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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS AND POSITIVE MAPS

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Abstract. In this paper we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see for instance [12] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function$f(t)=e^t$ is neither operator convex nor operator monotone.

In [4], see also [5, p. 60], we established the following Hermite-Hadamard type inequality for operator convex functions:

Theorem 1. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned} f\left(\frac{A+B}{2}\right)&\leq\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]\\ &\leq\int_0^1 f((1-t)A+tB)\,dt\\ &\leq\frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\leq\frac{f(A)+f(B)}{2}. \end{aligned}\tag{1.1}$$

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For recent related results on operator Hermite-Hadamard type inequalities, see [1]-[2], [5]-[10] and [13].

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^{+}(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [3] (see also [12, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A+\mu B)=\lambda\Phi(A)+\mu\Phi(B)$$

for any λ, $\mu\in\mathbb{C}$ and A, $B\in\mathcal{B}(H)$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A\in\mathcal{B}^{+}(H)$ then $\Phi(A)\in\mathcal{B}^{+}(K)$. We write $\Phi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(K)]$. The linear map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H)=1_K$. We write $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A\leq B \text{ implies } \Phi(A)\leq\Phi(B)$$

and preserves the adjoint operation $\Phi(A^{*})=\Phi(A)^{*}$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $\alpha 1_H\leq A\leq\beta 1_H$, then $\alpha 1_K\leq\Phi(A)\leq\beta 1_K$.

If the map $\Psi:\mathcal{B}(H)\to\mathcal{B}(K)$ is linear, positive and $\Psi(1_H)\in\mathcal{B}^{++}(K)$ then by putting $\Phi=\Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, namely it is also normalised.

The following Jensen's type result is well known [3]:

Theorem 2 (Davis-Choi-Jensen's Inequality)**.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A))\leq\Phi(f(A)). \tag{1.2}$$

We observe that if $\Psi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(K)]$ with $\Psi(1_H)\in\mathcal{B}^{++}(K)$, then by taking $\Phi=\Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.2) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\leq\Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H)f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H)\leq\Psi(f(A)). \tag{1.3}$$

In this paper, motivated by the above results, we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

2. Refinements of HH-Inequality

Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I and two selfadjoint operators A and B with spectra in I and $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$. We know that Φ

is continuous, see for instance [11, Proposition 2.8]. By taking the positive map Φ in (1.1) and using the continuity property of Φ, we have

$$\begin{aligned}\Phi\left(f\left(\frac{A+B}{2}\right)\right) &\leq \frac{1}{2}\left[\Phi\left(f\left(\frac{3A+B}{4}\right)\right)+\Phi\left(f\left(\frac{A+3B}{4}\right)\right)\right] \\ &\leq \int_0^1 \Phi\left(f\left(\left(1-t\right)A+tB\right)\right)dt \\ &\leq \frac{1}{2}\left[\Phi\left(f\left(\frac{A+B}{2}\right)\right)+\frac{\Phi\left(f\left(A\right)\right)+\Phi\left(f\left(B\right)\right)}{2}\right] \\ &\leq \frac{\Phi\left(f\left(A\right)\right)+\Phi\left(f\left(B\right)\right)}{2}.\end{aligned} \tag{2.1}$$

If we write the inequality (2.1) for $\Phi(A)$ and $\Phi(B)$ then we also have

$$\begin{aligned}f\left(\frac{\Phi\left(A\right)+\Phi\left(B\right)}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3\Phi\left(A\right)+\Phi\left(B\right)}{4}\right)+f\left(\frac{\Phi\left(A\right)+3\Phi\left(B\right)}{4}\right)\right] \\ &\leq \int_0^1 f\left(\left(1-t\right)\Phi\left(A\right)+t\Phi\left(B\right)\right)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{\Phi\left(A\right)+\Phi\left(B\right)}{2}\right)+\frac{f\left(\Phi\left(A\right)\right)+f\left(\Phi\left(B\right)\right)}{2}\right] \\ &\leq \frac{f\left(\Phi\left(A\right)\right)+f\left(\Phi\left(B\right)\right)}{2}.\end{aligned} \tag{2.2}$$

It is then natural to ask how the following integrals

$$\int_0^1 \Phi\left(f\left(\left(1-t\right)A+tB\right)\right)dt \text{ and } \int_0^1 f\left(\left(1-t\right)\Phi\left(A\right)+t\Phi\left(B\right)\right)dt$$

do compare?

