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HERMITE-HADMARD TYPE INTEGRAL INEQUALITIES FOR $(p,(s,m))$-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we define a new generalized class of p-convex functions which includes (s,m)-convex functions and harmonically (s,m)-convex functions as a special cases and establish some new Hermite-Hadamard type inequalities for (p,(s,m))-convex functions.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality for convex functions.It is important to note that some of the classical inequalities for means can be obtained from (1.1) with the appropriate selection of the function f. Both inequalities in (1.1) hold in reverse direction if f is concave. For the some results which generalize, improve and extend the inequalities (1.1), we refer the reader to the recent paper (see [3,5,7,8,10,11,15]).
In [5], Dragomir gave the following Lemma:

Lemma 1.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

By using Lemma 1.1, Dragomir obtained the following Hermite-Hadamard type inequalities for the convex functions.

Theorem 1.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$ is convex on $[a, b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|}{8}. \tag{1.3}$$

Theorem 1.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and $p > 1$. If $|f'|^q \in L[a, b]$ is convex on $[a, b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\frac{(|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}, \tag{1.4}$$

where, $\frac{1}{p}+\frac{1}{q}=1$.

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Let $A(a,b;t)=ta+(1-t)b$, $G(a,b;t)=a^tb^{1-t}$, $H(a,b;t)=\frac{ab}{ta+(1-t)b}$ and $M_p(a,b,;t)=(ta^p+(1-t)b^p)^{\frac{1}{p}}$, $p\in\mathbb{R}/\{0\}$, be the weighted arithmetic, geometric, harmonic, power of order p means of two positive real numbers a and b with $a\neq b$ for $t\in[0,1]$ respectively. $M_p(a,b,;t)$ is continuous and strictly increasing with respect to $t\in\mathbb{R}$ for the fixed $p\in\mathbb{R}/\{0\}$ and $a,b>0$ with $a\neq b$. In [20], Niculescu gave the definitions of convexities according to geometric mean as GA-convex, GG-convex and AG-convex. In [9], $\dot{I}$ şcan gave the definition of Harmonically convex and concave functions as follow

Definition 1.4. [9] A function $f:I\subset\mathbb{R}/\{0\}\to\mathbb{R}$ is said to be harmonically convex function if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq(1-t)f(x)+tf(y),\quad for\ all\ x,y\in K_h,\ t\in[0,1]. \tag{1.5}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.5) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

Theorem 1.5. [9] *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a harmonically convex mapping and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2} \tag{1.6}$$

the above inequalities are sharp.

Lemma 1.6. [9] *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $f'\in L[a,b]$ is convex on $[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx=\frac{ab(b-a)}{2}\int_0^1\frac{1-2t}{(tb+(1-t)a)^2}f'\left(\frac{1-2t}{tb+(1-t)a}\right)dt. \tag{1.7}$$

Using Lemma 1.6, the following inequalities hold.

Theorem 1.7. [9] *Let $f:I^\circ\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'|^q\in L[a,b]$ is harmonically convex on $[a,b]$ for $q\geq1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}[\lambda_2|f'(a)|^q+\lambda_3|f'(b)|^q]^{\frac{1}{q}}, \tag{1.8}$$

where

$$\lambda_1=\frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2=\frac{-1}{b(b-a)}+\frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_3=\frac{1}{a(b-a)}-\frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)=\lambda_1-\lambda_2$$

Theorem 1.8. [9] *Let $f:I^\circ\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$. If $|f'|^q\in L[a,b]$ is harmonically convex on $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}[\mu_1|f'(a)|^q+\mu_2|f'(b)|^q]^{\frac{1}{q}}, \tag{1.9}$$

where

$$\mu_1=\frac{[a^{2-2q}+b^{1-2q}[(b-a)(1-2q)-a]]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2=\frac{[b^{2-2q}-a^{1-2q}[(b-a)(1-2q)+b]]}{2(b-a)^2(1-q)(1-2q)}.$$

Definition 1.9. [13] The function $f:(0,\infty)\to\mathbb{R}$ is said to be harmonically (α,m)-convex, where $\alpha\in[0,1]$ and $m\in(0,1]$, if

$$(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y) \tag{1.10}$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$. If the inequality in (1.10) is reversed, then f is said to be harmonically (α,m)-concave.

