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# AN IMPROVED VERSION OF PERTURBED COMPANION OF OSTROWSKI TYPE INEQUALITIES

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Abstract. The purpose of this paper is to establish an improved version of perturbed companion of Ostrowski type integral inequalities for functions whose first derivatives are either bounded or of bounded variation.

## 1. Introduction

In 1938, Ostrowski first announced his inequality result for different differentiable mappings. Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many researchers have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example ([1]-[6], [8]-[11], [16],[17],[20]). Furthermore, several works were devoted to study of perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to ([7], [12]-[15],[19]). The structure of this paper is as follows: in Section 2 we present inequalities for mappings of bounded variation. In Section 3, we provide inequalities for functions whose derivatives are bounded. Finally, in Section 4 we extend inequalities for Lipschitzian mappings. Some previous results are recaptured as special cases.

Ostrowski proved a useful inequality, which gives an upper bound for the approximation of the integral average by the value of mapping at a certain point of the interval, which is given below:

**Theorem 1.** [18] *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty , \tag{1.1}$$

*for all $x \in [a,b]$.*

The constant $\frac{1}{4}$ is the best possible.

In order to prove our inequalities, we need the following identity which although of interest in itself.

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**Lemma 1.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$. Then for any $\lambda_i(x)$, $i=1,2,...,5$ complex numbers, we have*

$$\begin{aligned}
&\frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}(t-a)\left[f'(t)-\lambda_1(x)\right]dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2(x)\right]dt\right.\\
(1.2)\quad &+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3(x)\right]dt\\
&\left.+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4(x)\right]dt+\int_{\frac{a+2b-x}{2}}^{b}(t-b)\left[f'(t)-\lambda_5(x)\right]dt\right\}\\
=\;&\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[\lambda_5(x)-\lambda_1(x)\right]\right.\\
&\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right](\lambda_2(x)-\lambda_4(x))\right\}
\end{aligned}$$

*for all $x\in\left[a,\frac{a+b}{2}\right]$.*

*Proof.* Using the integration by parts for Riemann-Stieltjes integral for each integral, we can easily obtain the desired result (1.2). □

**Remark 1.** *By substituting $x=a$ in (1.2), we get*

$$\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3\right]dt=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt$$

*which was given by Dragomir in* [13].

**Corollary 1.** *Choosing $x=\frac{a+b}{2}$ in (1.2), we get*

$$\begin{aligned}
&\frac{1}{b-a}\left\{\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2\right]dt\right.\\
&\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4\right]dt+\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_5\right]dt\right\}\\
=\;&\frac{1}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&+\frac{1}{32}(b-a)\left[\lambda_5+\lambda_4-\lambda_1-\lambda_2\right]
\end{aligned}$$

**Corollary 2.** *If we substitute* $x = \frac{3a+b}{4}$ *in (1.2), we get*

$$\begin{aligned}
&\frac{1}{b-a}\left\{\int_a^{\frac{7a+b}{8}}(t-a)\left[f'(t)-\lambda_1\right]dt+\int_{\frac{7a+b}{8}}^{\frac{3a+b}{4}}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-\lambda_2\right]dt\right.\\
&+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_3\right]dt\\
&\left.+\int_{\frac{a+3b}{4}}^{\frac{a+7b}{8}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\lambda_4\right]dt+\int_{\frac{a+7b}{8}}^{b}(t-b)\left[f'(t)-\lambda_5\right]dt\right\}\\
=\;&\frac{1}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&+\frac{b-a}{128}\left[\lambda_5-\lambda_1+\lambda_2-\lambda_4\right]
\end{aligned}$$

Now using above lemma, we will present inequalities via three diferent cases.

## 2. Inequalities for Mappings of bounded variation

In this section we give some companion of perturbed Ostrowski inwqualities for function whose derivatives are of bounded variation.

Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on $I^\circ$($I^\circ$ is the interior of $I$) and $[a,b]\subset I^\circ$.Then, from , we have for

$$\lambda_1(x)=f'(a)$$

$$\lambda_2(x)=\frac{f'\left(\frac{a+x}{2}\right)+f'(x)}{2}$$

$$\lambda_3(x)=\frac{f'(x)+f'(a+b-x)}{2}$$

$$\lambda_4(x)=\frac{f'(a+b-x)+f'\left(\frac{a+2b-x}{2}\right)}{2}$$

$$\lambda_5(x)=f'(b)$$

$$
\begin{aligned}
(2.1)\quad & \frac{1}{b-a}\left\{\int_{a}^{\frac{a+x}{2}}(t-a)\left[f'(t)-f'(a)\right]dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)\right. \\
& \times\left[f'(t)-\frac{f'\left(\frac{a+x}{2}\right)+f'(x)}{2}\right]dt \\
& +\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right]dt \\
& +\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-\frac{f'(a+b-x)+f'\left(\frac{a+2b-x}{2}\right)}{2}\right]dt \\
& \left.+\int_{\frac{a+2b-x}{2}}^{b}(t-b)\left[f'(t)-f'(b)\right]dt\right\} \\
= \quad & \frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt \\
& +\frac{1}{8(b-a)}\left\{(x-a)^{2}\left[f'(b)-f'(a)\right]\right. \\
& +\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^{2}-4\left(x-\frac{3a+b}{4}\right)^{2}\right] \\
& \left.\times\left(f'\left(\frac{a+x}{2}\right)+f'(x)-f'(a+b-x)-f'\left(\frac{a+2b-x}{2}\right)\right)\right\}
\end{aligned}
$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

**Theorem 2.** *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on $I^{\circ}$ ($I^{\circ}$ is the interior of $I$) and $[a,b]\subset I^{\circ}$. If the second derivative $f'$ is of bounded variation on $[a,b]$, then,*

$$
\begin{aligned}
& \left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right. \\
& +\frac{1}{8(b-a)}\left\{(x-a)^{2}\left[f'(b)-f'(a)\right]\right. \\
& +\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^{2}-4\left(x-\frac{3a+b}{4}\right)^{2}\right] \\
& \left.\left.\times\left(f'\left(\frac{a+x}{2}\right)+f'(x)-f'(a+b-x)-f'\left(\frac{a+2b-x}{2}\right)\right)\right\}\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{1}{8\left(b-a\right)}\left\{\left(x-a\right)^{2}\bigvee_{a}^{\frac{a+x}{2}}\left(f'\right)\right.\\
& +sgn\left(\frac{3a+b}{4}-x\right)\left[4\left(x-\frac{3a+b}{4}\right)^{2}-\left(x-\frac{a+b}{2}\right)^{2}\right]\bigvee_{\frac{a+x}{2}}^{x}\left(f\,'\right)\\
& +4\left(x-\frac{a+b}{2}\right)^{2}\bigvee_{x}^{a+b-x}\left(f\,'\right)+\left[4\left(x-\frac{3a+b}{4}\right)^{2}-\left(x-\frac{a+b}{2}\right)^{2}\right]\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}\left(f\,'\right)\\
& \left.+\left(x-a\right)^{2}\bigvee_{\frac{a+2b-x}{2}}^{b}\left(f\,'\right)\right\}.
\end{aligned}
$$

*Proof.* From (2.1),

$$
\begin{aligned}
& \left|\frac{1}{4}\left[f\left(x\right)+f\left(a+b-x\right)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right.\\
& +\frac{1}{8\left(b-a\right)}\left\{\left(x-a\right)^{2}\left[f\,'\left(b\right)-f\,'\left(a\right)\right]\right.\\
& +\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^{2}-4\left(x-\frac{3a+b}{4}\right)^{2}\right]\\
& \left.\left.\times\left(f\,'\left(\frac{a+x}{2}\right)+f\,'\left(x\right)-f\,'\left(a+b-x\right)-f\,'\left(\frac{a+2b-x}{2}\right)\right)\right\}\right|\\
\leq\ & \frac{1}{b-a}\left\{\int_{a}^{\frac{a+x}{2}}\left|t-a\right|\left|f\,'\left(t\right)-f\,'\left(a\right)\right|dt\right.\\
& +\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f\,'\left(t\right)-\frac{f\,'\left(\frac{a+x}{2}\right)+f\,'\left(x\right)}{2}\right|dt\\
& +\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f\,'\left(t\right)-\frac{f\,'\left(x\right)+f\,'\left(a+b-x\right)}{2}\right|dt\\
& +\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f\,'\left(t\right)-\frac{f\,'\left(a+b-x\right)+f\,'\left(\frac{a+2b-x}{2}\right)}{2}\right|dt\\
& \left.+\int_{\frac{a+2b-x}{2}}^{b}\left|t-b\right|\left|f\,'\left(t\right)-f\,'\left(b\right)\right|dt\right\}.
\end{aligned}
$$