The following simple result holds:

Theorem 3. *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N\left[\mathcal{B}(H), \mathcal{B}(K)\right]$ we have*

$$\int_0^1 f\left(\left(1-t\right)\Phi\left(A\right)+t\Phi\left(B\right)\right)dt \leq \int_0^1 \Phi\left(f\left(\left(1-t\right)A+tB\right)\right)dt. \tag{2.3}$$

Proof. By (1.2) we have

$$f\left(\left(1-t\right)\Phi\left(A\right)+t\Phi\left(B\right)\right)=f\left(\Phi\left(\left(\left(1-t\right)A+tB\right)\right)\right)\leq \Phi\left(f\left(\left(1-t\right)A+tB\right)\right)$$

for any $t \in [0,1]$.

By integrating this inequality on $[0,1]$ and using the continuity property of Φ we get the desired result (2.3). □

We define by $\mathfrak{P}_I\left[\mathcal{B}(H), \mathcal{B}(K)\right]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is positive invertible operator in K.

Corollary 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and selfadjoint operators A and B with spectra in I. If $\Psi \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$, then we have*

$$\Psi^{1/2}(1_H) \left(\int_0^1 f\left(\Psi^{-1/2}(1_H)((1-t)\Psi(A) + t\Psi(B))\Psi^{-1/2}(1_H) \right) dt \right) \Psi^{1/2}(1_H)$$
$$\leq \int_0^1 \Psi(f((1-t)A + tB))\, dt. \tag{2.4}$$

Proof. If we write the inequality (2.3) for $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$, then we get

$$\int_0^1 f\left((1-t)\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H) + t\Psi^{-1/2}(1_H)\Psi(B)\Psi^{-1/2}(1_H) \right) dt$$
$$\leq \int_0^1 \Psi^{-1/2}(1_H)\Psi(f((1-t)A + tB))\Psi^{-1/2}(1_H)\, dt,$$

that can be written as

$$\int_0^1 f\left(\Psi^{-1/2}(1_H)((1-t)\Psi(A) + t\Psi(B))\Psi^{-1/2}(1_H) \right) dt$$
$$\leq \Psi^{-1/2}(1_H) \left(\int_0^1 \Psi(f((1-t)A + tB))\, dt \right) \Psi^{-1/2}(1_H).$$

Finally, if we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$, then we get the desired result (2.4). □

The following representation result holds.

Lemma 1. *Let $f : I \to \mathbb{C}$ be a continuous function on the interval I and two selfadjoint operators A and B with spectra in I. Then for any $\lambda \in [0,1]$ we have the representation*

$$\int_0^1 f((1-t)A + tB)\, dt = (1-\lambda)\int_0^1 f[(1-t)((1-\lambda)A + \lambda B) + tB]\, dt$$
$$+ \lambda \int_0^1 f[(1-t)A + t((1-\lambda)A + \lambda B)]\, dt. \tag{2.5}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (2.5) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 f[(1-t)(\lambda B + (1-\lambda)A) + tB]\, dt$$
$$= \int_0^1 f[((1-t)\lambda + t)B + (1-t)(1-\lambda)A]\, dt$$

and

$$\int_0^1 f[t(\lambda B + (1-\lambda)A) + (1-t)A]\, dt = \int_0^1 f[t\lambda B + (1-\lambda t)A]\, dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)du$. Then

