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# MORE OPERATOR INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. In this paper we present some new operator inequality for convex functions. We have obtained a number of Jensen's type inequalities for convex and operator convex functions of self-adjoint operators for positive linear maps. Some results are exemplified for power and logarithmic functions.

## 1. Introduction and Preliminaries

As is customary, we reserve $M, m$ for scalars. Other capital letters are used to denote general elements of the $C^*$-algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. An operator $A \in \mathcal{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathscr{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map establishes an isometrically $*$-isomorphism $\Phi$ between the set $C(sp(A))$ of all continuous functions on the spectrum of $A$, denoted $sp(A)$, and the $C^*$-algebra generated by $A$ and $I$ (see for instance [14, p. 15]). For any $f, g \in C(sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$;
- $\|\Phi(f)\| = \|f\| := \sup\limits_{t \in sp(A)} |f(t)|$;

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- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in sp(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(sp(A))$ and we call it the continuous functional calculus for a self-adjoint operator $A$. It is well known that, if $A$ is a self-adjoint operator and $f \in C(sp(A))$, then $f(t) \geq 0$ for any $t \in sp(A)$ implies that $f(A) \geq 0$. It is extendible for two real valued functions on $sp(A)$. A linear map $\phi$ is positive if $\phi(A) \geq 0$ whenever $A \geq 0$. It said to be normalized if $\phi(I) = I$. The set of all real valued continuous functions defined on an interval $[M, m]$ will be denoted by $C([M, m])$. For more studies in this direction, we refer to [2]. As is known to all, in [9, Theorem 1.2], the authors presented the operator version for the Jensen inequality. They proved the inequality

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle,$$

where $A$ is a self-adjoint operator with $sp(A) \subseteq [m, M]$ and $f(t)$ is a convex function on an interval $[m, M]$. Davis [4] and Choi [3] showed that if $\phi$ is a normalized positive linear map on $\mathcal{B}(\mathscr{H})$ and if $f$ is an operator convex function on an interval $[m, M]$, then the so-called Choi-Davis-Jensen inequality

$$f(\phi(A)) \leq \phi(f(A)), \tag{1.1}$$

holds for every self adjoint operator $A$ on $\mathscr{H}$ whose spectrum is contained in $[m, M]$. As a special case of inequality (1.1), the authors in [9, Theorem 1.21] established the following generalization of Jensen's inequality under the additional condition $p_i, (i \in \{1, \ldots, n\})$ are positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$

$$f\left(\sum_{i=1}^{n} p_i \phi_i(A_i)\right) \leq \sum_{i=1}^{n} p_i \phi_i(f(A_i)).$$

The following variant of Jensen's operator inequality for a convex function $f \in C([m, M])$, self-adjoint operators $A_i \in \mathcal{B}(\mathscr{H})$ with spectra in $[m, M]$ and normalized positive linear maps $\phi_i$ on $\mathcal{B}(\mathscr{H})$ was proved in [12]

$$f\left((m+M)1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(A_i)\right) \leq (f(m)+f(M))1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(f(A_i)).$$

Moreover, in the same paper the following series of inequalities was proved

$$\begin{aligned} & f\left(m+M\right)1_{\mathscr{H}}-\sum_{i=1}^{n}\phi_i\left(A_i\right) \\ & \le \frac{M1_{\mathscr{H}}-\sum\limits_{i=1}^{n}\phi_i\left(A_i\right)}{M-m}f\left(M\right)+\frac{\sum\limits_{i=1}^{n}\phi_i\left(A_i\right)-m1_{\mathscr{H}}}{M-m}f\left(M\right) \\ & \le \left(f\left(m\right)+f\left(M\right)\right)1_{\mathscr{H}}-\sum_{i=1}^{n}\phi_i\left(f\left(A_i\right)\right). \end{aligned}$$

There is considerable amount of literature devoting to the study of Jensen inequality, we refer to [11, 10] for a recent survey and references therein. This paper include but are not restricted to the positive linear map version of the Dragomir-Ionescu inequality and Slater's type inequalities for operators and its inverses.