Since $f\,'$ is of bounded variation on $\left[a,b\right]$, we get

$$
\left|f\,'\left(t\right)-f\,'\left(a\right)\right|\leq\bigvee_{a}^{t}\left(f\,'\right)
$$

for $t \in \left[a, \frac{a+x}{2}\right]$

$$\left| f'(t) - \frac{f'\left(\frac{a+x}{2}\right) + f'(x)}{2} \right| \leq \frac{1}{2} \bigvee_{\frac{a+x}{2}}^{x} (f') < \bigvee_{\frac{a+x}{2}}^{x} (f')$$

for $t \in \left[\frac{a+x}{2}, x\right]$

$$\left| f'(t) - \frac{f'(x) + f'(a+b-x)}{2} \right| \leq \frac{1}{2} \bigvee_{x}^{a+b-x} (f')$$

for $t \in [x, a+b-x]$

$$\left| f'(t) - \frac{f'(a+b-x) + f'\left(\frac{a+2b-x}{2}\right)}{2} \right| \leq \frac{1}{2} \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f') < \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f')$$

for $t \in \left[a+b-x, \frac{a+2b-x}{2}\right]$

$$|f'(t) - f'(b)| \leq \bigvee_{t}^{b} (f')$$

for $t \in \left[\frac{a+2b-x}{2}, b\right]$.

Thus, we have

$$\begin{aligned}
&\left| \frac{1}{4}\left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\
&+\frac{1}{8(b-a)}\left\{ (x-a)^2 \left[ f'(b) - f'(a) \right] \right. \\
&+\frac{1}{2}\left[ \left( x - \frac{a+b}{2} \right)^2 - 4\left( x - \frac{3a+b}{4} \right)^2 \right] \\
&\left. \left. \times \left( f'\left(\frac{a+x}{2}\right) + f'(x) - f'(a+b-x) - f'\left(\frac{a+2b-x}{2}\right) \right) \right\} \right| \\
\leq\ & \frac{1}{b-a}\left\{ \int_a^{\frac{a+x}{2}} (t-a) \bigvee_a^t (f')\,dt + \int_{\frac{a+x}{2}}^x \left| t - \frac{3a+b}{4} \right| \bigvee_{\frac{a+x}{2}}^x (f')\,dt \right. \\
&+\frac{1}{2}\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| \bigvee_x^{a+b-x} (f')\,dt + \int_{a+b-x}^{\frac{a+2b-x}{2}} \left( \frac{a+3b}{4} - t \right) \bigvee_{a+b-x}^{\frac{a+2b-x}{2}} (f')\,dt \\
&\left. + \int_{\frac{a+2b-x}{2}}^b (b-t) \bigvee_t^b (f')\,dt \right\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{b-a}\left\{\bigvee_{a}^{\frac{a+x}{2}}(f\,')\int_{a}^{\frac{a+x}{2}}(t-a)\,dt+\bigvee_{\frac{a+x}{2}}^{x}(f\,')\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|dt\right.\\
& +\frac{1}{2}\bigvee_{x}^{a+b-x}(f\,')\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|dt+\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f\,')\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(\frac{a+3b}{4}-t\right)dt\\
& \left.+\bigvee_{\frac{a+2b-x}{2}}^{b}(f\,')\int_{\frac{a+2b-x}{2}}^{b}(b-t)\,dt\right\}\\
=\ & \frac{1}{8\,(b-a)}\left\{(x-a)^2\bigvee_{a}^{\frac{a+x}{2}}(f\,')\right.\\
& +sgn\left(x-\frac{3a+b}{4}\right)\frac{1}{2}\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{\frac{a+x}{2}}^{x}(f\,')\\
& +4\left(x-\frac{a+b}{2}\right)^2\bigvee_{x}^{a+b-x}(f\,')+\frac{1}{2}\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f\,')\\
& \left.+(x-a)^2\bigvee_{\frac{a+2b-x}{2}}^{b}(f\,')\right\}.
\end{aligned}
$$

Here,

$$\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|dt=\frac{1}{8}\left(x-\frac{a+b}{2}\right)^2-\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2$$

for $x\in\left[a,\frac{3a+b}{4}\right]$ and

$$\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|dt=\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2-\frac{1}{8}\left(x-\frac{a+b}{2}\right)^2$$

for $x\in\left[\frac{3a+b}{4},\frac{a+b}{2}\right]$.