Definition 1.10. [1] The function $f:I\subset(0,\infty)\to\mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s\in(0,1]$ and $m\in(0,1]$ if

$$f(\frac{mxy}{mty+(1-t)x})=f\big((\frac{t}{x}+\frac{1-t}{my})^{-1}\big)\leq t^{s}f(x)+m(1-t)^{s}f(y)$$

$\forall x,y\in I$ and $t\in[0,1]$.

Remark 1.11. Note that for $s=1$, (s,m)-convexity reduces to harmonically m-convexity and for $m=1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense and for $s,m=1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity .

I. A. Baloch et al. established some Hermite-Hadamard type inequalities for Harmonically (s,m)-convex functions in second sense in [2].
In [21], Zhang and Wan gave definition of p-convex functions and we will consider the definition of p-convexity in the following reference:
İ. İşcan. Ostrowski type inequalities for p-convex functions, Researchgate doi: 10.13140/RG.2.1.1028.5209. Available online at https://www.researchgate.net/publication/299593487.

Definition 1.12. Let I be a p-convex set. A function $f:I\to\mathbb{R}$ is said to be a p-convex function or belong to the class $PC(I)$, if

$$f\left([\alpha x^{p}+(1-\alpha)y^{p}]^{\frac{1}{p}}\right)\leq\alpha f(x)+(1-\alpha)f(y) \tag{1.11}$$

for all $x,y\in I$, $\alpha\in[0,1]$ and $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2t+1$ and $k,r,t\in N$.

Example 1.13. Let $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^{p}$, $p\neq0$ and $g:(0,\infty)\to\mathbb{R}$, $g(x)=c$, $c\in\mathbb{R}$, then f and g are both p-convex and p-concave functions.

In [21], if we take $I\subset(0,\infty)$, $p\neq0$ and $h(t)=t$, then we have following theorem

Theorem 1.14. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}/\{0\}$ and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then we have*

$$f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}. \tag{1.12}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see [12,6,17,18,19,21].
In [17], if we take $I\subset(0,\infty)$ and $p\in\mathbb{R}/\{0\}$, then we have following lemma

Lemma 1.15. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $a,b\in I^{\circ}$ with $a<b$ and $p\in\mathbb{R}/\{0\}$. If $f'\in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx=\frac{b^{p}-a^{p}}{2p}\int_{0}^{1}\frac{1-2t}{[ta^{p}+(1-t)b^{p}]^{1-\frac{1}{p}}}f'([ta^{p}+(1-t)b^{p}]^{1-\frac{1}{p}})dt \tag{1.13}$$

Remark 1.16. In Lemma 1.15,
(i) If we take $p=1$, then we have equality (1.2) in Lemma 1.1.
(ii)If we take $p=-1$, then we have equality (1.7) in Lemma 1.6.

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are p-convex, İşcan [14] used Beta function, Gamma function and integral form of the hypergeometric function which are defined as follows

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\ \alpha,\beta>0$$

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha>0$$

and

$${}_2F_1(\alpha,\beta;\gamma,z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,\ \gamma>\beta>0,\ |z|<1$$

Theorem 1.17. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\frac{b^p-a^p}{2p}C_1^{1-\frac{1}{q}}\left[C_2|f'(a)|^q+C_3|f'(b)|^q\right]^{\frac{1}{q}},\tag{1.14}$$

where

$$\begin{aligned}C_1=C_1(a,b;p)\ &=\ \frac{1}{4}\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}-1}\\&\quad\times\left[{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)+{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)\right],\end{aligned}$$

$$\begin{aligned}C_2=C_2(a,b;p)\ &=\ \frac{1}{24}\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}-1}\\&\quad\times\left[{}_2F_1\left(1-\frac{1}{p},2;4;\frac{a^p-b^p}{a^p+b^p}\right)+6{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)\right.\\&\quad\left.+{}_2F_1\left(1-\frac{1}{p},2;4;\frac{b^p-a^p}{a^p+b^p}\right)\right],\end{aligned}$$

$$C_3=C_3(a,b;p)=C_1-C_2$$

Theorem 1.18. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\frac{b^p-a^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\Bigg)\left[C_4|f'(a)|^q+C_5|f'(b)|^q\right]^{\frac{1}{q}},\tag{1.15}$$

where

$$C_4=C_4(a,b;p;q)=\begin{cases}\frac{1}{2a^{qp-q}}\cdot{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{b}{a})^p), & p<0\\ \frac{1}{2b^{qp-q}}\cdot{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{a}{b})^p), & p>0\end{cases}$$

$$C_5=C_5(a,b;p;q)=\begin{cases}\frac{1}{2a^{qp-q}}\cdot{}_2F_1(q-\frac{q}{p},2;3;1-(\frac{b}{a})^p), & p<0\\ \frac{1}{2b^{qp-q}}\cdot{}_2F_1(q-\frac{q}{p},1;3;1-(\frac{a}{b})^p), & p>0\end{cases}$$