## 2. **Inequalities for Differentiable and Convex Functions**

The following result provides an operator version of the Dragomir-Ionescu inequality (see [8] for the original result):

**Theorem 2.1.** *Let $A$ be a self adjoint operator on the Hilbert space $\mathscr{H}$ with $sp\left(A\right)\subseteq I$ and $\phi$ be a normalized positive linear map on $\mathcal{B}\left(\mathscr{H}\right)$ and let $f:I\to\mathbb{R}$ be a convex and differentiable function on interval $I$ whose derivative $f'$ is continuous on $I$. Then*

$$\begin{aligned} 0 & \le \left\langle \phi\left(f\left(A\right)\right)x,x\right\rangle - f\left(\left\langle \phi\left(A\right)x,x\right\rangle\right) \\ & \le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle - \left\langle \phi\left(A\right)x,x\right\rangle\left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle \end{aligned} \tag{2.1}$$

*for each $x\in\mathscr{H}$ with $\|x\|=1$.*

*Proof.* First we remark that, the first inequality in (2.1) has been proven for convex functions with no differentiability assumption in [5]. As in [5] and for the sake of completeness, we give here a short proof of the second inequality from (2.1). Since $f$ is convex and differentiable, by the gradient inequality we have that

$$f\left(t\right)-f\left(s\right)\le f'\left(t\right)\left(t-s\right),$$

for any $t,s\in\left[m,M\right]$. Since $sp\left(A\right)\subseteq\left[m,M\right]$ by substitute $s=\left\langle \phi\left(A\right)x,x\right\rangle\in\left[m,M\right]$, we have

$$f\left(t\right)-f\left(\left\langle \phi\left(A\right)x,x\right\rangle\right)\le f'\left(t\right)\left(t-\left\langle \phi\left(A\right)x,x\right\rangle\right).$$

Now, by the functional calculus for $t \in [m, M]$ we deduce

$$f(A) - f(\langle \phi(A)x, x\rangle) 1_{\mathscr{H}} \le f'(A)(A - \langle \phi(A)x, x\rangle 1_{\mathscr{H}})$$

and since $\phi$ is a normalized positive linear map we have

$$\begin{aligned}&\langle \phi(f(A))x, x\rangle - f(\langle \phi(A)x, x\rangle)\\ &\le \langle \phi(f'(A)A)x, x\rangle - \langle \phi(A)x, x\rangle \langle \phi(f'(A))x, x\rangle\end{aligned}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$. $\square$

*Remark* 2.2. There are several examples of normalized positive linear maps. But for our application we consider among them $\phi : \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$, which $\phi(A) = \left(\frac{1}{n}tr(A)\right) 1_{\mathscr{H}}$ for all Hermitian matrices $A \in \mathscr{M}_2(\mathbb{C})$. From inequality (2.1), we have

$$0 \le \frac{1}{n}tr(f(A)) - f\left(\frac{1}{n}tr(A)\right) \le \frac{1}{n}tr(f'(A)A) - \frac{1}{n^2}tr(A)\,tr(f'(A)).$$

If we take $f(t) = t^2$, then we get

$$\frac{1}{n}tr^2(A) \le tr\left(A^2\right).$$

In the following we give more interesting situations.

**Corollary 2.3.** *Let $A_i$ are self-adjoint operators with $sp(A_i) \subseteq I$, $i \in \{1, \ldots, n\}$ and $\phi_i$ are normalized positive linear map on $\mathcal{B}(\mathscr{H})$ ($i \in \{1, \ldots, n\}$) and assume that $f$ is as in the Theorem 2.1. Then*

$$\begin{aligned}0 &\le \left\langle \sum_{i=1}^{n} \phi_i(f(A_i))x, x\right\rangle - f\left\langle \sum_{i=1}^{n} \phi_i(A_i)x, x\right\rangle\\ &\le \left\langle \sum_{i=1}^{n} \phi_i(f'(A_i)A_i)x, x\right\rangle - \left\langle \sum_{i=1}^{n} \phi_i(A_i)x, x\right\rangle \qquad (2.2)\\ &\quad\times \left\langle \sum_{i=1}^{n} \phi_i(f'(A_i))x, x\right\rangle,\end{aligned}$$

*for each $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* If we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & A_n\end{pmatrix},$$

then we have $sp\left(\widetilde{A}\right) \subseteq [m, M] \subseteq \overset{o}{I}$, moreover, define

$$\phi : \mathcal{B}(\mathscr{H}) \oplus \ldots \oplus \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$$

as

$$\phi\left(A_1 \oplus \ldots \oplus A_n\right) = \sum_{i=1}^{n} \phi_i\left(A_i\right)$$

in this case, we have

$$\phi\left(f\left(\widetilde{A}\right)\right) = \sum_{i=1}^{n} \phi_i\left(f\left(A_i\right)\right),\ \phi\left(\widetilde{A}\right) = \sum_{i=1}^{n} \phi_i\left(A_i\right).$$

