ for $t < a$.

Now we give some representation:

Weighted version of the functional defined by Dragomir and Fedotov:

$$D(w, f, u) = m(a, b) \int_a^b f(t)du(t) - [u(b) - u(a)] \int_a^b w(t)f(t)dt.$$

Weighted Chebysev functional:

$$\begin{aligned} T(w, f, g) &= \frac{1}{m(a, b)} \int_a^b w(t)f(t)g(t)dt \\ &\quad - \left(\frac{1}{m(a, b)} \int_a^b w(t)f(t)dt \right) \left(\frac{1}{m(a, b)} \int_a^b w(t)g(t)dt \right). \end{aligned}$$

Weighted Ostrowski transform:

$$\Theta_{f,w}(t) = m(a, b)f(t) - \int_a^b w(s)f(s)ds.$$

Weighted generalized trapezoid transform:

$$\Phi_{g,w}(t) = m(t, b)g(a) + m(a, t)g(b) - m(a, b)g(t).$$

Before we start our main results, we state and prove following lemmas:

Lemma 2. *If $f, u : [a, b] \to \mathbb{R}$ are bounded functions such that the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ and the Riemann integral $\int_a^b w(t)f(t)dt$ exist, then we have*

$$D(w, f, u) = \int_a^b \Theta_{f,w}(s)du(s) = \int_a^b \left(\int_a^b Q_w(t, s)df(t) \right) du(s), \tag{2.1}$$

where

$$Q_w(t, s) = \begin{cases} \int_b^t w(s)ds, & a \le s \le t \le x \\ \\ \int_a^t w(s)ds, & x < t < s \le b. \end{cases}$$

Proof. Using the integrating the by parts in Riemann-Stieltjes integral, we have

$$
\begin{aligned}
& \int_a^b \left(\int_a^b Q_w(t,s)df(t) \right) du(s) \\
= & \int_a^b \left[\int_a^s \left(\int_a^t w(\xi)d\xi \right) df(t) + \int_s^b \left(\int_b^t w(\xi)d\xi \right) df(t) \right] du(s) \\
= & \int_a^b \left[\left(\int_a^t w(\xi)d\xi \right) f(t)\Big|_a^s - \int_a^s w(t)f(t)dt \right. \\
& \left. \left(\int_b^t w(\xi)d\xi \right) f(t)_s^b - \int_s^b w(t)f(t)dt \right] du(s) \\
= & \int_a^b \left[\left(\int_a^b w(\xi)d\xi \right) f(s) + \int_a^b w(t)f(t)dt \right] du(s) \\
= & \int_a^b \Theta_{f,w}(s)du(s).
\end{aligned}
$$

This completes the proof of the second equality. The first identity is obvious. □

Corollary 1. *Let $g : [a,b] \to \mathbb{R}$ a function such that g is Riemann integrable on $[a,b]$. If we choose $u(t) = \int_a^t w(s)g(s)$ in Lemma 2, then we have*

$$
T(w,f,g) = \frac{1}{m^2(a,b)} \int_a^b \Theta_{f,w}(s)w(s)g(s)ds = \frac{1}{m^2(a,b)} \int_a^b \left(\int_a^b Q_w(t,s)df(t) \right) w(s)g(s)ds.
$$

Lemma 3. *With the assumptions in Lemma 2, we have*

$$
D(w,f,u) = \int_a^b \Phi_{u,w}(t)df(t) = \int_a^b \left(\int_a^b Q_w(t,s)du(s) \right) df(t), \tag{2.2}
$$

where the mapping $Q_w(t,s)$ is defined by as in Lemma 2.

Proof. By the Fubini type theorem for the Riemann-Stieltjes integral, we get

$$
\int_a^b \left(\int_a^b Q_w(t,s)du(s) \right) df(t) = \int_a^b \left(\int_a^b Q_w(t,s)df(t) \right) du(s).
$$

This completes the proof of the between the first and the last term in (2.2).

