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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY $(p,(s,m))$-CONVEX FUNCTIONS

IMRAN ABBAS BALOCH, İMDAT İSCAN

Abstract. In this article, we define the class of harmonically p-convex functions which is generalization of convex and harmonically convex functions and Harmonically p-quasiconvex functions. We also define the class of harmonically $(p,(s,m))$-convex functions which is generalization of harmonically (s,m)-convex functions. Furthermore, we establish Hermite-Hadamard inequalities for these classes of functions.

1. **Introduction**

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex function. Note that some of the classical inequalities for mean can be derived from (1.1) for the appropriate particular selection of mapping f.Both inequalities in (1.1) hold in the reverse direction if f is concave.
In [5], Imdat Iscan introduced the concept of harmonically convex function, and established a variant od Hermite-Hadamard type inequalities which holds for these classes of functions as follows:

Definition 1.1. Let $I \subset \mathbb{R}/\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If inequality in (1.2) is reversed, then f is said to be harmonically concave.

Theorem 1.2. *(see [5]) Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically convex and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.3}$$

Theorem 1.3. *(see [5]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I°,$a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left(\lambda_2|f'(a)|^q + \lambda_3|f'(b)|^q\right)^{\frac{1}{q}}, \tag{1.4}$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$

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$$\lambda_3 = \frac{1}{a(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right).$$

Theorem 1.4. *(see [5]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1|f'(a)|^q + \mu_2|f'(b)|^q\right)^{\frac{1}{q}}, \tag{1.5}$$

where

$$\mu_1 = \frac{[a^{2-2q} + b^{1-2q}[(b-a)(1-2q)-a]]}{2(b-a)^2(1-q)(1-2q)},$$

$$\mu_2 = \frac{[b^{2-2q} - a^{1-2q}[(b-a)(1-2q)+b]]}{2(b-a)^2(1-q)(1-2q)}.$$

In [6], Imdat Iscan introduced the concept of harmonically s-convex function in second sense as follow:

Definition 1.5. A function $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically s-convex in second sense , if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le t^s f(y) + (1-t)^s f(x) \tag{1.6}$$

for all $x,y \in I$ and $t \in [0,1]$. If inequality in (1.6) is reversed, then f is said to be harmonically s-concave.

Remark 1.6. Note that for $s = 1$, harmonically s-convexity reduces to ordinary harmonically convexity.

In [4], Feixiang Chen and Shanhe Wu generalized Hermite-Hadamard type inequalities given in [5] which hold for harmonically s-convex functions in second sense.

Theorem 1.7. *(see [4]). Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically s- convex in second sense on $[a,b]$ for some fixed $s \in (0,1]$, $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}C_1^{1-\frac{1}{q}}(a,b)[C_2(s;a,b)|f'(a)|^q + C_3(s;a,b)|f'(b)|^q]^{\frac{1}{q}},$$

where

$$C_1(a,b) = b^{-2}\left({}_2F_1(2,2;3,1-\frac{a}{b}) - {}_2F_1(2,1;2,1-\frac{a}{b}) + \frac{1}{2}{}_2F_1(2,1;3,\frac{1}{2}(1-\frac{a}{b})\right)$$

$$C_2(s,a,b) = b^{-2}\left(\frac{2}{s+2}{}_2F_1(2,s+2;s+3,1-\frac{a}{b}) - \frac{1}{s+1}{}_2F_1(2,s+1;s+2,1-\frac{a}{b}) + \frac{1}{2^s(s+1)(s+2)}{}_2F_1(2,s+1;s+3,\frac{1}{2}(1-\frac{a}{b}))\right)$$

$$C_3(s,a,b) = b^{-2}\left(\frac{2}{(s+1)(s+2)}{}_2F_1(2,2;s+3,1-\frac{a}{b}) - \frac{1}{s+1}{}_2F_1(2,1;s+2,1-\frac{a}{b}) + \frac{1}{2}{}_2F_1(2,1;3,\frac{1}{2}(1-\frac{a}{b}))\right)$$

Remark 1.8. Note that for $s = 1$, $C_1(a,b) = \lambda_1$, $C_2(1,a,b) = \lambda_2$ and $C_3(1,a,b) = \lambda_3$. Hence, Theorem 1.3 is particular case of theorem 1.7 for $s = 1$.