Applying Theorem 2.1 for $\widetilde{A}$ we deduce the desired result (2.2). □

**Corollary 2.4.** *Notation as in above. If $p_i$ are positive real numbers with $\sum\limits_{i=1}^{n} p_i = 1$, then*

$$\begin{aligned} 0 \leq & \left\langle \sum_{i=1}^{n} p_i f\left(\phi_i\left(A_i\right)\right)x, x \right\rangle - f\left(\left\langle \sum_{i=1}^{n} p_i \phi_i\left(A_i\right)x, x \right\rangle\right) \\ & \left\langle \sum_{i=1}^{n} p_i f'\left(\phi_i\left(A_i\right) A_i\right)x, x \right\rangle - \left\langle \sum_{i=1}^{n} p_i \left(\phi_i\left(A_i\right)\right)x, x \right\rangle \\ & \times \left\langle \sum_{i=1}^{n} p_i f'\left(\phi_i\left(A_i\right)\right)x, x \right\rangle, \end{aligned}$$

*for each $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Remark* 2.5. The following particular cases are of interest (see also [5]):

(a) If $p \geq 1$ and $A$ is a positive operator on $\mathscr{H}$, then we have the inequality

$$\begin{aligned} 0 &\leq \left\langle \phi\left(A^p\right)x, x\right\rangle - \left\langle \phi\left(A\right)x, x\right\rangle^p \\ &\leq p\left[\left\langle \phi\left(A^p\right)x, x\right\rangle - \left\langle \phi\left(A\right)x, x\right\rangle \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle\right] \end{aligned} \tag{2.3}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

(b) If $A$ is a positive definite operator on $\mathscr{H}$, then the inequality (2.3) holds for $p < 0$.

(c) If $0 < p < 1$ and $A$ is a positive definite operator on $\mathscr{H}$, then

$$\begin{aligned} &\left\langle \phi\left(A^p\right)x, x\right\rangle - \left\langle \phi\left(A\right)x, x\right\rangle^p \\ &\geq p\left[\left\langle \phi\left(A^p\right)x, x\right\rangle - \left\langle \phi\left(A\right)x, x\right\rangle \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle\right] \geq 0 \end{aligned}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Now, from a different view point we may state:

**Theorem 2.6.** *All assumptions as in Theorem 2.1. Then*

$$\begin{aligned}&\left\langle \phi\left(f\left(A\right)-f\left(B\right)\right)x,x\right\rangle \\ &\le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle - \left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle \left\langle \phi\left(B\right)x,x\right\rangle .\end{aligned} \tag{2.4}$$

*for each $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* Since $f$ is convex and differentiable we have

$$f\left(t\right)-f\left(s\right)\le f'\left(t\right)\left(t-s\right)$$

for any $t, s \in I$. Fix $s \in I$ and apply the functional calculus for the operator $A$, then

$$f\left(A\right)-f\left(s\right)1_{\mathscr{H}}\le f'\left(A\right)\left(A-s\right).$$

Since $\phi$ is normalized positive linear map we get

$$\phi\left(f\left(A\right)\right)-f\left(s\right)1_{\mathscr{H}}\le \phi\left(f'\left(A\right)A\right)-s\phi\left(f'\left(A\right)\right) \tag{2.5}$$

therefore

$$\left\langle \phi\left(f\left(A\right)\right)x,x\right\rangle - f\left(s\right)\le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle - s\left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$. Apply again functional calculus for the operator $B$ to obtain

$$\begin{aligned}&\left\langle \phi\left(f\left(A\right)\right)x,x\right\rangle 1_{\mathscr{H}} - f\left(B\right) \\ &\le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle 1_{\mathscr{H}} - \left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle B.\end{aligned} \tag{2.6}$$