Integrating the by parts, we obtain

$$\begin{aligned}
\int_a^b Q_w(t,s)du(s) &= \int_a^t\left(\int_b^t w(\xi)d\xi\right)du(s)+\int_t^b\left(\int_a^t w(\xi)d\xi\right)du(s)\\
&= \left(\int_b^t w(\xi)d\xi\right)[u(t)-u(a)]+\left(\int_a^t w(\xi)d\xi\right)[u(b)-u(a)]\\
&= m(t,b)u(a)+m(a,t)u(b)-m(a,b)u(t)\\
&= \Phi_{u,w}(t).
\end{aligned}$$

This completes the proof. □

Corollary 2. *Assume that* $g:[a,b]\to\mathbb{R}$ *Riemann integrable on* $[a,b]$, *then we have*

$$T(w,f,g)=\frac{1}{m^2(a,b)}\int_a^b \widetilde{\Phi}_{g,w}(t)df(t)=\frac{1}{m^2(a,b)}\int_a^b\left(\int_a^b Q_w(t,s)w(s)g(s)ds\right)df(t)$$

where

$$\widetilde{\Phi}_{g,w}(t)=\Phi_{\int_a^{\cdot} g,w}(t)=m(a,t)\int_a^b w(s)g(s)ds-m(a,b)\int_a^t w(s)g(s)ds.$$

Remark 1. *If we choose* $w(t)=t$ *in Lemma 2 and Lemma 3, then our results reduce Lemma 1 and Lemma 2 in* [22], *respectively.*

3. Inequalities in the Case when u is of bounded variation

Now using above identities, we state and prove the following inequalities in the case when u is of bounded variation.

Theorem 4. *Let* $w:[a,b]\to\mathbb{R}$ *be nonnegative and continuous on* $[a,b]$. *If* $f,u:[a,b]\to\mathbb{R}$ *are of bounded variation on* $[a,b]$, *then we have the inequalities*

$$\begin{aligned}
&|D(w,f,u)| \\
&\le \int_a^b[m(a,t)-m(t,b)]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)\\
&\quad -\left(\int_a^b w(t)\bigvee_a^t(f)dt\right)\bigvee_a^b(u)+2\int_a^b w(t)\bigvee_a^t(f)\bigvee_a^t(u)dt\\
&\le \int_a^b[m(a,t)-m(t,b)]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)+\int_a^b w(t)\bigvee_a^t(f)\bigvee_a^t(u)dt\\
&\le \int_a^b[m(a,t)-m(t,b)]\bigvee_a^t(f)d\left(\bigvee_a^t(u)\right)+m(a,b)\bigvee_a^b(f)\bigvee_a^b(u).
\end{aligned}\tag{3.1}$$

Proof. Taking the madulus in Lemma 2 and using the Lemma 1, we have

$$
\begin{aligned}
(3.2)\qquad & |D(w,f,u)| \\
= & \left| \int_a^b \left(\int_a^b Q_w(t,s)\,df(t) \right) du(s) \right| \\
\leq & \int_a^b \left| \int_a^b Q_w(t,s)\,df(t) \right| d\left(\bigvee_a^s (u) \right) \\
= & \int_a^b \left| \int_a^s \left(\int_a^t w(\xi)d\xi \right) df(t) + \int_s^b \left(\int_b^t w(\xi)d\xi \right) df(t) \right| d\left(\bigvee_a^s (u) \right) \\
\leq & \int_a^b \left[\left| \int_a^s \left(\int_a^t w(\xi)d\xi \right) df(t) \right| + \left| \int_s^b \left(\int_b^t w(\xi)d\xi \right) df(t) \right| \right] d\left(\bigvee_a^s (u) \right).
\end{aligned}
$$

Since f is of bounded variation, using again the Lemma 1, we obtain

$$
\begin{aligned}
(3.3)\qquad \left| \int_a^s \left(\int_a^t w(\xi)d\xi \right) df(t) \right| & \leq \int_a^s \left(\int_a^t w(\xi)d\xi \right) d\left(\bigvee_a^t (f) \right) \\
& = \left(\int_a^s w(\xi)d\xi \right) \bigvee_a^t (f) - \int_a^s w(t) \bigvee_a^t (f)dt \\
& = m(a,s) \bigvee_a^s (f) - \int_a^s w(t) \bigvee_a^t (f)dt
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\qquad \left| \int_s^b \left(\int_b^t w(\xi)d\xi \right) df(t) \right| & \leq \int_s^b \left| \int_b^t w(\xi)d\xi \right| d\left(\bigvee_a^t (f) \right) \\
& = \int_s^b \left(\int_t^b w(\xi)d\xi \right) d\left(\bigvee_a^t (f) \right) \\
& = \int_s^b w(t) \bigvee_a^t (f)dt - m(s,b) \bigvee_a^s (f).
\end{aligned}
$$