$$\int_0^1 f[((1-t)\lambda + t)B + (1-t)(1-\lambda)A]\, dt = \frac{1}{1-\lambda}\int_\lambda^1 f[uB + (1-u)A]\, du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f[t\lambda B + (1-\lambda t) A]\, dt = \frac{1}{\lambda}\int_0^\lambda f[uB + (1-u) A]\, du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 f[(1-t)(\lambda B + (1-\lambda) A) + tB]\, dt \\
&+ \lambda \int_0^1 f[t(\lambda B + (1-\lambda) A) + (1-t) A]\, dt \\
&= \int_\lambda^1 f[uB + (1-u) A]\, du + \int_0^\lambda f[uB + (1-u) A]\, du \\
&= \int_0^1 f[uB + (1-u) A]\, du
\end{aligned}$$

and the identity (2.5) is proved. □

We have now the following generalization of (1.1):

Theorem 4. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we have the inequalities*

$$\begin{aligned}
(2.6)\qquad & f\left(\frac{A+B}{2}\right) \\
&\le (1-\lambda) f\left[\frac{(1-\lambda) A + (1+\lambda) B}{2}\right] + \lambda f\left[\frac{(2-\lambda) A + \lambda B}{2}\right] \\
&\le \int_0^1 f((1-t) A + tB)\, dt \\
&\le \frac{1}{2}[f((1-\lambda) A + \lambda B) + (1-\lambda) f(B) + \lambda f(A)] \\
&\le \frac{f(A) + f(B)}{2}.
\end{aligned}$$

Proof. Using the Hermite-Hadamard inequality (1.1) we have

$$\begin{aligned}
(2.7)\qquad f\left[\frac{(1-\lambda) A + (1+\lambda) B}{2}\right] &\le \int_0^1 f[(1-t)((1-\lambda) A + \lambda B) + tB]\, dt \\
&\le \frac{f((1-\lambda) A + \lambda B) + f(B)}{2}
\end{aligned}$$

and

$$\begin{aligned}
(2.8)\qquad f\left[\frac{(2-\lambda) A + \lambda B}{2}\right] &\le \int_0^1 f[(1-t) A + t((1-\lambda) A + \lambda B)]\, dt \\
&\le \frac{f(A) + f((1-\lambda) A + \lambda B)}{2}
\end{aligned}$$

for any $\lambda \in [0,1]$.

If we multiply inequality (2.7) by $1-\lambda$ and (2.8) by λ, add the obtained inequalities and use representation (2.5), then we get

$$\begin{aligned}&(1-\lambda) f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right]\\&\leq \int_0^1 f((1-t) A+t B) d t\\&\leq(1-\lambda) \frac{f((1-\lambda) A+\lambda B)+f(B)}{2}+\lambda \frac{f(A)+f((1-\lambda) A+\lambda B)}{2},\end{aligned}$$

which proves the second and third inequalities in (2.6).

By the operator convexity of f we have

$$\begin{aligned}&(1-\lambda) f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right]\\&\geq f\left[(1-\lambda) \frac{(1-\lambda) A+(1+\lambda) B}{2}+\lambda \frac{(2-\lambda) A+\lambda B}{2}\right]=f\left(\frac{A+B}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}[f((1-\lambda) A+\lambda B)+(1-\lambda) f(B)+\lambda f(A)]\\&\leq \frac{1}{2}[(1-\lambda) f(A)+\lambda f(B)+(1-\lambda) f(B)+\lambda f(A)]=\frac{f(A)+f(B)}{2}\end{aligned}$$

that prove the first and last inequality in (2.6). □

We have:

Corollary 2. *Let $f: I \rightarrow \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned}&\Phi\left(f\left(\frac{A+B}{2}\right)\right)\\&\leq(1-\lambda) \Phi\left(f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]\right)+\lambda \Phi\left(f\left[\frac{(2-\lambda) A+\lambda B}{2}\right]\right)\\&\leq \int_0^1 \Phi(f((1-t) A+t B)) d t\\&\leq \frac{1}{2}[\Phi(f((1-\lambda) A+\lambda B))+(1-\lambda) \Phi(f(B))+\lambda \Phi(f(A))]\\&\leq \frac{\Phi(f(A))+\Phi(f(B))}{2}\end{aligned}\tag{2.9}$$

for any $\lambda \in[0,1]$.

