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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR p-PREINVEX FUNCTIONS

IMRAN ABBAS BALOCH, İMDAT İSCAN

Abstract. In this article, we define the class of p-preinvex functions which is generalization of preinvex and harmonically preinvex functions. We also define the notion of p-prequasiinvex function. Finally, we establish Hermite-Hadamard type inequalities when the power of the absolute value of the derivative of the integrand is p-preinvex.

1. Introduction

In this section, we recall various concepts and known results,see [16],[20] and references theirin.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function, if

$$f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + m(1-\lambda)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.2. A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if for every $x, y \in S$ and $t \in [0,1]$, we have

$$x + t\eta(y,x) \in S$$

Remark 1.3. Note that definition of invex set has a clear geometric interpretation.This definition essentially says that there is a path starting from a point x which is contain in S.We do not require that the point y should be the one of the end points of path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x, y \in S$, then $\eta(y,x) = y - x$ and corresponding invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y,x) = y - x$, but converse is not necessarily true, see [4],[7] and references theirin

Definition 1.4. Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η if for every $x, y \in S$ and $t \in [0,1]$, we have

$$f(x + t\eta(y,x)) \leq tf(x) + (1-t)f(y).$$

Note that every convex function is a preinvex function, but converse is not true.
We need the following assumption regarding the function η which is due to Mohan and Neogy [].
Condition C: Let $S \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : S \times S \to \mathbb{R}$. For any $x, y \in S$ and any $t \in [0,1]$,

$$\eta(y, y + t\eta(y,x)) = -t\eta(y,x)$$

$$\eta(x, y + t\eta(y,x)) = (1-t)\eta(y,x).$$

Note that for every $x, y \in S$ and $t_1, t_2 \in [0,1]$, from Condition C, we have

$$\eta(y + t_2\eta(y,x), y + t_1\eta(y,x)) = (t_2 - t_1)\eta(y,x).$$

In [5], Noor proved the Hermite-Hadamard inequality for the preinvex function as follow:

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Theorem 1.5. *Let $f : S = [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval S° and $a, b \in S^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(a+\eta(b,a))}{2} \leq \frac{f(a)+f(b)}{2}$$

In [16], M. Z. Sarikaya et all. gave the following refinements of Hermite-Hadamard inequality for preinvex functions.

Theorem 1.6. *Let $S \subseteq \mathbb{R}$ be an open invex set with respect to eta : $S \times S \to \mathbb{R}$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on S, then for every $a, b \in S$, the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \frac{\eta(b,a)}{8}[|f'(a)|+|f'(b)|] \tag{1.1}$$

Theorem 1.7. *Let $S \subseteq \mathbb{R}$ be an open invex set with respect to eta : $S \times S \to \mathbb{R}$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on S, then for every $a, b \in S$, the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

$$\leq \frac{\eta(b,a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}[(3|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}})^{\frac{p}{p-1}}+(|f'(a)|^{\frac{p}{p-1}}+3|f'(b)|^{\frac{p}{p-1}})^{\frac{p}{p-1}}] \tag{1.2}$$

Theorem 1.8. *Let $S \subseteq \mathbb{R}$ be an open invex set with respect to eta : $S \times S \to \mathbb{R}$. Suppose that $f : S \to \mathbb{R}$ is a differentiable function. Assume that $\in \mathbb{R}$ with $p \geq 1$. If $|f'|^q$ is preinvex on S, then for every $a, b \in S$, the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - f\left(\frac{2a+\eta(b,a)}{2}\right)\right| \leq \frac{\eta(b,a)}{8}[\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+2|f'(b)|^q}{3}\right)^{\frac{1}{q}}]. \tag{1.3}$$

2. **Main Results**

Now, we define the class of the p-preinvex functions which is a generalization of preinvex functions and harmonically preinvex functions:

Definition 2.1. Let $p \in \mathbb{R}/\{0\}$.The set $A_{\eta,p} \subseteq (0,\infty)$is said to be p-invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$, we have

$$[(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}} \in A.$$

The p-invex set $A_{\eta,p}$ is also call a (p,η)-connected set.

Remark 2.2. Note that for $p = 1$, p-invex set becomes invex set and for $p = -1$, p-invex set become to harmonic invex-set.

Definition 2.3. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-preinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p + t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y), \tag{2.1}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

Remark 2.4. Note that for $p = 1$ p-preinvex functions becomes preinvex functions and for $p = -1$, p-preinvex functions become harmonically preinvex functions

Theorem 2.5. *Let $f : S = [a, a+\eta(b,a)] \to (0,\infty)$ be a p-preinvex function on the interval S° and $a, b \in S^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds:*

$$f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}$$

Proof. Since f is p-preinvex function on $S = [a, a+\eta(b,a)]$. Then

$$f\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right) \le tf(a)+(1-t)f(b)$$

for all $t \in [0,1]$. integrating, we have

$$\begin{aligned}\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx &= \int_0^1 f\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt \\ &\le \int_0^1 [tf(a)+(1-t)f(b)]dt \\ &= \frac{f(a)+f(b)}{2},\end{aligned}$$