If we substitute the inequalities (3.3) and (3.4) in (3.2), we establish

$$
\begin{aligned}
&|D(w,f,u)| \qquad (3.5)\\
\leq\ & \int_a^b \left[[m(a,s)-m(s,b)] \bigvee_a^s (f) \right.\\
&\left. - \int_a^s w(t) \bigvee_a^t (f) dt + \int_s^b w(t) \bigvee_a^t (f) dt \right] d\left(\bigvee_a^s (u) \right)\\
=\ & \int_a^b \left[[m(a,s)-m(s,b)] \bigvee_a^s (f) \right] d\left(\bigvee_a^s (u) \right)\\
&+ \int_a^b \left[\int_a^b w(t) \bigvee_a^t (f) dt - 2 \int_a^s w(t) \bigvee_a^t (f) dt \right] d\left(\bigvee_a^s (u) \right)\\
=\ & \int_a^b \left[[m(a,s)-m(s,b)] \bigvee_a^s (f) \right] d\left(\bigvee_a^s (u) \right)\\
&+ \int_a^b w(t) \bigvee_a^t (f) dt \bigvee_a^b (u) - 2 \int_a^b \left(\int_a^s w(t) \bigvee_a^t (f) dt \right) d\left(\bigvee_a^s (u) \right).
\end{aligned}
$$

In last line of (3.5), we have

$$
\begin{aligned}
& \int_a^b \left(\int_a^s w(t) \bigvee_a^t (f) dt \right) d\left(\bigvee_a^s (u) \right) \qquad (3.6)\\
=\ & \left(\int_a^s w(t) \bigvee_a^t (f) dt \right) \bigvee_a^s (u) \Bigg|_a^b - \int_a^b w(s) \bigvee_a^s (f) \bigvee_a^s (u) ds\\
=\ & \left(\int_a^b w(t) \bigvee_a^t (f) dt \right) \bigvee_a^b (u) - \int_a^b w(s) \bigvee_a^s (f) \bigvee_a^s (u) ds.
\end{aligned}
$$

If we put the equality (3.6) in (3.5), we obtain the first inequality in (3.1).

The other inequalities are obvious from the fact that

$$
\begin{aligned}
\int_a^b w(s) \bigvee_a^s (f) \bigvee_a^s (u) ds &\leq \bigvee_a^b (u) \int_a^b w(s) \bigvee_a^s (f) ds\\
&\leq m(a,b) \bigvee_a^b (f) \bigvee_a^b (u).
\end{aligned}
$$

□

Remark 2. *If we choose $w(t) = t$ in Theorem 4, then we obtain Theorem 1 in* [22].

Theorem 5. *Let $w:[a,b]\to\mathbb{R}$ be nonnegative and continuous on $[a,b]$. If $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then we have the inequality*

$$\begin{aligned}
&|D(w,f,u)| \\
\leq\ & \int_a^b [m(a,t)-m(t,b)]\, f(t)d\left(\bigvee_a^t(u)\right) \\
&+2\int_a^b w(t)f(t)\bigvee_a^t(u)dt-\left(\int_a^b w(t)f(t)dt\right)\bigvee_a^b(u) \\
\leq\ & \int_a^b [m(a,t)-m(t,b)]\, f(t)d\left(\bigvee_a^t(u)\right)+\left(\int_a^b w(t)f(t)dt\right)\bigvee_a^b(u).
\end{aligned}\tag{3.7}$$

Proof. It is well known that if the stieltjes integrals $\int_\alpha^\beta p(t)dv(t)$ and $\int_\alpha^\beta |p(t)|\,dv(t)$ exist and v is monotonic nondecreasing on $[\alpha,\beta]$, then

$$\int_\alpha^\beta p(t)dv(t)\leq\int_\alpha^\beta |p(t)|\,dv(t).\tag{3.8}$$

Using the inequality (3.8), we have

$$\begin{aligned}
\left|\int_a^s\left(\int_a^t w(\xi)d\xi\right)df(t)\right| &\leq \int_a^s\left(\int_a^t w(\xi)d\xi\right)df(t) \\
&= m(a,s)f(s)-\int_a^s w(t)f(t)dt
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
\left|\int_s^b\left(\int_b^t w(\xi)d\xi\right)df(t)\right| &\leq \int_s^b\left|\int_b^t w(\xi)d\xi\right|df(t) \\
&= \int_s^b\left(\int_t^b w(\xi)d\xi\right)df(t) \\
&= \int_s^b w(t)f(t)dt-m(s,b)f(s).
\end{aligned}\tag{3.10}$$