Theorem 1.9. *(see [4]) Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is harmonically s- convex in second sense on $[a,b]$ for some fixed $s \in (0,1]$, $q > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$

$$\le \frac{a(b-a)}{2b}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{s+1}[{}_2F_1(2q,s+1;s+2,1-\frac{a}{b})|f'(b)|^q + {}_2F_1(2q,1;s+2,1-\frac{a}{b})|f'(a)|^q]\right]^{\frac{1}{q}}$$

Remark 1.10. Note that for $s=1$

$$\mu_1=\frac{1}{2b^{2q}}\cdot {}_2F_1(2q,2,3,1-\frac{a}{b}),$$

and

$$\mu_2=\frac{1}{2b^{2q}}\cdot {}_2F_1(2q,1,3,1-\frac{a}{b}).$$

Hence, Theorem 1.4 is particular case of theorem 1.9 for $s=1$.

In ([8]), Jaekeun Park considered the class of (s,m)-convex functions in second sense. This class of function is defined as follow

Definition 1.11. For some fixed $s\in(0,1]$ and $m\in[0,1]$ a mapping $f:I\subset[0,\infty)\to\mathbb{R}$ is said to be (s,m)-convex in the second sense on I if

$$f(tx+m(1-t)y)\le t^s f(x)+m(1-t)^s f(y)$$

holds, for all $x,y\in I$ and $t\in[0,1]$

In ([7]), Imdat Iscan introduced the concept of harmonically (α,m)-convex functions and established some Hermite-Hadamard type inequalities for this class of function. This class of functions is defined as follow

Definition 1.12. The function $f:(0,\infty)\to\mathbb{R}$ is said to be harmonically (α,m)-convex, where $\alpha\in[0,1]$ and $m\in(0,1]$, if

$$(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y) \tag{1.7}$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$. If the inequality in (1.7) is reversed, then f is said to be harmonically (α,m)-concave.

In [1] , authors introduce the concept of Harmonically (s,m)-convex functions in second sense which generalize the notion of Harmonically convex and Harmonically s-convex functions in second sense introduced by Imdat Iscan in [5],[6].
In [2], we established some results connected with the right side of new inequality similar to (1.1) for this class of functions such that results given by Imdat Iscan [5] , Feixiang Chen and Shanhe Wu [4] are obtained for the particular values of s,m as follows:

Definition 1.13. The function $f:I\subset(0,\infty)\to\mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s\in(0,1]$ and $m\in(0,1]$ if

$$f(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\le t^s f(x)+m(1-t)^s f(y)$$

$\forall x,y\in I$ and $t\in[0,1]$.

Remark 1.14. Note that for $s=1$, (s,m)-convexity reduces to harmonically m-convexity and for $m=1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense (see [6]) and for $s,m=1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity (see [5]).

The following result of the Hermite-Hadamard type holds.

Theorem 1.15. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a harmonically (s,m)-convex function in second sense with $s\in[0,1]$ and $m\in(0,1]$. If $0<a<b<\infty$ and $f\in L[a,b]$, then one has following inequality*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\le\min\left[\frac{f(a)+mf(\frac{b}{m})}{s+1},\frac{f(b)+mf(\frac{a}{m})}{s+1}\right]$$

Corollary 1.16. *If we take $m=1$ in theorem 1.15, then we get*

$$\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\le\frac{f(a)+f(b)}{s+1}$$

Corollary 1.17. *If we take $s=1$ in theorem 1.15, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \min\left[\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf(\frac{a}{m})}{2}\right]$$