3. Bounds for HH-Difference

We consider the difference functional

$$J_n(\mathbf{p} ; \mathbf{A}, f, I):=\sum_{j=1}^n p_j f(A_j)-P_n f\left(\frac{1}{P_n} \sum_{j=1}^n p_j A_j\right)\tag{3.1}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$, $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$ for $j \in \{1, ..., n\}$ and $f : I \to \mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, ..., n\}$.

In [7] we established the following properties of the functional $J_n(\cdot; \mathbf{A}, f, I)$:

Theorem 5. *Assume that $f : I \to \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$, then for any $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, I) \geq J_n(\mathbf{p}; \mathbf{A}, f, I) + J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{3.2}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p}; \mathbf{A}, f, I) \geq J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{3.3}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a monotonic functional in the operator order.

The following boundedness property also holds:

Corollary 3. *Assume that the function $f : I \to \mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ satisfies the condition $\operatorname{Sp}(A_j) \subseteq I$ for any $j \in \{1, ..., n\}$. If $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n^+$ and there exists the positive constants m, M such that*

$$m\mathbf{q} \leq \mathbf{p} \leq M\mathbf{q}, \tag{3.4}$$

then

$$mJ_n(\mathbf{q}; \mathbf{A}, f, I) \leq J_n(\mathbf{p}; \mathbf{A}, f, I) \leq MJ_n(\mathbf{q}; \mathbf{A}, f, I) \tag{3.5}$$

in the operator order.

We observe that if all $q_j > 0$, $j \in \{1, ..., n\}$, then we have the inequality

$$\begin{aligned} \min_{j \in \{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) &\leq J_n(\mathbf{p}; \mathbf{A}, f, I) \\ &\leq \max_{j \in \{1,...,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) \end{aligned} \tag{3.6}$$

in the operator order.

In particular, by (3.6) for $n = 2$, $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ we get

$$\begin{aligned} &\min\left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) f(A) + qf(B) - f((1-q) A + qB)] \\ &\leq [(1-p) f(A) + pf(B) - f((1-p) A + pB)] \\ &\leq \max\left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) f(A) + qf(B) - f((1-q) A + qB)] \end{aligned} \tag{3.7}$$

for any selfadjoint operators A and B with spectra in I.

If we take $q=\frac{1}{2}$ in (1.1), then we get

$$2\min\{t,1-t\}\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right] \le [(1-t)f(A)+tf(B)-f((1-t)A+tB)] \le 2\max\{t,1-t\}\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right] \tag{3.8}$$

for any selfadjoint operators A and B with spectra in I and $t\in[0,1]$.

If we take in (3.7) the map Φ, then we have

$$\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}[(1-q)\Phi(f(A))+q\Phi(f(B))-\Phi(f((1-q)A+qB))] \le [(1-p)\Phi(f(A))+p\Phi(f(B))-\Phi(f((1-p)A+pB))] \le \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}[(1-q)\Phi(f(A))+q\Phi(f(B))-\Phi(f((1-q)A+qB))] \tag{3.9}$$

for any $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$.

The following result provides some upper and lower bounds for the *HH-difference*

$$\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt.$$

Theorem 6. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\frac{1}{2}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \le \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt \le \frac{1}{2}\frac{q^2-q+1}{q(1-q)}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \tag{3.10}$$

for any $q\in(0,1)$.

Proof. From (3.7) we have

$$\min\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \le [(1-t)f(A)+tf(B)-f((1-t)A+tB)] \le \max\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \tag{3.11}$$

with $t\in[0,1]$ and $q\in(0,1)$.