This completes the proof. □

**Corollary 3.** *If we choose $x=a$ in Theorem 2, then we get the Liu's result* [17].

**Corollary 4.** *Under assumption of Theorem 2 with* $x = \frac{a+b}{2}$, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&\left.+\frac{(b-a)}{32}\left[f'(b)-f'(a)+\frac{f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)}{2}\right]\right|\\\leq\;&\frac{(b-a)}{32}\bigvee_a^b(f').\end{aligned}$$

## 3. Inequalities for Functions Whose Derivatives are Bounded

Now, we obtain some inequalities for bounded function using the identity (1.2). Recall the sets of complex-valued functions:

$$\begin{aligned}&\overline{U}_{[a,b]}(\gamma,\Gamma)\\:\;&=\left\{f:[a,b]\to\mathbb{C}|\,Re\left[(\Gamma-f(t))\left(\overline{f(t)}\right)-\overline{\gamma}\right]\geq 0\text{ for almast every }t\in[a,b]\right\}\end{aligned}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}|\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\text{ for a.e. }t\in[a,b]\right\}.$$

**Proposition 1.** *For any* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *we have that* $\overline{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty and closed sets and*

$$\overline{U}_{[a,b]}(\gamma,\Gamma)=\overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Let $I_1=\left[a,\frac{a+x}{2}\right]$, $I_2=\left[\frac{a+x}{2},x\right]$ $I_3=[x,a+b-x]$ $I_4=\left[a+b-x,\frac{a+2b-x}{2}\right]$ and $I_5=\left[\frac{a+2b-x}{2},b\right]$.

**Theorem 3.** *Let* $f:[a,b]\to\mathbb{C}$ *be a differantiable function on* $(a,b)$. *Suppose that* $\gamma_i(x),\Gamma_i(x)\in\mathbb{C}$, $\gamma_i(x)\neq\Gamma_i(x)$, $i=1,2,3,4,5$ *and*

$$f'\in\bigcap_{i=1}^5\overline{U}_{I_i}(\gamma_i,\Gamma_i)$$

*then we have the inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&+\frac{1}{16(b-a)}\left\{(x-a)^2[\gamma_5(x)+\Gamma_5(x)-\gamma_1(x)-\Gamma_1(x)]\right.\\&\left.\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right](\gamma_2(x)+\Gamma_2(x)-\gamma_4(x)-\Gamma_4(x))\right\}\right|\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{8\left(b-a\right)}\left\{\left(x-a\right)^{2}\left|\Gamma_{1}(x)-\gamma_{1}(x)\right|\right. \\
& +sgn\left(x-\frac{3a+b}{4}\right)\left[4\left(x-\frac{3a+b}{4}\right)^{2}-\left(x-\frac{a+b}{2}\right)^{2}\right]\left|\Gamma_{2}(x)-\gamma_{2}(x)\right| \\
& +4\left(x-\frac{a+b}{2}\right)^{2}\left|\Gamma_{3}(x)-\gamma_{3}(x)\right| \\
& +\left[4\left(x-\frac{3a+b}{4}\right)^{2}-\left(x-\frac{a+b}{2}\right)^{2}\right]\left|\Gamma_{4}(x)-\gamma_{4}(x)\right| \\
& \left.+\left(x-a\right)^{2}\left|\Gamma_{5}(x)-\gamma_{5}(x)\right|\right\}.
\end{aligned}
$$

*for all* $x\in\left[a,\frac{a+b}{2}\right]$.