Now, take $t=\frac{1}{2}$ in inequality (2.1) and by setting $x^p = ta^p+(1-t)(a+\eta(b,a))^p$ and $(x+\eta(y,x))^p = (1-t)a^p+t(a+\eta(b,a))^p$, we have

$$\begin{aligned}f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{x^p+(x+\eta(y,x))^p}{2}\right]^{\frac{1}{p}}\right) \\ &\le \frac{f(x)+f(y)}{2}\end{aligned}$$

Since, above inequality holds for all x, y. So, it must holds for $y = x+\eta(y,x)$, then

$$f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{f\left([ta^p+(1-t)(a+\eta(b,a))^p]^{\frac{1}{p}}\right)+f\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)}{2}$$

Now by integrating for $t \in [0,1]$, we obtain

$$\begin{aligned}2f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) &\le \int_0^1 f\left([ta^p+(1-t)(a+\eta(b,a))^p\right)dt \\ &+ \int_0^1 f\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt \\ &= \frac{2p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\end{aligned}$$

This completes the proof. □

Definition 2.6. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-prequasiinvex function with respect to η if, where $p \in \mathbb{R}/\{0\}$, , if

$$f\left([(1-t)x^p+t(x+\eta(y,x))^p]^{\frac{1}{p}}\right) \le \max\{f(x), f(y)\}, \tag{2.2}$$

for all $x, y \in A_{\eta,p}$ and $t \in [0,1]$.

Lemma 2.7. *Let S be an invex set with respect to η and $a, a+\eta(b,a) \in S$. Then, we have following identity*

$$\frac{p}{[(a+\eta(b,a))^p-a^p]}f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right) - \frac{p^2}{[(a+\eta(b,a))^p-a^p]^2}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx$$

$$= \Bigg[\int_{\frac{1}{2}}^{1}(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt$$

$$- \int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt\Bigg]$$

for all $p\in\mathbb{R}/\{0\}$.

Proof. Let

$$I_1=\frac{1}{p}[(a+\eta(b,a))^p-a^p]\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt$$

integrating by parts, we get

$$I_1=\left|tf\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}f\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt,$$

by setting $x=[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}$ and $dt=\frac{p}{[(a+\eta(b,a))^p-a^p]x^{1-p}}dx$, we have

$$I_1=\frac{1}{2}f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{[\frac{a^p+(a+\eta(b,a))^p}{2}]^{\frac{1}{p}}}\frac{f(x)}{x^{1-p}}dx,$$

and let

$$I_2=\frac{1}{p}[(a+\eta(b,a))^p-a^p]\int_{\frac{1}{2}}^{1}(t-1)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)dt.$$

Similarly, by integrating by parts and after simplification we have

$$I_2=\frac{1}{2}f\left(\left[\frac{a^p+(a+\eta(b,a))^p}{2}\right]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_{[\frac{a^p+(a+\eta(b,a))^p}{2}]^{\frac{1}{p}}}^{a+\eta(b.a)}\frac{f(x)}{x^{1-p}}dx,$$

now, by adding I_1 and I_2 we get required result. □

Theorem 2.8. *Let* $S\subseteq\mathbb{R}$ *be an open invex set with respect to* $\eta:S\times S\to\mathbb{R}$ *and* $p\in\mathbb{R}/\{0\}$ *. Suppose that* $f:S\to\mathbb{R}$ *is a differentiable function. If* $|f'|$ *is p-preinvex on* S*, then for every* $a,b\in S$*, the following inequality holds*

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$

$$\le[(a+\eta(b,a))^p-a^p]\Big[(S_1+S_3)|f'(a)|+(S_2+S_4)|f'(b)|\Big], \tag{2.3}$$

where

$$S_1=\int_0^{\frac{1}{2}}t^2[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}dt$$

$$S_2=\int_0^{\frac{1}{2}}t(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}dt$$

$$S_3=\int_{\frac{1}{2}}^{1}t(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}dt$$

$$S_4=\int_{\frac{1}{2}}^{1}(1-t)(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}dt$$

Proof. Using Lemma 2.7 and $|f'|$ is p-preinvex on S, we have

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq [(a+\eta(b,a))^p-a^p]\Bigg[\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt$$
$$+\int_{\frac{1}{2}}^1(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt\Bigg]$$
$$\leq [(a+\eta(b,a))^p-a^p]\Bigg[\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt$$
$$+\int_{\frac{1}{2}}^1(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\Big(t|f'(a)|+(1-t)|f'(b)|\Big)dt\Bigg]$$
$$=[(a+\eta(b,a))^p-a^p]\Big[(S_1+S_3)|f'(a)|+(S_2+S_4)|f'(b)|\Big].$$