Theorem 1.19. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\frac{b^p-a^p}{2p}C_6^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}},\tag{1.16}$$

where

$$C_6 = C_6(a,b;p;q) = \begin{cases} \frac{1}{a^{rp-r}} \cdot {}_2F_1(r-\frac{r}{p},1;2;1-(\frac{b}{a})^p), & p<0 \\ \frac{1}{b^{rp-r}} \cdot {}_2F_1(r-\frac{r}{p},1;2;1-(\frac{a}{b})^p), & p>0 \end{cases}$$

2. **Main Results**

Now, we define the class of $(p,(s,m))$-convex functions which unifies different type of convexities and using Lemma 1.15, we introduce more general Hermite-Hadamard type inequalities for this class of functions.

Definition 2.1. The function $f:(0,b^*]\to\mathbb{R}$, $b^*>0$ is said to be $(p,(s,m))$-convex function in first sense, where $s\in(0,1]$ and $m\in(0,1]$, if

$$f\left([tx^p+(1-t)(my)^p]^{\frac{1}{p}}\right)\leq t^s f(x)+m(1-t^s)f(y) \tag{2.1}$$

for all $x,y\in(0,b^*]$, $t\in[0,1]$.

If the inequality (2.1) is reversed,, then f is said to be $(p,(s,m))$-concave function in first sense.

Definition 2.2. The function $f:(0,b^*]\to\mathbb{R}$, $b^*>0$ is said to be $(p,(s,m))$-convex function in second sense, where $s\in(0,1]$ and $m\in(0,1]$, if

$$f\left([tx^p+(1-t)(my)^p]^{\frac{1}{p}}\right)\leq t^s f(x)+m(1-t)^s f(y) \tag{2.2}$$

for all $x,y\in(0,b^*]$, $t\in[0,1]$.

If the inequality (2.2) is reversed,, then f is said to be $(p,(s,m))$-concave function in second sense.

Remark 2.3. Note that for $p=1$, $(p,(s,m))$-convexity in first and second sense reduce to (s,m)-convexity in first and second sense respectively and for $p=-1$, $(p,(s,m))$-convexity in first and second sense reduce to harmonically (α,m)-convexity and harmonically (s,m)-convexity in second sense respectively.

Proposition 2.4. *Let $I\subset(0,\infty)$ be a real interval, $p\in\mathbb{R}/\{0\}$ and $f:I\to\mathbb{R}$ is a function, then;*
(1) if $p\leq 1$ and f is (s,m)-convex and nondecreasing function then f is $(p,(s,m))$-convex.
(2) if $p\geq 1$ and f is $(p,(s,m))$-convex and nondecreasing function then f is (s,m)-convex .
(3) if $p\leq 1$ and f is $(p,(s,m))$-concave and nondecreasing function then f is (s,m)-concave.
(4) if $p\geq 1$ and f is (s,m)-concave and nondecreasing function then f is $(p,(s,m))$-concave.
(5) if $p\geq 1$ and f is (s,m)-convex and nonincreasing function then f is $(p,(s,m))$-convex.
(6) if $p\leq 1$ and f is $(p,(s,m))$-convex and nonincreasing function then f is (s,m)-convex .
(7) if $p\geq 1$ and f is $(p,(s,m))$-concave and nonincreasing function then f is (s,m)-concave.
(8) if $p\leq 1$ and f is (s,m)-concave and nonincreasing function then f is $(p,(s,m))$-concave.