*Proof.* Taking the modulus identity (1.2) for $\lambda_{i}(x)=\frac{\gamma_{i}(x)+\Gamma_{i}(x)}{2}$, $i=1,2,...,5$, since $f'\in\bigcap_{i=1}^{5}\overline{U}_{I_{i}}\left(\gamma_{i},\Gamma_{i}\right)$, we have

$$
\begin{aligned}
& \left|\frac{1}{4}\left[f\left(x\right)+f\left(a+b-x\right)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right. \\
& +\frac{1}{16\left(b-a\right)}\left\{\left(x-a\right)^{2}\left[\gamma_{5}(x)+\Gamma_{5}(x)-\gamma_{1}(x)-\Gamma_{1}(x)\right]\right. \\
& \left.\left.+\left[\left(x-\frac{a+b}{2}\right)^{2}-4\left(x-\frac{3a+b}{4}\right)^{2}\right]\left(\gamma_{2}(x)+\Gamma_{2}(x)-\gamma_{4}(x)-\Gamma_{4}(x)\right)\right\}\right| \\
\leq\ & \frac{1}{b-a}\left\{\int_{a}^{\frac{a+x}{2}}\left|t-a\right|\left|f'\left(t\right)-\frac{\gamma_{1}(x)+\Gamma_{1}(x)}{2}\right|dt\right. \\
& +\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f'\left(t\right)-\frac{\gamma_{2}(x)+\Gamma_{2}(x)}{2}\right|dt \\
& +\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'\left(t\right)-\frac{\gamma_{3}(x)+\Gamma_{3}(x)}{2}\right|dt \\
& +\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f'\left(t\right)-\frac{\gamma_{4}(x)+\Gamma_{4}(x)}{2}\right|dt \\
& \left.+\int_{\frac{a+2b-x}{2}}^{b}\left|t-b\right|\left|f'\left(t\right)-\frac{\gamma_{5}(x)+\Gamma_{5}(x)}{2}\right|dt\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2(b-a)}\left\{|\Gamma_1(x)-\gamma_1(x)|\int_a^{\frac{a+x}{2}}(t-a)\,dt+|\Gamma_2(x)-\gamma_2(x)|\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|dt\right.\\
&\quad+|\Gamma_3(x)-\gamma_3(x)|\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt\\
&\quad\left.+|\Gamma_4(x)-\gamma_4(x)|\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(\frac{a+3b}{4}-t\right)dt+|\Gamma_5(x)-\gamma_5(x)|\int_{\frac{a+2b-x}{2}}^{b}(b-t)dt\right\}\\
&= \frac{1}{8(b-a)}\left\{(x-a)^2|\Gamma_1(x)-\gamma_1(x)|\right.\\
&\quad+sgn\left(x-\frac{3a+b}{4}\right)\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]|\Gamma_2(x)-\gamma_2(x)|\\
&\quad+4\left(x-\frac{a+b}{2}\right)^2|\Gamma_3(x)-\gamma_3(x)|\\
&\quad\left.\left[4\left(x-\frac{3a+b}{4}\right)^2-\left(x-\frac{a+b}{2}\right)^2\right]|\Gamma_4(x)-\gamma_4(x)|+(x-a)^2|\Gamma_5(x)-\gamma_5(x)|\right\}.
\end{aligned}
$$

This completes the proof. □

**Remark 2.** *If we choose $x=a$ in Theorem 3, then we get a new result, proved by Dragomir* [13].

**Corollary 5.** *Under assumption of Theorem 3 with $x=\frac{a+b}{2}$, we get the inequality*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&+\frac{1}{64}(b-a)\left[\gamma_5(x)+\Gamma_5(x)-\gamma_1(x)-\Gamma_1(x)\right.\\
&\left.+\gamma_2(x)+\Gamma_2(x)-\gamma_4(x)+\Gamma_4(x)\right]|\\
&\leq \frac{b-a}{16}\left[|\Gamma_1(x)-\gamma_1(x)|+|\Gamma_2(x)-\gamma_2(x)|\right].
\end{aligned}
$$

*In particular, we have*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\
&\left.+\frac{1}{64}(b-a)\left[\gamma_1(x)+\Gamma_1(x)+\gamma_2(x)+\Gamma_2(x)\right]\right|\\
&\leq \frac{b-a}{16}\left[|\Gamma_1(x)-\gamma_1(x)|+|\Gamma_2(x)-\gamma_2(x)|\right].
\end{aligned}
$$

## 4. Inequalities for Lipschitzian Mappings

In this section, we obtain some inequalities for function whose derivatives are Lipschitzian.

We say that the function $g:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ if

$$|g(t)-g(s)|\leq L\,|t-s|$$

for any $t,s\in[a,b]$.