If we substitute the inequalities (3.9) and (3.10) in (3.2), we obtain

$$
\begin{aligned}
(3.11)\qquad & |D(w,f,u)| \\
&\leq \int_a^b [m(a,s)-m(s,b)]\, f(s) d\left(\bigvee_a^s (u)\right) \\
&\quad \int_a^b \left[\int_s^b w(t)f(t)dt - \int_a^s w(t)f(t)dt\right] d\left(\bigvee_a^s (u)\right) \\
&= \int_a^b [m(a,s)-m(s,b)]\, f(s) d\left(\bigvee_a^s (u)\right) \\
&\quad + \int_a^b \left(\int_s^b w(t)f(t)dt\right) d\left(\bigvee_a^s (u)\right) \\
&\quad - \int_a^b \left(\int_a^s w(t)f(t)dt\right) d\left(\bigvee_a^s (u)\right).
\end{aligned}
$$

Using the integtation by parts in Riemann-Stieltjes integral, we have

$$
(3.12)\qquad \int_a^b \left(\int_s^b w(t)f(t)dt\right) d\left(\bigvee_a^s (u)\right) = \int_a^b w(s)f(s)\bigvee_a^s (u) ds
$$

and

$$
\begin{aligned}
(3.13)\qquad & \int_a^b \left(\int_a^s w(t)f(t)dt\right) d\left(\bigvee_a^s (u)\right) \\
&= \int_a^b w(t)f(t)dt \bigvee_a^b (u) - \int_a^b w(s)f(s)\bigvee_a^s (u) ds.
\end{aligned}
$$

Putting the equalities (3.12) and (3.13) in (3.11), we completes the proof first inequality in (3.7).

The second inequality is obvious. □

Remark 3. *If we choose $w(t) = t$ in Theorem 5, then the first inequality in (3.7) reduces the inequality (3.7) in* [22].

4. Inequalities in the Case when f is of bounded variation

In this section, we give same inequality in the case when f is of bounded variation using the identities presented in Section 2.

Theorem 6. *Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If $u : [a,b] \to \mathbb{R}$ is continuous such that there exist constant $\alpha, \beta > 0$ and $L_a, L_b > 0$ with*

$$
(4.1)\qquad |u(t) - u(a)| \leq L_a (t-a)^{\alpha}
$$

and

$$|u(b)-u(t)| \leq L_b\,(b-t)^{\beta} \tag{4.2}$$

for all $t\in[a,b]$, *then we have*

$$\begin{aligned}
&|D(w,f,u)| \\
&\leq L_a\left[\int_a^b w(t)(t-a)^{\alpha}\bigvee_a^t(f)dt-\alpha\int_a^b m(t,b)(t-a)^{\alpha-1}\bigvee_a^t(f)dt\right] \\
&\quad +L_b\left[\beta\int_a^b m(a,t)\,(b-t)^{\beta-1}\bigvee_a^t(f)dt-\int_a^b w(t)\,(b-t)^{\beta}\bigvee_a^t(f)dt\right].
\end{aligned} \tag{4.3}$$

Proof. Taking the madulus in Lemma 3 and using the Lemma 1, we have

$$\begin{aligned}
&|D(w,f,u)| \\
&\leq \int_a^b\left|\int_a^b Q_w(t,s)du(s)\right| d\left(\bigvee_a^t(f)\right) \\
&\leq \int_a^b\left[\left|\int_a^t\left(\int_b^t w(\xi)d\xi\right)du(s)\right|+\left|\int_t^b\left(\int_a^t w(\xi)d\xi\right)du(s)\right|\right] d\left(\bigvee_a^t(f)\right) \\
&\leq \int_a^b\left[m(t,b)\,|u(t)-u(a)|+m(a,t)\,|u(b)-u(t)|\right] d\left(\bigvee_a^t(f)\right).
\end{aligned} \tag{4.4}$$

Using the properties (4.1) and (4.2) in (4.4), we obtain

$$\begin{aligned}
&|D(w,f,u)| \\
&\leq \int_a^b\left[L_a m(t,b)\,(t-a)^{\alpha}+L_b m(a,t)\,(b-t)^{\beta}\right] d\left(\bigvee_a^t(f)\right) \\
&= L_a\int_a^b m(t,b)\,(t-a)^{\alpha}\, d\left(\bigvee_a^t(f)\right)+L_b\int_a^b m(a,t)\,(b-t)^{\beta}\, d\left(\bigvee_a^t(f)\right).
\end{aligned} \tag{4.5}$$