This completes the proof. □

Theorem 2.9. *Let $S\subseteq\mathbb{R}$ be an open invex set with respect to $\eta:S\times S\to\mathbb{R}$ and $p\in\mathbb{R}/\{0\}$. Suppose that $f:S\to\mathbb{R}$ is a differentiable function.Assume that $q>1$ and if $|f'|^{\frac{q}{q-1}}$ is p-preinvex on S, then for every $a,b\in S$, the following inequality holds*

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq\frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\left(S_5|f(a)|^{\frac{q}{q-1}}+S_6|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}+\left(S_7|f(a)|^{\frac{q}{q-1}}+S_8|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\right],$$

where

$$S_5=\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$
$$S_6=\int_0^{\frac{1}{2}}(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$
$$S_7=\int_{\frac{1}{2}}^1 t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$
$$S_8=\int_{\frac{1}{2}}^1(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}dt$$

Proof. Using Lemma 2.7, Holder's inequality and p-preinvexity of $|f'|^{\frac{q}{q-1}}$ on S, we have

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq [(a+\eta(b,a))^p-a^p]\Bigg[\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt$$
$$+\int_{\frac{1}{2}}^1(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt\Bigg]$$
$$\leq[(a+\eta(b,a))^p-a^p]\Bigg[\left(\int_0^{\frac{1}{2}}t^q dt\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^q dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\Bigg]$$

$$\leq [(a+\eta(b,a))^p-a^p]\Bigg[\left(\int_0^{\frac{1}{2}}t^q dt\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}(t|f(a)|^{\frac{q}{q-1}}+(1-t)|f(b)|^{\frac{q}{q-1}})dt\right)^{\frac{q-1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^q dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p(q-1)}}(t|f(a)|^{\frac{q}{q-1}}+(1-t)|f(b)|^{\frac{q}{q-1}})dt\right)^{\frac{q-1}{q}}\Bigg]$$

$$=\frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\left[\left(S_5|f(a)|^{\frac{q}{q-1}}+S_6|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}+\left(S_7|f(a)|^{\frac{q}{q-1}}+S_8|f(b)|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\right]$$

The proof is completed. □

Theorem 2.10. *Let $S\subseteq\mathbb{R}$ be an open invex set with respect to $\eta: S\times S\to\mathbb{R}$ and $p\in\mathbb{R}/\{0\}$. Suppose that $f: S\to\mathbb{R}$ is a differentiable function.Assume that $q\geq 1$ such that $\frac{1}{q}+\frac{1}{r}=1$ and if $|f'|^q$ is p-preinvex on S, then for every $a,b\in S$, the following inequality holds*

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq\frac{[(a+\eta(b,a))^p-a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}}\left[\left(S_9|f(a)|^q+S_{10}|f(b)|^q\right)^{\frac{1}{q}}+\left(S_{11}|f(a)|^q+S_{12}|f(b)|^q\right)^{\frac{1}{q}}\right],$$

where

$$S_9=\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}dt$$
$$S_{10}=\int_0^{\frac{1}{2}}(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}dt$$
$$S_{11}=\int_{\frac{1}{2}}^{1}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}dt$$
$$S_{12}=\int_{\frac{1}{2}}^{1}(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}dt$$

Proof. Using Lemma 2.7, Holder's inequality and p-preinvexity of $|f'|^q$ on S, we have

$$\left|f\left([(a+\eta(b,a))^p-a^p]^{\frac{1}{p}}\right)-\frac{p}{[(a+\eta(b,a))^p-a^p]}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{1-p}}dx\right|$$
$$\leq[(a+\eta(b,a))^p-a^p]\Bigg[\int_0^{\frac{1}{2}}t[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt$$
$$+\int_{\frac{1}{2}}^{1}(1-t)[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1-p}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|dt\Bigg]$$

$$\leq[(a+\eta(b,a))^p-a^p]\Bigg[\left(\int_0^{\frac{1}{2}}t^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}[(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}\left|f'\left([(1-t)a^p+t(a+\eta(b,a))^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq [(a+\eta(b,a))^p - a^p]\Bigg[\left(\int_0^{\frac{1}{2}} t^r dt\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}} [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}(t|f(a)|^q + (1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1 (1-t)^r dt\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1 [(1-t)a^p + t(a+\eta(b,a))^p]^{\frac{q(1-p)}{p}}(t|f(a)|^q + (1-t)|f(b)|^q)dt\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{[(a+\eta(b,a))^p - a^p]}{2^{1+\frac{1}{r}}(r+1)^{\frac{1}{r}}}\left[\left(S_9|f(a)|^q + S_{10}|f(b)|^q\right)^{\frac{1}{q}} + \left(S_{11}|f(a)|^q + S_{12}|f(b)|^q\right)^{\frac{1}{q}}\right]$$

The proof is completed. □

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Imran Abbas Baloch, Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
E-mail address: iabbasbaloch@gmail.com, iabbasbaloch@sms.edu.pk

İmdat Iscan, Department of Mathematics, Faculty of Arts and Sciences,, Giresun University, 28200, Giresun, TURKEY
E-mail address: imdat.iscan@giresun.edu.tr