Theorem 1.18. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, \frac{b}{m} \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a, \frac{b}{m}]$ for $q \ge 1$ with $s \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2^{2-\frac{1}{q}}}\left[\rho_1(s,q;a,b)|f'(a)|^q + m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

where

$$\begin{aligned}\rho_1(s,q;a,b) &= \frac{\beta(1,s+2)}{b^{2q}}.{}_2F_1\big(2q,1;s+3;1-\frac{a}{b}\big) - \frac{\beta(2,s+1)}{b^{2q}}.{}_2F_1\big(2q,2;s+3;1-\frac{a}{b}\big)\\ &+ \frac{2^{2q-s}\beta(2,s+1)}{(a+b)^{2q}}.{}_2F_1\big(2q,2;s+3;1-\frac{2a}{a+b}\big)\end{aligned}$$

$$\begin{aligned}\rho_2(s,q;a,b) &= \frac{\beta(s+1,2)}{2^s b^{2q}}.{}_2F_1\big(2q,s+1;s+3,\frac{1}{2}(1-\frac{a}{b})\big) - \frac{\beta(s+1,2)}{b^{2q}}.{}_2F_1\big(2q,s+1;s+3,1-\frac{a}{b}\big)\\ &+ \frac{\beta(s+2,1)}{b^{2q}}.{}_2F_1\big(2q,s+2;s+3,1-\frac{a}{b}\big)\end{aligned}$$

β is Euler Beta function defined by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0$$

and${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b;c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1$$

Theorem 1.19. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I, $ma, b \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a, \frac{b}{m}]$ for $q \ge 1$ with $s \in [0,1]$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\ &\le \frac{ab(b-a)}{2}\rho_1^{1-\frac{1}{q}}(0,q;a,b)\left[\rho_1(s,q;a,b)|f'(a)|^q + m\rho_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}\end{aligned}$$

Theorem 1.20. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $ma, b \in I^\circ$ with $a < b$, $m \in (0,1]$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically (s,m)-convex in second sense on $[a, \frac{b}{m}]$ for $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$ with $s \in [0,1]$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\ &\le \frac{ab(b-a)}{2}\big(\frac{1}{p+1}\big)^{\frac{1}{p}}\left[\nu_1(s,q;a,b)|f'(a)|^q + m\nu_2(s,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}\end{aligned}$$

where

$$\nu_1(s,q;a,b) = \frac{\beta(1,s+1)}{b^{2q}}.{}_2F_1(2q,1;s+2,1-\frac{a}{b})$$

and

$$\nu_2(s,q;a,b) = \frac{\beta(s+1,1)}{b^{2q}}.{}_2F_1(2q,s+1;s+2,1-\frac{a}{b})$$

2. Main Results

Now, we define the class of the harmonically p-convex functions which is a generalization of convex functions and harmonically convex functions:

Definition 2.1. The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically p-convex function, where $p \in \mathbb{R}/\{0\}$, if

$$f\left(\frac{xy}{[ty^p + (1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p} + \frac{1-t}{y^p}\right)^{\frac{-1}{p}}\right) \le tf(x) + (1-t)f(y), \tag{2.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2.3) is reversed, then f is said to be harmonically p-concave.

Remark 2.2. Note that for $p = -1$, harmonic p-convexity reduce to convexity and for $p = 1$ harmonic p-convexity reduce to harmonic convexity.

Theorem 2.3. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically p-convex and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then following inequalities hold*

$$f\left(\frac{2ab}{[a^p + b^p]^{\frac{1}{p}}}\right) \le \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx \le \frac{f(a)+f(b)}{2}. \tag{2.2}$$

Definition 2.4. The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically p-quasiconvex function, where $p \in \mathbb{R}/\{0\}$, if

$$f\left(\frac{xy}{[ty^p + (1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p} + \frac{1-t}{y^p}\right)^{\frac{-1}{p}}\right) \le \max\{f(x), f(y)\}, \tag{2.3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2.3) is reversed, then f is said to be harmonically p-quasiconcave.

Now, we define the class of the harmonically $(p,(s,m))$-convex functions:

Definition 2.5. The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically $(p,(s,m))$-convex function, where $p \in \mathbb{R}/\{0\}$, $s, m \in (0,1]$, if

$$f\left(\frac{mxy}{[t(my)^p + (1-t)x^p]^{\frac{1}{p}}}\right) = f\left(\left(\frac{t}{x^p} + \frac{1-t}{(my)^p}\right)^{\frac{-1}{p}}\right) \le t^s f(x) + m(1-t)^s f(y), \tag{2.4}$$

for all $x, y \in I$ with $my \in I$ and $t \in [0,1]$. If the inequality in (2.4) is reversed, then f is said to be harmonically $(p,(s,m))$-concave.