Integration by parts, we have

$$\begin{aligned}
&\int_a^b m(t,b)\,(t-a)^{\alpha}\, d\left(\bigvee_a^t(f)\right) \\
&= m(t,b)\,(t-a)^{\alpha}\left(\bigvee_a^t(f)\right)\Bigg|_a^b \\
&\quad -\int_a^b\left[-w(t)\,(t-a)^{\alpha}+\alpha m(t,b)\,(t-a)^{\alpha-1}\bigvee_a^t(f)\right]dt \\
&= \int_a^b w(t)(t-a)^{\alpha}\bigvee_a^t(f)dt-\alpha\int_a^b m(t,b)(t-a)^{\alpha-1}\bigvee_a^t(f)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b m(a,t)\,(b-t)^{\beta}\,d\left(\bigvee_a^t(f)\right) \\
=\;& m(a,t)\,(b-t)^{\beta}\left(\bigvee_a^t(f)\right)\Bigg|_a^b \\
&-\int_a^b\left[w(t)\,(b-t)^{\beta}+\beta m(a,t)\,(b-t)^{\beta-1}\bigvee_a^t(f)\right]dt \\
=\;& \beta\int_a^b m(a,t)\,(b-t)^{\beta-1}\bigvee_a^t(f)dt-\int_a^b w(t)\,(b-t)^{\beta}\bigvee_a^t(f)dt.
\end{aligned}$$

These equalities completes the proof. □

Remark 4. *If we choose $w(t)=t$ in Theorem 6, then we obtain Theorem 4 in* [22].

Corollary 3. *Let f and w be as in Theorem 6. If u is of $r-H-$Hölder type, i.e.,*

$$|u(t)-u(s)|\leq H\,|t-s|^{r}\ \text{for any}\ t,s\in[a,b]\,,$$

where $H>0$ and $r\in(0,1)$ are given, then

$$\begin{aligned}
&|D(w,f,u)| \\
\leq\;& H\left[\int_a^b w(t)\left[(t-a)^{r}-(b-t)^{r}\right]\bigvee_a^t(f)dt\right. \\
&\left.+r\int_a^b\left[m(a,t)\,(b-t)^{r-1}-m(t,b)(t-a)^{r-1}\right]\bigvee_a^t(f)dt\right].
\end{aligned}\tag{4.6}$$

Corollary 4. *If u is Lipschitzian with the constant $L>0$, then we have*

$$\begin{aligned}
&|D(w,f,u)| \\
\leq\;& 2L\left[\int_a^b w(t)\left(t-\frac{a+b}{2}\right)\bigvee_a^t(f)dt+\int_a^b\left[\frac{m(a,t)-m(t,b)}{2}\right]\bigvee_a^t(f)dt\right].
\end{aligned}$$

5. Inequalities for $(l,L)-$Lipschitzan Fuctions

The following lemma was given by Dragomir in [22].

Lemma 4. *Let $u:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) The function $u-\frac{l+L}{2}.e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)-$Lipschitzan;

(ii) We have the inequalities

$$l\leq\frac{u(t)-u(s)}{t-s}\leq L\ \text{for each}\ t,s\in[a,b]\,,\ t\neq s;$$

(iii) We have the inequalities

$$l\,(t-s)\leq u(t)-u(s)\leq L\,(t-s)\ \text{for each}\ t,s\in[a,b]\,,\ t>s.$$

Definition 3. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i)-(iii) from Lemma 4 is said to be $(l,L)-$Lipschitzan on $[a,b]$. If $L > 0$ and $l = -L$, then $(-L,L)-$Lipschitzan means $L-$Lipschitzan in the classical sense.*

Theorem 7. *Let $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. If $u : [a,b] \to \mathbb{R}$ is an $(l,L)-$Lipschitzan fuction, then we have the inequality*

$$\begin{aligned} &\left| D(w,f,u) - \frac{l+L}{2} \int_a^b \int_a^b w(s) \left[f(t) - f(s) \right] dsdt \right| \\ \leq \quad &(L-l) \left[\int_a^b w(t) \left(t - \frac{a+b}{2} \right) \bigvee_a^t (f) dt + \int_a^b \left[\frac{m(a,t) - m(t,b)}{2} \right] \bigvee_a^t (f) dt \right]. \end{aligned}$$