If we integrate over $t\in[0,1]$ the inequality (3.11), then we get

$$[(1-q)f(A)+qf(B)-f((1-q)A+qB)]\int_0^1\min\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}dt \le \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt \le [(1-q)f(A)+qf(B)-f((1-q)A+qB)]\int_0^1\max\left\{\frac{t}{q},\frac{1-t}{1-q}\right\}dt \tag{3.12}$$

for any A, B with spectra in I and $q \in (0,1)$.

Observe that

$$\frac{t}{q} - \frac{1-t}{1-q} = \frac{t-q}{q(1-q)}$$

showing that

$$\min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} = \begin{cases} \frac{t}{q} \text{ if } 0 \le t \le q \le 1 \\ \\ \frac{1-t}{1-q} \text{ if } 0 \le q \le t \le 1 \end{cases}$$

and

$$\max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} = \begin{cases} \frac{1-t}{1-q} \text{ if } 0 \le t \le q \le 1 \\ \\ \frac{t}{q} \text{ if } 0 \le q \le t \le 1. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 \min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} dt &= \int_0^q \frac{t}{q} dt + \int_q^1 \frac{1-t}{1-q} dt \\ &= \frac{q^2}{2q} + \frac{1}{1-q}\left(1 - q - \left(\frac{1-q^2}{2}\right)\right) = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} dt &= \int_0^q \frac{1-t}{1-q} dt + \int_q^1 \frac{t}{q} dt \\ &= \frac{1}{1-q}\left(q - \frac{q^2}{2}\right) + \frac{1-q^2}{2q} \\ &= \frac{q^2 - q + 1}{2q(1-q)} \end{aligned}$$

and by (3.12) we obtain the desired result (3.10). □

Corollary 4. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned} (3.13) \quad & \frac{1}{2}\left[(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))\right] \\ & \le \frac{\Phi(f(A)) + \Phi(f(B))}{2} - \int_0^1 \Phi(f((1-t)A + tB))\, dt \\ & \le \frac{1}{2}\frac{q^2 - q + 1}{q(1-q)}\left[(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))\right]. \end{aligned}$$

We also have the following bounds for the other *HH-difference*

$$\int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right).$$

Theorem 7. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\begin{aligned}(3.14)\quad & \frac{1}{2q(1-q)} \min\{1-q,q\} \\ & \times \left[\int_0^1 f((1-t)A+tB)\,dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A+sB)\,ds\right] \\ & \leq \int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q,q\} \\ & \times \left[\int_0^1 f((1-t)A+tB)\,dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A+sB)\,ds\right]\end{aligned}$$

or, equivalently

$$\begin{aligned}(3.15)\quad & \frac{2q(1-q)}{\max\{1-q,q\}}\left[\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right] \\ & \leq \int_0^1 f((1-t)A+tB)\,dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A+sB)\,ds \\ & \leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right]\end{aligned}$$

for any $q \in (0,1)$, $q \neq \frac{1}{2}$.

Proof. If we take in (3.7) $p = \frac{1}{2}$, then we have

$$\begin{aligned}(3.16)\quad & \frac{1}{2q(1-q)} \min\{1-q,q\}\left[(1-q)f(A)+qf(B)-f((1-q)A+qB)\right] \\ & \leq \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q,q\}\left[(1-q)f(A)+qf(B)-f((1-q)A+qB)\right]\end{aligned}$$

for any A, B with spectra in I and $q \in (0,1)$.

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ in (3.16), then we get

$$\begin{aligned}(3.17)\quad & \frac{1}{2q(1-q)} \min\{1-q,q\} \\ & \times [(1-q)f((1-t)A+tB)+qf(tA+(1-t)B) \\ & -f((1-q)[(1-t)A+Bt]+q[tA+(1-t)B])] \\ & \leq \frac{f((1-t)A+tB)+f(tA+(1-t)B)}{2} - f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2q(1-q)} \max\{1-q,q\} \\ & \times [(1-q)f((1-t)A+tB)+qf(tA+(1-t)B) \\ & -f((1-q)[(1-t)A+Bt]+q[tA+(1-t)B])]\end{aligned}$$

for any A, $B \in C$, $t \in [0,1]$ and $q \in (0,1)$.