Remark 2.6. Note that for $p = 1$ harmonic $(p,(s,m))$-convexity reduce to harmonic (s,m)-convexity.

Theorem 2.7. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonically $(p,(s,m))$-convex and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then following inequality hold*

$$\frac{p(mab)^p}{(mb)^p - a^p}\int_a^{mb} \frac{f(x)}{x^{p+1}}dx \le \frac{f(a)+mf(b)}{s+1}. \tag{2.5}$$

For finding some new more general inequalities of Hermite-Hadamard type for the functions whose derivatives are harmonically (s,m)-convex in second sense, we need the following lemma:

Lemma 2.8. *Let $f : I \subset \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on I° with $a < b$. If $f \in L[a,b]$ and $p \in \mathbb{R}/\{0\}$, then*

$$\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx = \frac{ab(b^p - a^p)}{2p}\int_0^1 \frac{1-2t}{[tb^p + (1-t)a^p]^{\frac{p+1}{p}}} f'\left(\frac{ab}{[tb^p + (1-t)a^p]^{\frac{1}{p}}}\right)dt$$

Proof. Let

$$I=\frac{ab(b^p-a^p)}{2p}\int_0^1\frac{1-2t}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt$$

Integrating by parts, we have

$$I=\frac{1}{2}\left|(2t-1)f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|_0^1-\int_0^1 f\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)dt$$

Setting $x=\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}$, we have $dt=\frac{p(ab)^p}{a^p-b^p}\frac{dx}{x^{p+1}}$, hence we obtain

$$I=\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx$$

this completes the proof. □

Theorem 2.9. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,\frac{b}{m}\in I^\circ$ with $a<b$, $m\in(0,1]$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$
$$\leq\frac{ab(b^p-a^p)}{2p}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\sigma_1(s,p,q;a,b)|f'(a)|^q+m\sigma_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

where

$$\sigma_1(s,p,q;a,b)=\begin{cases}\frac{1}{b^{(p+1)q}}\beta(1,s+2).{}_2F_1(\frac{(p+1)q}{p},1;s+3;1-(\frac{a}{b})^p)\\ -\frac{1}{b^{(p+1)q}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;1-(\frac{a}{b})^p)\\ +\frac{1}{2^s}\left(\frac{2}{a^p+b^p}\right)^{\frac{(p+1)q}{p}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;\frac{b^p-a^p}{b^p+a^p}), & p>0\\ \\ \frac{1}{a^{(p+1)q}}\beta(s+2,1).{}_2F_1(\frac{(p+1)q}{p},s+2;s+3;1-(\frac{b}{a})^p)\\ -\frac{1}{a^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;1-(\frac{b}{a})^p)\\ +\frac{1}{2^s a^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;\frac{a^p-b^p}{2a^p}), & p<0\end{cases}$$

and

$$\sigma_2(s,p,q;a,b)=\begin{cases}\frac{1}{b^{(p+1)q}}\beta(s+2,1).{}_2F_1(\frac{(p+1)q}{p},s+2;s+3;1-(\frac{a}{b})^p)\\ -\frac{1}{b^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;1-(\frac{a}{b})^p)\\ +\frac{1}{2^s b^{(p+1)q}}\beta(s+1,2).{}_2F_1(\frac{(p+1)q}{p},s+1;s+3;\frac{b^p-a^p}{2b^p}), & p>0\\ \\ \frac{1}{a^{(p+1)q}}\beta(1,s+2).{}_2F_1(\frac{(p+1)q}{p},1;s+3;1-(\frac{b}{a})^p)\\ -\frac{1}{a^{(p+1)q}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;1-(\frac{b}{a})^p)\\ +\frac{1}{2^s}\left(\frac{2}{a^p+b^p}\right)^{\frac{(p+1)q}{p}}\beta(2,s+1).{}_2F_1(\frac{(p+1)q}{p},2;s+3;\frac{a^p-b^p}{a^p+b^p}), & p<0\end{cases}$$