Proof. Since $g(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)$ is a convex function on $(0,\infty)$ and $g(x)=x^p$, $p\in(0,1]$ is concave function on $(0,1)$, the proof is obvious from the following power mean inequalities

$$[tx^p+(1-t)(my)^p]^{\frac{1}{p}}\geq tx+(1-t)(my),\ \ p\geq 1$$

and

$$[tx^p+(1-t)(my)^p]^{\frac{1}{p}}\leq tx+(1-t)(my),\ \ p\leq 1$$

for all $x,y\in(0,\infty)$ and $t\in[0,1]$. □

Proposition 2.5. *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ and if we consider the function $g:[a^p,b^p]\to\mathbb{R}$ defined by $g(t)=f(t^{\frac{1}{p}})$, $p\neq 0$, then f is $(p,(s,m))$-convex if and only if g is (s,m)-convex on $[a^p,b^p]$.*

Theorem 2.6. *Let function $f:(0,b^*]\to\mathbb{R}$, $b^*>0$ be $(p,(s,m))$-convex function in second sense, where $s\in(0,1]$ and $m\in(0,1]$. Then, following inequality holds for all $a,b\in(0,b^*]$ with $a<mb<b$.*

$$\frac{p}{(mb)^p-a^p}\int_a^{mb}\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+mf(b)}{s+1}\tag{2.3}$$

Proof. Since, f is $(p,(s,m))$-convex in second sense. Hence inequality (2.2) holds for all $a,b\in(0,b^*]$ with $a<mb<b$ and by integrating it we obtain

$$\begin{aligned}\frac{p}{(mb)^p-a^p}\int_a^{mb}\frac{f(x)}{x^{1-p}}dx &= \int_0^1 f\Big([ta^p+(1-t)(mb)^p]^{\frac{1}{p}}\Big)dt\\ &\leq \int_0^1[t^sf(a)+m(1-t)^sf(b)]dt\\ &= \frac{f(a)+mf(b)}{s+1}\end{aligned}$$

□

Theorem 2.7. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is $(p,(s,m))$-convex on $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\frac{b^p-a^p}{2p}\eta_1^{1-\frac{1}{p}}[\eta_2|f(a)|^q+m\eta_3|f(b)|^q]^{\frac{1}{q}}.\tag{2.4}$$

where

$$\eta_1(a,b;p)=\begin{cases}b^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;1-(\frac{a}{b})^p)\\ -b^{1-p}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;1-(\frac{a}{b})^p)\\ +\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;\frac{b^p-a^p}{b^p+a^p}), & p>0\\ \\ a^{1-p}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;1-(\frac{b}{a})^p)\\ -a^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;1-(\frac{b}{a})^p)\\ +2^{\frac{2}{p}-2}a^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;\frac{1}{2}(1-(\frac{b}{a})^p)), & p<0\end{cases},$$

$$\eta_1(a,b;p)=\begin{cases}\frac{1}{2^s}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;\frac{1}{2}(1-(\frac{a}{b})^p))\\ -b^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;1-(\frac{a}{b})^p)\\ +b^{1-p}\beta(s+2,1).{}_2F_1(1-\frac{1}{p},s+2;s+3;1-(\frac{a}{b})^p), & p>0\\ \\ a^{1-p}\beta(1,s+2).{}_2F_1(1-\frac{1}{p},1;s+3;1-(\frac{b}{a})^p)\\ -a^{1-p}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;1-(\frac{b}{a})^p)\\ +\frac{1}{2^s(a^p+b^p)^{1-\frac{1}{p}}}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;\frac{a^p-b^p}{a^p+b^p}), & p<0\end{cases}$$

and

$$\eta_1(a,b;p)=\begin{cases}\frac{1}{2^s}(\frac{2}{a^p+b^p})^{1-\frac{1}{p}}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;\frac{b^p-a^p}{a^p+b^p})\\ -b^{1-p}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;1-(\frac{a}{b})^p)\\ +b^{1-p}\beta(1,s+2).{}_2F_1(1-\frac{1}{p},1;s+3;1-(\frac{a}{b})^p), & p>0\\ \\ a^{1-p}\beta(s+2,1).{}_2F_1(1-\frac{1}{p},s+2;s+3;1-(\frac{b}{a})^p)\\ -a^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;1-(\frac{b}{a})^p)\\ +\frac{1}{2^s}a^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;\frac{a^p-b^p}{2a^p}), & p<0\end{cases}$$