**Theorem 4.** *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on $(a,b)$. If the second derivative $f''$ is a Lipschitzian mapping with the constant $L>0$,then we have the inequality*

$$\begin{aligned}(4.1)\quad &\left|\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right.\\&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b)-f'(a)\right]\right.\\&\left.\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\right\}\right|\\\leq\;&\frac{L}{12(b-a)}\left[(x-a)^3+8\left(x-\frac{3a+b}{4}\right)^3+7\left(x-\frac{a+b}{2}\right)^3\right]\end{aligned}$$

*for all $x\in\left[a,\frac{a+b}{2}\right]$.*

*Proof.* If we take the $\lambda_1=f'(a)$, $\lambda_2=f'\left(\frac{3a+b}{4}\right)$, $\lambda_3=f'\left(\frac{a+b}{2}\right)$, $\lambda_4=f'\left(\frac{a+3b}{4}\right)$ and $\lambda_5=f'(b)$ in equality (1.2), we have

$$\begin{aligned}(4.2)\quad &\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\&+\frac{1}{8(b-a)}\left\{(x-a)^2\left[f'(b)-f'(a)\right]\right.\\&\left.+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\right\}\\=\;&\frac{1}{b-a}\left\{\int_a^{\frac{a+x}{2}}(t-a)\left[f'(t)-f'(a)\right]dt\right.\\&+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)\left[f'(t)-f'\left(\frac{3a+b}{4}\right)\right]dt\\&\left.+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt\right.\end{aligned}$$

$$+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)\left[f'(t)-f'\left(\frac{a+3b}{4}\right)\right]dt$$
$$+\int_{\frac{a+2b-x}{2}}^{b}(t-b)\left[f'(t)-f'(b)\right]dt\Bigg\}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Since $f''$ is Lipschitzian, taking the madulus in (4.2), we have

$$\begin{aligned}
&\frac{1}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&+\frac{1}{8(b-a)}\Big\{(x-a)^2\left[f'(b)-f'(a)\right]\\
&+\left[\left(x-\frac{a+b}{2}\right)^2-4\left(x-\frac{3a+b}{4}\right)^2\right]\left(f'\left(\frac{3a+b}{4}\right)-f'\left(\frac{a+3b}{4}\right)\right)\Big\}\\
\leq\ &\frac{1}{b-a}\Bigg\{\int_a^{\frac{a+x}{2}}|t-a|\,|f'(t)-f'(a)|\,dt+\int_{\frac{a+x}{2}}^{x}\left|t-\frac{3a+b}{4}\right|\left|f'(t)-f'\left(\frac{3a+b}{4}\right)\right|dt\\
&+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt\\
&+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left|t-\frac{a+3b}{4}\right|\left|f'(t)-f'\left(\frac{a+3b}{4}\right)\right|dt+\int_{\frac{a+2b-x}{2}}^{b}|t-b|\,|f'(t)-f'(b)|\,dt\Bigg\}\\
\leq\ &\frac{L}{b-a}\Bigg\{\int_a^{\frac{a+x}{2}}(t-a)^2\,dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^2dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2dt\\
&+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^2dt+\int_{\frac{a+2b-x}{2}}^{b}(t-b)^2\,dt\Bigg\}\\
=\ &\frac{L}{b-a}\left[\frac{(x-a)^3}{12}+\frac{2}{3}\left(x-\frac{3a+b}{4}\right)^3+\frac{7}{12}\left(x-\frac{a+b}{2}\right)^3\right]
\end{aligned}$$

which completes the proof. □

**Remark 3.** *If we choose $x=a$ in Theorem 3, then we get another inequality which was proved by Dragomir* [14].

**Corollary 6.** *Under assumption of Theorem 4 with* $x = \frac{a+b}{2}$, *we get the inequality*

$$\begin{aligned} &\left| \frac{1}{2}\left[ f\left(\frac{a+b}{2}\right) + \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\ &\left. \frac{1}{32}(b-a)\left[ f'(b) - f'(a) + f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right] \right| \\ \leq\ & \frac{(b-a)^2}{48} L. \end{aligned}$$

**Corollary 7.** *Under assumption of Theorem 4 with* $x = \frac{3a+b}{4}$, *we get the inequality*

$$\begin{aligned} &\left| \frac{1}{4}\left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right. \\ &\left. \frac{1}{128}(b-a)\left[ f'(b) - f'(a) + f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right] \right| \\ \leq\ & \frac{(b-a)^2}{96} L. \end{aligned}$$

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