If we take the integral over $t \in [0,1]$ in (3.17) and take into account that

$$\int_0^1 f((1-t)A + tB)\,dt = \int_0^1 f(tA + (1-t)B)\,dt$$

we get

$$\begin{aligned}
(3.18)\qquad & \frac{1}{2q(1-q)} \min\{1-q, q\} \left[\int_0^1 f((1-t)A + tB)\,dt \right. \\
& \left. - \int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B])\,dt\right] \\
& \le \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \\
& \le \frac{1}{2q(1-q)} \max\{1-q, q\} \left[\int_0^1 f((1-t)A + tB)\,dt \right. \\
& \left. - \int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B])\,dt\right]
\end{aligned}$$

or any A, B with spectra in I and $q \in (0,1)$.

Observe that for any A, B with spectra in I, $t \in [0,1]$ and $q \in (0,1)$ we have

$$\begin{aligned}
& (1-q)[(1-t)A + tB] + q[tA + (1-t)B] \\
& = [(1-q)(1-t) + qt]A + [(1-q)t + (1-t)q]B
\end{aligned}$$

and by putting $s := (1-q)t + (1-t)q$, for $q \neq \frac{1}{2}$ we have

$$[(1-q)(1-t) + qt]A + [(1-q)t + (1-t)q]B = (1-s)A + sB.$$

If $q \neq \frac{1}{2}$, then s is a change of variable, $ds = (1-2q)\,dt$ and we have for any A, B with spectra in I that

$$\begin{aligned}
& \int_0^1 f((1-q)[(1-t)A + tB] + q[tA + (1-t)B])\,dt \\
& = \frac{1}{1-2q} \int_q^{1-q} f((1-s)A + sB)\,ds.
\end{aligned}$$

On making use of (3.18) we get the desired result (3.14). □

Corollary 5. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned}
(3.19)\qquad & \frac{2q(1-q)}{\max\{1-q, q\}} \left[\int_0^1 \Phi(f((1-t)A + tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right] \\
& \le \int_0^1 \Phi(f((1-t)A + tB))\,dt - \frac{1}{1-2q} \int_q^{1-q} \Phi(f((1-s)A + sB))\,ds \\
& \le \frac{2q(1-q)}{\min\{1-q, q\}} \left[\int_0^1 \Phi(f((1-t)A + tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right]
\end{aligned}$$

for any $q \in (0,1)$, $q \neq \frac{1}{2}$.

Remark 1. *If we take* $q=\frac{1}{4}$ *in (3.15) and (3.19), then we get*

$$\begin{aligned}(3.20)\quad &\frac{1}{2}\left[\int_0^1 f\left((1-t)A+tB\right)dt-f\left(\frac{A+B}{2}\right)\right]\\ &\leq\int_0^1 f\left((1-t)A+tB\right)dt-2\int_{1/4}^{3/4} f\left((1-s)A+sB\right)ds\\ &\leq\frac{3}{2}\left[\int_0^1 f\left((1-t)A+tB\right)dt-f\left(\frac{A+B}{2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}(3.21)\quad &\frac{1}{2}\left[\int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt-\Phi\left(f\left(\frac{A+B}{2}\right)\right)\right]\\ &\leq\int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt-2\int_{1/4}^{3/4} \Phi\left(f\left((1-s)A+sB\right)\right)ds\\ &\leq\frac{3}{2}\left[\int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt-\Phi\left(f\left(\frac{A+B}{2}\right)\right)\right]\end{aligned}$$

for any A, B *with spectra in* I *and* $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right].$

4. Some Examples

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$.

If we write the inequality (2.3) for the power $1\leq r\leq 2$ (or $-1\leq r\leq 0$) we have

$$(4.1)\quad \int_0^1\left((1-t)\Phi(A)+t\Phi(B)\right)^r dt\leq\int_0^1\Phi\left(\left((1-t)A+tB\right)^r\right)dt,$$

where $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and A, $B\in\mathcal{B}^+(H)$ $(A,B\in\mathcal{B}^{++}(H))$. In the case $0\leq r\leq 1$ the inequalities reverse in (4.1).