Proof. From above Lemma and using power mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$
$$\leq\frac{ab(b^p-a^p)}{2p}\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt$$
$$\leq\frac{ab(b^p-a^p)}{2p}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}$$

$$\times\bigg(\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}\bigg|f'\bigg(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\bigg)\bigg|^q dt\bigg)^{\frac{1}{q}}$$

Since, $|f'|^q$ is harmonically $(p,(s,m))$-convex function in second sense, we have

$$\leq \frac{ab(b^p-a^p)}{2p}\bigg(\int_0^1 |1-2t|dt\bigg)^{1-\frac{1}{q}}$$

$$\times\bigg(\int_0^1 \frac{|1-2t|[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}dt\bigg)^{\frac{1}{q}}$$

$$=\frac{ab(b^p-a^p)}{2p}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\bigg[\sigma_1(s,p,q;a,b)|f'(a)|^q+m\sigma_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\bigg]^{\frac{1}{q}}$$

□

Corollary 2.10. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(1,1))$-convex in second sense on $[a,b]$ for $q\geq 1$, then*

$$\bigg|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\bigg|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\bigg[\sigma_1(1,p,q;a,b)|f'(a)|^q+\sigma_2(1,p,q;a,b)|f'(b)|^q\bigg]^{\frac{1}{q}}$$

Corollary 2.11. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,\frac{b}{m}\in I^\circ$ with $a<b$, $m\in(0,1]$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(1,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\bigg|\frac{f(a)+f(b)}{2}-\frac{(ab)}{b-a}\int_a^b\frac{f(x)}{x^2}dx\bigg|$$

$$\leq \frac{ab(b-a)}{2p}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\bigg[\sigma_1(s,1,q;a,b)|f'(a)|^q+m\sigma_2(s,1,q;a,b)|f'(\frac{b}{m})|^q\bigg]^{\frac{1}{q}}$$

Corollary 2.12. *If we take $m=1$ in Theorem 2.9, then we get*

$$\bigg|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\bigg|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\bigg[\sigma_1(s,p,q;a,b)|f'(a)|^q+\sigma_2(s,p,q;a,b)|f'(b)|^q\bigg]^{\frac{1}{q}}$$

Corollary 2.13. *If we take $s=1$ in Theorem 2.9, then we get*

$$\bigg|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\bigg|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\bigg[\sigma_1(1,p,q;a,b)|f'(a)|^q+m\sigma_2(1,p,q;a,b)|f'(\frac{b}{m})|^q\bigg]^{\frac{1}{q}}$$

Theorem 2.14. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I, $a,\frac{b}{m}\in I^\circ$ with $a<b$, $m\in(0,1]$, $f'\in L[a,b]$ and $p\in\mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a,\frac{b}{m}]$ for $q\geq 1$ with $s\in[0,1]$, then*

$$\bigg|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\bigg|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\bigg[\sigma_1(s,p,1;a,b)|f'(a)|^q+m\sigma_2(s,p,1;a,b)|f'(\frac{b}{m})|^q\bigg]^{\frac{1}{q}}$$

Proof. From Lemma, Power mean inequality and harmonically $(p,(s,m))$-convexity in second sense of$|f'|^q$ on $[a,\frac{b}{m}]$,we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt$$

$$\leq\frac{ab(b^p-a^p)}{2p}\left(\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{ab(b^p-a^p)}{2p}\left(\int_0^1\frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\frac{|1-2t|[t^s|f'(a)|^q+m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}dt\right)^{\frac{1}{q}}$$

$$=\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(s,p,1;a,b)|f'(a)|^q+m\sigma_2(s,p,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