Since $\phi$ is normalized positive linear map we have

$$\begin{aligned}&\left\langle \phi\left(f\left(A\right)\right)x,x\right\rangle 1_{\mathscr{H}} - \phi\left(f\left(B\right)\right) \\ &\le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle 1_{\mathscr{H}} - \left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle \phi\left(B\right)\end{aligned} \tag{2.7}$$

therefore

$$\begin{aligned}&\left\langle \phi\left(f\left(A\right)\right)x,x\right\rangle - \left\langle \phi\left(f\left(B\right)\right)y,y\right\rangle \\ &\le \left\langle \phi\left(f'\left(A\right)A\right)x,x\right\rangle - \left\langle \phi\left(f'\left(A\right)\right)x,x\right\rangle \left\langle \phi\left(B\right)y,y\right\rangle .\end{aligned} \tag{2.8}$$

for each $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$. This is an inequality of interest in itself. Finally, on making $y = x$ in (2.8) we deduce the desired result (2.4). $\square$

By setting $\phi\left(A\right) = \left(\frac{1}{n}tr\left(A\right)\right)1_{\mathscr{H}}$ in (2.4), we find the following result.

*Remark* 2.7. Let $A, B \in \mathscr{M}_n\left(\mathbb{C}\right)$ be two Hermitian matrices, then

$$\frac{1}{n}tr\left(f\left(A\right)-f\left(B\right)\right)\le \frac{1}{n}tr\left(f'\left(A\right)A\right)-\frac{1}{n^2}tr\left(f'\left(A\right)\right)tr\left(B\right).$$

*Remark* 2.8. If we choose $A = B$ in (2.4), we get

$$\langle \phi(f'(A)A)x, x\rangle \geq \langle \phi(f'(A))x, x\rangle \langle \phi(A)x, x\rangle \tag{2.9}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

The following result is known in the literature as the Kadison inequality.

*Remark* 2.9. If we take $f(t) = t^2$ in (2.9), we deduce

$$\left\langle \phi\left(A^2\right)x, x\right\rangle \geq \langle \phi(A)x, x\rangle^2$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

The following result concerning Slater type inequality holds:

**Theorem 2.10.** *Let $A$ be a self-adjoint operator on the Hilbert space $\mathscr{H}$ with $sp(A) \subseteq [m, M] \subseteq I$ and $\phi$ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$ and let $f : I \to \mathbb{R}$ be a convex and differentiable function on interval $I$ whose derivative $f'$ is continuous and strictly positive on $I$. Then*

$$\begin{aligned} 0 &\leq f\left(\frac{\langle \phi(Af'(A))x, x\rangle}{\langle \phi(f'(A))x, x\rangle}\right) - \langle \phi(f(A))x, x\rangle \\ &\leq f'\left(\frac{\langle \phi(Af'(A))x, x\rangle}{\langle \phi(f'(A))x, x\rangle}\right)\left[\frac{\langle \phi(Af'(A))x, x\rangle}{\langle \phi(f'(A))x, x\rangle} - \langle \phi(A)x, x\rangle\right] \end{aligned} \tag{2.10}$$

*for each $x \in \mathscr{H}$ with $\|x\| = 1$.*

*Proof.* Since $f$ is convex and differentiable on $I$, then we have that

$$f'(s)(t - s) \leq f(t) - f(s) \leq f'(t)(t - s)$$

for any $t, s \in [m, M]$. If we fix $t \in [m, M]$ and apply the functional calculus for the operator $A$, then we obtain

$$f'(A)(t - A) \leq f(t)1_{\mathscr{H}} - f(A) \leq f'(t)(t - A) \tag{2.11}$$

for any $t \in [m, M]$. Since $\phi$ is normalized positive linear map we have

$$t\phi(f'(A)) - \phi(f'(A)A) \leq f(t)1_{\mathscr{H}} - \phi(f(A)) \leq f'(t)t1_{\mathscr{H}} - f'(t)\phi(A) \tag{2.12}$$

for any $t \in [m, M]$. The inequality (2.12) is equivalent with

$$\begin{aligned} t\langle \phi(f'(A))x, x\rangle - \langle \phi(f'(A)A)x, x\rangle &\leq f(t) - \langle \phi(f(A))x, x\rangle \\ &\leq f'(t)t - f'(t)\langle \phi(A)x, x\rangle \end{aligned} \tag{2.13}$$

for any $t \in [m, M]$. Now, since $A$ is self-adjoint with $m1_{\mathscr{H}} \le A \le M1_{\mathscr{H}}$ and $f'(A)$ is strictly positive, then $mf'(A) \le Af'(A) \le Mf'(A)$. Also, $\phi$ is a unital positive linear map, therefore

$$m \langle \phi(f'(A)) x, x\rangle \le \langle \phi(Af'(A)) x, x\rangle \le M \langle \phi(f'(A)) x, x\rangle,$$

hence $\langle \phi(Af'(A)) x, x\rangle \langle \phi(f'(A)) x, x\rangle^{-1} \in [m, M]$. If we put

$$t := \frac{\langle \phi(Af'(A)) x, x\rangle}{\langle \phi(f'(A)) x, x\rangle}$$

in the equation (2.13), we get the desired result (2.10). □

For the concave case, we have the following result.