Proof. From Lemma 1.15, by using power mean integral inequality and by $(p,(s,m))$-convexity in second sense of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| &\leq \frac{b^p-a^p}{2p}\int_0^1\left|\frac{1-2t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt\\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\frac{|1-2t|[t^s|f(a)|^q+m(1-t)^s|f(b)|^q]}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right)^{\frac{1}{q}}\\
&= \frac{b^p-a^p}{2p}\eta_1^{1-\frac{1}{p}}[\eta_2|f(a)|^q+m\eta_3|f(b)|^q]^{\frac{1}{q}}.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt &= \int_0^{\frac{1}{2}}\frac{(1-2t)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt+\int_{\frac{1}{2}}^1\frac{(2t-1)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\\
&= \int_0^1\frac{(1-t)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt-\int_0^1\frac{t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\\
&+ 2\int_{\frac{1}{2}}^1\frac{(2t-1)}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\\
&= \begin{cases}
b^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;1-(\frac{a}{b})^p)\\
-\,b^{1-p}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;1-(\frac{a}{b})^p)\\
+\,\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;\frac{b^p-a^p}{b^p+a^p}), & p>0\\
\\
a^{1-p}\beta(2,1).{}_2F_1(1-\frac{1}{p},2;3;1-(\frac{b}{a})^p)\\
-\,a^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;1-(\frac{b}{a})^p)\\
+\,2^{\frac{2}{p}-2}a^{1-p}\beta(1,2).{}_2F_1(1-\frac{1}{p},1;3;\frac{1}{2}(1-(\frac{b}{a})^p)), & p<0
\end{cases}\\
&= \eta_1(a,b;p),
\end{aligned}$$

$$\begin{aligned}
\int_0^1\frac{|1-2t|t^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt &= 2\int_0^{\frac{1}{2}}\frac{(1-2t)t^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt+\int_0^1\frac{(2t-1)t^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\\
&= \begin{cases}
\frac{1}{2^s}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;\frac{1}{2}(1-(\frac{a}{b})^p))\\
-\,b^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;1-(\frac{a}{b})^p)\\
+\,b^{1-p}\beta(s+2,1).{}_2F_1(1-\frac{1}{p},s+2;s+3;1-(\frac{a}{b})^p), & p>0\\
\\
a^{1-p}\beta(1,s+2).{}_2F_1(1-\frac{1}{p},1;s+3;1-(\frac{b}{a})^p)\\
-\,a^{1-p}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;1-(\frac{b}{a})^p)\\
+\,\frac{1}{2^s(a^p+b^p)^{1-\frac{1}{p}}}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;\frac{a^p-b^p}{a^p+b^p}), & p<0
\end{cases}\\
&= \eta_2(a,b;p),
\end{aligned}$$

$$\int_0^1 \frac{|1-2t|(1-t)^s}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt = \begin{cases} \frac{1}{2^s}\left(\frac{2}{a^p+b^p}\right)^{1-\frac{1}{p}}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;\frac{b^p-a^p}{a^p+b^p}) \\ -\, b^{1-p}\beta(2,s+1).{}_2F_1(1-\frac{1}{p},2;s+3;1-(\frac{a}{b})^p) \\ +\, b^{1-p}\beta(1,s+2).{}_2F_1(1-\frac{1}{p},1;s+3;1-(\frac{a}{b})^p), & p>0 \\ \\ a^{1-p}\beta(s+2,1).{}_2F_1(1-\frac{1}{p},s+2;s+3;1-(\frac{b}{a})^p) \\ -\, a^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;1-(\frac{b}{a})^p) \\ +\, \frac{1}{2^s}a^{1-p}\beta(s+1,2).{}_2F_1(1-\frac{1}{p},s+1;s+3;\frac{a^p-b^p}{2a^p}), & p<0 \end{cases}$$
$$= \eta_3(a,b;p),$$

□

Remark 2.8. If we take $s=m=1$ in Theorem 2.7, we obtain the inequality (1.12). If we take $s=m=1$ and $p=-1$ in Theorem 2.7, we obtain the inequality (1.8).

Corollary 2.9. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|$ is $(p,(s,m))$-convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\le\frac{b^p-a^p}{2p}[\eta_2|f(a)|+m\eta_3|f(b)|]. \tag{2.5}$$

where η_2 and η_3 are defined as in Theorem 2.6.

Remark 2.10. If we take $s=m=p=1$ in Corollary 2.9, we obtain the inequality (1.3).