□

Corollary 2.15. *If we take $s=m=1$ in Theorem 2.14, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(1,p,1;a,b)|f'(a)|^q+\sigma_2(1,p,1;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.16. *If we take $s=1$ in Theorem 2.14, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(1,p,1;a,b)|f'(a)|^q+m\sigma_2(1,p,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Corollary 2.17. *If we take $m=1$ in Theorem 2.14, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq\frac{ab(b^p-a^p)}{2p}\sigma_1^{1-\frac{1}{q}}(1,p,1;a,b)\left[\sigma_1(s,p,1;a,b)|f'(a)|^q+\sigma_2(s,p,1;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.18. *If we take $p=1$ in Theorem 2.14, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$\leq\frac{ab(b-a)}{2}\sigma_1^{1-\frac{1}{q}}(1,1,1;a,b)\left[\sigma_1(s,1,1;a,b)|f'(a)|^q+m\sigma_2(s,1,1;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Theorem 2.19. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, \frac{b}{m} \in I^\circ$ with $a < b$, $m \in (0,1]$, $f' \in L[a,b]$ and $p \in \mathbb{R}/\{0\}$. If $|f'|^q$ is harmonically $(p,(s,m))$-convex in second sense on $[a, \frac{b}{m}]$ for $q > 1$, $\frac{1}{r} + \frac{1}{q}$ with $s \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$= \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q + m\varpi_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

where

$$\varpi_1(s,p,q;a,b) = \begin{cases} \frac{1}{b^{(p+1)q}}\beta(1,s+1).{}_2F_1(\frac{(p+1)q}{p},1;s+2;1-(\frac{a}{b})^p), & p > 0 \\ \frac{1}{a^{(p+1)q}}\beta(s+1,1).{}_2F_1(\frac{(p+1)q}{p},s+1;s+2;1-(\frac{b}{a})^p), & p < 0 \end{cases}$$

and

$$\varpi_2(s,p,q;a,b) = \begin{cases} \frac{1}{b^{(p+1)q}}\beta(s+1,1).{}_2F_1(\frac{(p+1)q}{p},s+1;s+2;1-(\frac{a}{b})^p), & p > 0 \\ \frac{1}{a^{(p+1)q}}\beta(1,s+1).{}_2F_1(\frac{(p+1)q}{p},1;s+2;1-(\frac{b}{a})^p), & p < 0 \end{cases}.$$

Proof. From Lemma, Power mean inequality and harmonically $(p,(s,m))$-convexity in second sense of$|f'|^q$ on $[a, \frac{b}{m}]$,we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$\leq \frac{ab(b^p-a^p)}{2p}\int_0^1 \frac{|1-2t|}{[tb^p+(1-t)a^p]^{\frac{p+1}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|dt$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \frac{1}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}\left|f'\left(\frac{ab}{[tb^p+(1-t)a^p]^{\frac{1}{p}}}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b^p-a^p)}{2p}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \frac{[t^s|f'(a)|^q + m(1-t)^s|f'(\frac{b}{m})|^q]}{[tb^p+(1-t)a^p]^{\frac{(p+1)q}{p}}}dt\right)^{\frac{1}{q}}$$

$$= \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q + m\varpi_2(s,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

□

Corollary 2.20. *If we take $s = m = 1$ in Theorem 2.19, we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p(ab)^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{p+1}}dx\right|$$

$$= \frac{ab(b^p-a^p)}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\varpi_1(1,p,q;a,b)|f'(a)|^q + \varpi_2(1,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.21. *If we take $s=1$ in Theorem 2.19, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$=\frac{ab(b^p-a^p)}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\varpi_1(1,p,q;a,b)|f'(a)|^q+m\varpi_2(1,p,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

Corollary 2.22. *If we take $m=1$ in Theorem 2.19, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p(ab)^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{p+1}}dx\right|$$

$$=\frac{ab(b^p-a^p)}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\varpi_1(s,p,q;a,b)|f'(a)|^q+\varpi_2(s,p,q;a,b)|f'(b)|^q\right]^{\frac{1}{q}}$$

Corollary 2.23. *If we take $p=1$ in Theorem 2.19, we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(ab)}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$=\frac{ab(b-a)}{2}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[\varpi_1(s,1,q;a,b)|f'(a)|^q+m\varpi_2(s,1,q;a,b)|f'(\frac{b}{m})|^q\right]^{\frac{1}{q}}$$

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Imran Abbas Baloch, Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
E-mail address: iabbasbaloch@gmail.com, iabbasbaloch@sms.edu.pk

İmdat Iscan, Department of Mathematics, Faculty of Arts and Sciences,, Giresun University, 28200, Giresun, TURKEY
E-mail address: imdat.iscan@giresun.edu.tr