**Corollary 2.11.** *Let $A$ and $\phi$ be as in Theorem 2.10 and $f : I \to \mathbb{R}$ be a concave and differentiable function on interval $I$ whose derivative $f'$ is continuous and strictly positive on $I$. Then*

$$\begin{aligned}0 &\le \langle \phi(f(A)) x, x\rangle - f\left(\langle \phi(Af'(A)) x, x\rangle \langle \phi(f'(A)) x, x\rangle^{-1}\right)\\ &\le f'\left(\langle \phi(Af'(A)) x, x\rangle \langle \phi(f'(A)) x, x\rangle^{-1}\right)\\ &\times \left[\left(\langle \phi(A) x, x\rangle \langle \phi(f'(A)) x, x\rangle - \langle \phi(Af'(A)) x, x\rangle\right) \langle \phi(f'(A)) x, x\rangle^{-1}\right]\end{aligned} \tag{2.14}$$

*for each $x \in \mathscr{H}$ with $\|x\| = 1$.*

The following results follows from the inequalities (2.10) and (2.14) for the convex (concave) function $f(t) = t^p, p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0, 1))$ by performing the required calculation. The details are omitted.

*Remark* 2.12. Let $\phi$ be a positive linear map on $\mathcal{B}(\mathscr{H})$ and let $A$ is a strictly positive operator on $\mathscr{H}$.

(a) If $p \ge 1$, we have that

$$\begin{aligned}0 &\le \langle \phi(A^p) x, x\rangle^{p-1} - \left\langle \phi\left(A^{p-1}\right) x, x\right\rangle^p\\ &\le p\langle \phi(A^p) x, x\rangle^{p-2} \left[\langle \phi(A^p) x, x\rangle - \langle \phi(A) x, x\rangle \left\langle \phi\left(A^{p-1}\right) x, x\right\rangle\right].\end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

(b) If $0 < p < 1$, we have that

$$\begin{aligned}0 &\le \left\langle \phi\left(A^{p-1}\right) x, x\right\rangle^p - \langle \phi(A^p) x, x\rangle^{p-1}\\ &\le p\langle \phi(A^p) x, x\rangle^{p-2} \left[\langle \phi(A) x, x\rangle - \left\langle \phi\left(A^{p-1}\right) x, x\right\rangle \langle \phi(A^p) x, x\rangle\right].\end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

(c) If $p < 0$, we have that

$$0 \le \langle \phi(A^p)x, x\rangle^{p-1} - \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle^p$$
$$\le (-p)\langle \phi(A^p)x, x\rangle^{p-2}\left[\langle \phi(A)x, x\rangle - \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle \langle \phi(A^p)x, x\rangle\right].$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

*Remark* 2.13. Notice that we can obtain other results by taking various functions. The details are omitted.

## 3. **Some Inequalities Involving States**

In this section, we apply the continuous functional calculus to convex and differentiable functions and present some reverses Jensen's inequalities involving states on $C^*$-algebras. Our main result of this section reads as follows.

**Theorem 3.1.** *Let $\tau$ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval $I$ whose derivative $f'$ is continuous on $I$. Then*

$$0 \le \tau(f(A)) - f(\tau(A)) \le \tau(f'(A)A) - \tau(A)\tau(f'(A)). \tag{3.1}$$

*Proof.* Since $f$ is convex and differentiable, we have that

$$f(t) - f(s) \le f'(t)(t - s)$$

if we chose in this inequality $s = \tau(A)$ we obtain

$$f(t) - f(\tau(A)) \le f'(t)(t - \tau(A)).$$

Applying functional calculus for the operator $A$

$$f(A) - f(\tau(A)) \le f'(A)A - f'(A)\tau(A).$$

For the state $\tau$ we have

$$\tau(f(A)) - f(\tau(A)) \le \tau(f'(A)A) - \tau(A)\tau(f'(A)).$$

Notice that, the first inequality in (3.1) can be proved in a similar way, however the details are omitted. ☐

**Corollary 3.2.** *Apply Theorem 3.1 to the state $\tau$ defined by $\tau(A) = \langle Ax, x\rangle$ for fixed unit vector $x \in \mathscr{H}$. Then*

$$0 \le \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \le \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle.$$

This inequality was obtained by Dragomir in [1] (see also [7]).