If we write the inequality (2.9) for the power $1\leq r\leq 2$ (or $-1\leq r\leq 0$) we have

$$\begin{aligned}(4.2)\quad &\Phi\left(\left(\frac{A+B}{2}\right)^r\right)\\ &\leq(1-\lambda)\Phi\left(\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]^r\right)+\lambda\Phi\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^r\right)\\ &\leq\int_0^1\Phi\left(\left((1-t)A+tB\right)^r\right)dt\\ &\leq\frac{1}{2}\left[\Phi\left(\left((1-\lambda)A+\lambda B\right)^r\right)+(1-\lambda)\Phi(B^r)+\lambda\Phi(A^r)\right]\\ &\leq\frac{\Phi(A^r)+\Phi(B^r)}{2},\end{aligned}$$

where $\lambda\in[0,1]$, $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and A, $B\in\mathcal{B}^+(H)$ $(A,B\in\mathcal{B}^{++}(H))$. In the case $0\leq r\leq 1$ the inequalities reverse in (4.2).

If we write the inequality (3.9) for the power $1\leq r\leq 2$ (or $-1\leq r\leq 0$) we get for $p\in[0,1]$, $q\in(0,1)$ that

$$\begin{aligned}(4.3)\qquad & \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[(1-q)\,\Phi\left(A^{r}\right)+q\Phi\left(B^{r}\right)-\Phi\left(\left((1-q)\,A+qB\right)^{r}\right)\right]\\ & \leq\left[(1-p)\,\Phi\left(A^{r}\right)+p\Phi\left(B^{r}\right)-\Phi\left(\left((1-p)\,A+pB\right)^{r}\right)\right]\\ & \leq\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[(1-q)\,\Phi\left(A^{r}\right)+q\Phi\left(B^{r}\right)-\Phi\left(\left((1-q)\,A+qB\right)^{r}\right)\right]\end{aligned}$$

where $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and $A,B\in\mathcal{B}^{+}(H)$ $(A,\ B\in\mathcal{B}^{++}(H))$.

From (3.13) we have for $1\leq r\leq 2$ (or $-1\leq r\leq 0$) that

$$\begin{aligned}(4.4)\qquad & \frac{1}{2}\left[(1-q)\,\Phi\left(A^{r}\right)+q\Phi\left(B^{r}\right)-\Phi\left(\left((1-q)\,A+qB\right)^{r}\right)\right]\\ & \leq\frac{\Phi\left(A^{r}\right)+\Phi\left(B^{r}\right)}{2}-\int_{0}^{1}\Phi\left(\left((1-t)\,A+tB\right)^{r}\right)dt\\ & \leq\frac{1}{2}\frac{q^{2}-q+1}{q\,(1-q)}\left[(1-q)\,\Phi\left(A^{r}\right)+q\Phi\left(B^{r}\right)-\Phi\left(\left((1-q)\,A+qB\right)^{r}\right)\right]\end{aligned}$$

while from(3.19) we have that

$$\begin{aligned}(4.5)\qquad & \frac{2q\,(1-q)}{\max\{1-q,q\}}\left[\int_{0}^{1}\Phi\left(\left((1-t)\,A+tB\right)^{r}\right)dt-\Phi\left(\left(\frac{A+B}{2}\right)^{r}\right)\right]\\ & \leq\int_{0}^{1}\Phi\left(\left((1-t)\,A+tB\right)^{r}\right)dt-\frac{1}{1-2q}\int_{q}^{1-q}\Phi\left(\left((1-s)\,A+sB\right)^{r}\right)ds\\ & \leq\frac{2q\,(1-q)}{\min\{1-q,q\}}\left[\int_{0}^{1}\Phi\left(\left((1-t)\,A+tB\right)^{r}\right)dt-\Phi\left(\left(\frac{A+B}{2}\right)^{r}\right)\right],\end{aligned}$$

where $p\in[0,1]$, $q\in(0,1)$, $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and $A,\ B\in\mathcal{B}^{+}(H)$ $(A,\ B\in\mathcal{B}^{++}(H))$.