Theorem 2.11. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is $(p,(s,m))$-convex on $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\le\frac{b^p-a^p}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}[\eta_4|f(a)|^q+\eta_5|f(b)|^q]^{\frac{1}{q}} \tag{2.6}$$

where

$$\eta_4(a,b;p;q)=\begin{cases} b^{q-pq}\beta(s+1,1).{}_2F_1(q-\frac{q}{p},s+1;s+2;1-(\frac{a}{b})^p), & p>0 \\ \\ a^{q-pq}\beta(1,s+1).{}_2F_1(q-\frac{q}{p},1;s+2;1-(\frac{b}{a})^p), & p<0 \end{cases},$$

and

$$\eta_4(a,b;p;q)=\begin{cases} b^{q-pq}\beta(1,s+1).{}_2F_1(q-\frac{q}{p},1;s+2;1-(\frac{a}{b})^p), & p>0 \\ \\ a^{q-pq}\beta(s+1,1).{}_2F_1(q-\frac{q}{p},s+1;s+2;1-(\frac{b}{a})^p), & p<0 \end{cases}$$

Proof. From Lemma 1.15, Holder's inequality and the $(p,(s,m))$-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| &\le \frac{b^p-a^p}{2p}\int_0^1\left|\frac{1-2t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt \\
&\le \frac{b^p-a^p}{2p}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{q-\frac{q}{p}}}\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{b^p-a^p}{2p}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{q-\frac{q}{p}}}[t^s|f(a)|^q+m(1-t)^s|f(b)|^q]dt\right)^{\frac{1}{q}} \\
&= \frac{b^p-a^p}{2p}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}[\eta_4|f(a)|^q+\eta_5|f(b)|^q]^{\frac{1}{q}},
\end{aligned}$$

where an easy calculation shows that

$$\int_0^1 \frac{t^s}{[ta^p+(1-t)b^p]^{q-\frac{q}{p}}}dt=\eta_4(a,b;p;q)$$

$$\int_0^1 \frac{(1-t)^s}{[ta^p+(1-t)b^p]^{q-\frac{q}{p}}}dt=\eta_5(a,b;p;q)$$

□

Remark 2.12. In Theorem 2.11
(1)If we take $s=m=1$, we obtain the inequality (1.13).
(2)If we take $s=m=p=1$, we obtain the inequality (1.4).
(3)If we take $s=m=$, $p=-1$, we obtain the inequality (1.9).

Theorem 2.13. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}/\{0\}$ and $f'\in L[a,b]$. If $|f'|^q$ is $(p,(s,m))$-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*
(2.7)

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right|\le\frac{b^p-a^p}{2p}\eta_6^{\frac{1}{r}}\left(\frac{1}{2^s(s+1)(s+2)}+\frac{1}{s+2}\right)^{\frac{1}{q}}\left(\frac{|f(a)|^q+m|f(b)|^q}{2}\right)^{\frac{1}{q}},$$

where

$$\eta_6(a,b;p;r)=\begin{cases} b^{r-pr}\beta(1,2).{}_2F_1(r-\frac{r}{p},1;3;1-(\frac{a}{b})^p) \\ \quad -b^{r-pr}\beta(2,1).{}_2F_1(r-\frac{r}{p},2;3;1-(\frac{a}{b})^p) \\ \quad +(\frac{2}{a^p+b^p})^{r-\frac{r}{p}}\beta(2,1).{}_2F_1(r-\frac{r}{p},2;3;\frac{b^p-a^p}{b^p+a^p}), & p>0 \\ \\ a^{r-pr}\beta(2,1).{}_2F_1(r-\frac{r}{p},2;3;1-(\frac{b}{a})^p) \\ \quad -a^{r-pr}\beta(1,2).{}_2F_1(r-\frac{r}{p},1;3;1-(\frac{b}{a})^p) \\ \quad +a^{r-pr}\beta(1,2).{}_2F_1(r-\frac{r}{p},1;3;\frac{a^p-b^p}{2a^p}), & p<0 \end{cases}$$

Proof. From Lemma 1.15, Holder's inequality and the $(p,(s,m))$-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| &\leq \frac{b^p-a^p}{2p}\int_0^1\left|\frac{1-2t}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}\right|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|dt \\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|1-2t|}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{\frac{1}{r}} \\
&\quad \times\left(\int_0^1 |1-2t|\left|f'\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|1-2t|}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt\right)^{\frac{1}{r}} \\
&\quad \times\left(\int_0^1 |1-2t|[t^s|f(a)|^q+m(1-t)^s|f(b)|^q]dt\right)^{\frac{1}{q}} \\
&= \frac{b^p-a^p}{2p}\eta_6^{\frac{1}{r}}\left(\frac{1}{2^s(s+1)(s+2)}+\frac{1}{s+2}\right)^{\frac{1}{q}}\left(\frac{|f(a)|^q+m|f(b)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

Where an easy calculation gives

$$\int_0^1 \frac{|1-2t|}{[ta^p+(1-t)b^p]^{r-\frac{r}{p}}}dt=\eta_6(a,b;p;r)$$

□

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