By Theorem 2.6 and making use of a similar argument to the one in the proof of Theorem 3.1 we can state the following result as well. However, the details are nor provided here.

**Theorem 3.3.** *Let $\tau$ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval $I$ whose derivative $f'$ is continuous on $I$. Then*

$$\tau(f(A)) - \tau(f(B)) \le \tau(f'(A)A) - \tau(f'(A))\tau(B). \tag{3.2}$$

*Remark* 3.4. If we choose $A = B$ in (3.2), we obtain

$$\tau(f'(A))\tau(A) \le \tau(f'(A)A).$$

*Remark* 3.5. Let $\tau$ be a state on $\mathcal{B}(\mathscr{H})$ and $f(t) = t^2$ in (3.2), then

$$\tau(A)^2 \le \tau\left(A^2\right) \ \ (A \ge 0).$$

In a similar fashion, for self-adjoint operators $A \in \mathcal{B}(\mathscr{H})$

$$\tau\left(e^{\alpha A}\right)^2 \le \tau\left(e^{2\alpha A}\right) \ \ (\alpha \ge 0).$$

By applying Theorem 2.10 and using the same strategy as in the proof of Theorem 3.1 we get the next result.

**Theorem 3.6.** *Let $\tau$ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval $I$ whose derivative $f'$ is continuous and strictly positive on $I$. Then*

$$\begin{aligned} 0 &\le f\left(\frac{\tau(Af'(A))}{\tau(f'(A))}\right) - \tau(f(A)) \\ &\le f'\left(\frac{\tau(Af'(A))}{\tau(f'(A))}\right)\left[\frac{\tau(Af'(A))}{\tau(f'(A))} - \tau(A)\right]. \end{aligned}$$

## References


[1] R.P. Agarwal, and S.S. Dragomir. *A survey of Jensen type inequalities for functions of self-adjoint operators in Hilbert spaces*, Computers & Mathematics with Applications **59**(12) (2010): 3785-3812.

[2] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, (2007).

[3] M.D. Choi, *A Schwarz inequality for positive linear maps on $C^*$-algebras*, Illinois J. Math. **18** (1974): 565-574.

[4] C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc., **8** (1957): 42-44.

[5] S.S. Dragomir, *A weaken version of Davis-Choi-Jensen's inequality for normalized positive linear maps*, Preprint RGMIA Res. Rep. Coll. **19** (2016), Art 61.

[6] S.S.Dragomir, *Inequalities for functions of self-adjoint operators on Hilbert spaces*, arXiv preprint arXiv:1203.1667 (2012).

[7] S.S. Dragomir, *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Inequal. Appl. 2010, Art. ID 496821, 15 pp.

[8] S.S. Dragomir and N.M. Ionescu, *Some converse of Jensen's inequality and applications*. Rev. Anal. Numér. Théor. Approx. **23**(1) (1994): 71-78.

[9] T. Furuta, J. Mićić, J. Pečarić, Y. Seo, *Mond-Pečarić Method in Operator Inequalities, Inequalities for Bounded Self-adjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] S. Ivelić, A. Matković, and J.E. Pečarić. *On a Jensen-Mercer operator inequality.* Banach J. Math. Anal **5**(1) (2011): 19-28.

[11] M. Khosravi, J.S. Aujla, S.S. Dragomir, and M. S. Moslehian. *Refinements of Choi-Davis-Jensen's inequality.* Bull. Math. Anal. Appl, **3**(2) (2011): 127-133.

[12] A. Matković, J. Pečarić and I. Perić, *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl. **418**(2-3) (2006): 551-564.

[13] H.R. Moradi, M.E. Omidvar and S.S. Dragomir, *An operator extension of Čebyšev inequality*, Preprint RGMIA Res. Rep. Coll. **19** (2016), Art. submitted.

[14] G. J. Murphy. *$C^*$-Algebras and Operator Theory*, Academic Press, San Diego (1990).