Proof. From the Lemma 2, we have

$$\begin{aligned} &D\left(w, f, u - \frac{l+l}{2}.e \right) \\ = \ &\int_a^b \left[\left(\int_a^b w(s)ds \right) f(t) + \int_a^b w(s)f(s)ds \right] d\left[u(t) - \frac{l+l}{2} t \right] \\ = \ &\int_a^b \left[\left(\int_a^b w(s)ds \right) f(t) + \int_a^b w(s)f(s)ds \right] du(t) \\ &- \frac{l+l}{2} \int_a^b \left[\left(\int_a^b w(s)ds \right) f(t) + \int_a^b w(s)f(s)ds \right] dt \\ = \ &D(w,f,u) - \frac{l+L}{2} \int_a^b \int_a^b w(s) \left[f(t) - f(s) \right] dsdt. \end{aligned}$$

Applying the Corollary 4 for the function $u - \frac{l+l}{2}.e$, which is $\frac{1}{2}(L-l)-$Lipschitzian, we have

$$\begin{aligned} &\left| D\left(w, f, u - \frac{l+l}{2}.e \right) \right| \\ \leq \quad &(L-l) \left[\int_a^b w(t) \left(t - \frac{a+b}{2} \right) \bigvee_a^t (f) dt + \int_a^b \left[\frac{m(a,t) - m(t,b)}{2} \right] \bigvee_a^t (f) dt \right] \end{aligned}$$

which completes the proof. □

Remark 5. *If we choose $w(t) = t$ in Theorem 7, then we obtain Theorem 5 in* [22].

6. Bounds For Weighted Chebysev Functional

In this section, we apply the our results for the weighted Chebsev functional. From Section 2, we know that

$$T(w,f,g) = \frac{1}{m^2(a,b)} D(w,f,u) \tag{6.1}$$

choosing the $u(t)=\int_a^t w(s)g(s)$ in Lemma 2.

Moreover, u is of bounded variation on any subinterval $[a,s]$, $s\in[a,b]$, and g is continuous on $[a,b]$, then we have

$$\bigvee_a^s(u)=\int_a^s w(t)\left|g(t)\right|dt,\ \ s\in[a,b]. \tag{6.2}$$

Proposition 1. *If f is of bounded variation on $[a,b]$, then we have the inequality*

$$\begin{aligned}
&|T(w,f,g)| \\
\leq\ & \frac{1}{m^2(a,b)}\left[\int_a^b [m(a,t)-m(t,b)]\,w(t)\left|g(t)\right|\bigvee_a^t(f)dt\right. \\
&-\left(\int_a^b w(t)\bigvee_a^t(f)dt\right)\left(\int_a^b w(t)\left|g(t)\right|dt\right) \\
&\left.+2\int_a^b w(t)\bigvee_a^t(f)\left(\int_a^t w(s)\left|g(s)\right|ds\right)dt\right] \\
\leq\ & \frac{1}{m^2(a,b)}\left[\int_a^b [m(a,t)-m(t,b)]\,w(t)\left|g(t)\right|\bigvee_a^t(f)dt\right. \\
&\left.+\int_a^b w(t)\bigvee_a^t(f)\left(\int_a^t w(s)\left|g(s)\right|ds\right)dt\right]
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{1}{m^2(a,b)}\left[\int_a^b [m(a,t)-m(t,b)]\,w(t)\left|g(t)\right|\bigvee_a^t(f)dt\right. \\
&\left.+m(a,b)\left(\int_a^b w(t)\left|g(t)\right|dt\right)\bigvee_a^b(f)\right].
\end{aligned}$$

Proof. If choose $u(t)=\int_a^t w(s)g(s)$ in Theorem 4 and use the identity (6.1) and (6.2), we prove the required result easily. □

Proposition 2. *If f is monotonic nondecreasing on $[a,b]$, then we have the inequality*

$$\begin{aligned}
&\left|T(w,f,g)\right| \\
\leq\ & \frac{1}{m^2(a,b)}\left[\int_a^b \left[m(a,t)-m(t,b)\right] f(t)w(t)\left|g(t)\right| dt\right. \\
&\left. +2\int_a^b w(t)f(t)\left(\int_a^t w(s)\left|g(s)\right| ds\right)dt-\left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)\left|g(t)\right| dt\right)\right] \\
\leq\ & \frac{1}{m^2(a,b)}\left[\int_a^b \left[m(a,t)-m(t,b)\right] f(t)w(t)\left|g(t)\right| dt\right. \\
&\left. +\left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)\left|g(t)\right| dt\right)\right].
\end{aligned}$$

Proof. The proof is obvious from Theorem 5. □

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