The function $f(t)=-\ln t$ is operator convex on $(0,\infty)$. Then by (2.3) we have

$$(4.6)\qquad \int_{0}^{1}\ln\left((1-t)\,\Phi(A)+t\Phi(B)\right)dt\geq\int_{0}^{1}\Phi\left(\ln\left((1-t)\,A+tB\right)\right)dt$$

while by (2.9) we have, for $\lambda\in[0,1]$ that

$$\begin{aligned}(4.7)\qquad & \Phi\left(\ln\left(\frac{A+B}{2}\right)\right)\\ & \geq(1-\lambda)\,\Phi\left(\ln\left[\frac{(1-\lambda)\,A+(1+\lambda)\,B}{2}\right]\right)+\lambda\Phi\left(\ln\left[\frac{(2-\lambda)\,A+\lambda B}{2}\right]\right)\\ & \geq\int_{0}^{1}\Phi\left(\ln\left((1-t)\,A+tB\right)\right)dt\\ & \geq\frac{1}{2}\left[\Phi\left(\ln\left((1-\lambda)\,A+\lambda B\right)\right)+(1-\lambda)\,\Phi\left(\ln(B)\right)+\lambda\Phi\left(\ln(A)\right)\right]\\ & \geq\frac{\Phi\left(\ln(A)\right)+\Phi\left(\ln(B)\right)}{2},\end{aligned}$$

where $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and $A,\ B\in\mathcal{B}^{++}(H)$.

From (3.9) we have for $p \in [0,1]$, $q \in (0,1)$ that

$$\begin{aligned}
&\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[\Phi\left(\ln\left((1-q)A+qB\right)\right)-(1-q)\Phi\left(\ln(A)\right)-q\Phi\left(\ln(B)\right)\right] \\
&\leq \left[\Phi\left(\ln\left((1-p)A+pB\right)\right)-(1-p)\Phi\left(\ln(A)\right)-p\Phi\left(\ln(B)\right)\right] \\
&\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[\Phi\left(\ln\left((1-q)A+qB\right)\right)-(1-q)\Phi\left(\ln(A)\right)-q\Phi\left(\ln(B)\right)\right],
\end{aligned} \tag{4.8}$$

from (3.13) we have

$$\begin{aligned}
&\frac{1}{2}\left[\Phi\left(\ln\left((1-q)A+qB\right)\right)-(1-q)\Phi\left(\ln(A)\right)-q\Phi\left(\ln(B)\right)\right] \\
&\leq \int_0^1 \Phi\left(\ln\left((1-t)A+tB\right)\right)dt-\frac{\Phi\left(\ln(A)\right)+\Phi\left(\ln(B)\right)}{2} \\
&\leq \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[\Phi\left(\ln\left((1-q)A+qB\right)\right)-(1-q)\Phi\left(\ln(A)\right)-q\Phi\left(\ln(B)\right)\right].
\end{aligned} \tag{4.9}$$

while from (3.19)

$$\begin{aligned}
&\frac{2q(1-q)}{\max\{1-q,q\}}\left[\Phi\left(\ln\left(\frac{A+B}{2}\right)\right)-\int_0^1 \Phi\left(\ln\left((1-t)A+tB\right)\right)dt\right] \\
&\leq \frac{1}{1-2q}\int_q^{1-q}\Phi\left(\ln\left((1-s)A+sB\right)\right)ds-\int_0^1 \Phi\left(\ln\left((1-t)A+tB\right)\right)dt \\
&\leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[\Phi\left(\ln\left(\frac{A+B}{2}\right)\right)-\int_0^1 \Phi\left(\ln\left((1-t)A+tB\right)\right)dt\right]
\end{aligned} \tag{4.10}$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^{++}(H)$.

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