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INTEGRAL INEQUALITIES FOR DIFFERENTIABLE HARMONICALLY (s,m)-PREINVEX FUNCTIONS

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Abstract. In this paper, we define a new generalized class of preinvex functions which includes harmonically (s,m)-convex functions as a special case and establish a new identity. Using this identity, we introduce some new integral inequalities for harmonically (s,m)-preinvex functions.

1. Introduction

In this section, we recall some basic concepts, properties and results in the convex analysis. For more details, see [25, 50] and the references theirin. Let K be a set in the finite dimensional Euclidean space $\mathbb{R}^n$,whose inner product and norm are denoted by $\langle .,.\rangle$ and $\|.\|$ respectively.

Definition 1.1. A set K in $\mathbb{R}^n$ is said to be a convex set, if and only if,

$$(1-t)u+tv\in K,\ \ for\ all\ u,v\ \in K\ ,\ t\ \in[0,1].$$

Definition 1.2. A function f on the convex set K is said to be a convex function if and only if

$$f((1-t)u+tv)\leq(1-t)f(u)+tf(v),\ \ for\ all\ u,v\ \in K\ ,\ t\ \in[0,1].$$

For the differentiable convex function, we have the following interesting result.

Theorem 1.3. [55] *Let K be a nonempty convex set in $\mathbb{R}^n$, and let f be a differentiable convex function on the set K. Then $u\in K$ is the minimum of f if and only if $u\in K$ satisfies the inequality*

$$\langle f'(u),v-u\rangle\geq 0,\ for\ all\ v\in K.$$

Definition 1.4. [13] A set $K_\eta\subseteq\mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(.,.)$ if and only if

$$x+t\eta(y,x)\in K_\eta,\ for\ all\ x,y\in K_\eta,\ t\in[0,1].$$

The invex set K_η is also called η-connected set. Note that, if $\eta(b,a)=b-a$, then invex set becomes the convex set. Clearly, every convex set is an invex set but converse is not true in general.

Definition 1.5. [57] Let K_η be an invex set in $\mathbb{R}$. Then, a function $f:\mathbb{R}\to\mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(.,.)$ if and only if

$$f(x+t\eta(y,x))\leq(1-t)f(x)+tf(y)\ \ for\ all\ x,y\in K_\eta,\ t\in[0,1].$$

Theorem 1.6. [28] *Let K_η be an invex set in $\mathbb{R}$ and let f be a differentiable preinvex function on set K_η. Then $u\in K_\eta$ is the minimum of f if and only if $u\in K_\eta$ satisfies the inequality*

$$\langle f'(u),\eta(v,u)\rangle\geq 0,\ for\ all\ v\in K_\eta.$$

Definition 1.7. [15] A set $K_h\subset\mathbb{R}/\{0\}\to\mathbb{R}$ is said to be a harmonically convex set if and only if

$$\frac{uv}{v+t(u-v)}\in K_h,\ \ for\ all\ u,v\in K_h,\ t\in[0,1].$$

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Definition 1.8. [15] A function $f : K_h \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically convex function if and only if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(y), \quad for\ all\ x, y \in K_h,\ t \in [0,1].$$

Definition 1.9. [4] The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s \in (0,1]$ and $m \in (0,1]$ if

$$f\big(\frac{mxy}{mty+(1-t)x}\big) = f\big((\frac{t}{x} + \frac{1-t}{my})^{-1}\big) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0,1]$.

Remark 1.10. Note that for $s = 1$, (s,m)-convexity reduces to harmonically m-convexity and for $m = 1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense and for $s, m = 1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity .

Definition 1.11. [49] A set $I = [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(,)$ if and only if

$$\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)} \in I, \quad for\ all\ x, y \in I,\ t \in [0,1]$$

Definition 1.12. [50] Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I \to [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is relative harmonic preinvex function with respect to an arbitrary nonnegative function h and an arbitrary bifunction $\eta(,)$ if

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq h(1-t)f(x) + h(t)f(y), \quad for\ all\ x, y \in I,\ t \in [0,1]$$

Condition C: Let $I \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(,) : I \times I \to \mathbb{R}$. For any $x, y \in I$ and $t \in [0,1]$, we have

$$\eta(y, y+t\eta(x,y)) = -t\eta(x,y)$$

$$\eta(x, y+t\eta(x,y)) = (1-t)\eta(x,y)$$

Note that for every $x, y \in I$, $t_1, t_2 \in [0,1]$ and from condition C, we have

$$\eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) = (t_2-t_1)\eta(x,y)$$

2. **Main Results**

Now, we define the class of harmonically (s,m)-preinvex functions which is motivated by the definition of harmonically (s,m)-convex functions defined by I. A. Baloch et al. [4].

Definition 2.1. A function $f : [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically (s,m)-preinvex functions with respect to the bifunction $\eta(,)$, if

$$f\left(\frac{x(x+\eta(my,x))}{x+t\eta(my,x)}\right) = f\left(\frac{t}{x} + \frac{1-t}{x+\eta(my,x)}\right)^{-1} \leq t^s f(x) + m(1-t)^s f(y)$$

for all $x, y \in [a, a+\eta(b,a)]$, with $x < my$, $t \in [0,1]$, $s \in (0,1]$, $m \in (0,1]$.

Note: if $\eta(y,x) = y - x$, then harmonic (s,m)-preinvexity reduce to harmonic (s,m)-convexity.
We need the following identity, which plays an important role in the derivations of our main results.

Lemma 2.2. *Let $f:[a,a+\eta(mb,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be a differentiable function on the interior of I° of I. If $f'\in[a,a+\eta(mb,a)]$ and $\lambda\in[0,1]$, then*

$$\Upsilon_f(a,a+\eta(mb,a);\lambda)$$
$$=\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\int_0^{\frac{1}{2}}\frac{\lambda-2t}{(a+t\eta(mb,a))^2}f'\bigg(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\bigg)dt$$
$$+\int_{\frac{1}{2}}^{1}\frac{2-2t-\lambda}{(a+t\eta(mb,a))^2}f'\bigg(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\bigg)dt\Bigg]$$

where

$$\Upsilon_f(a,a+\eta(mb,a);\lambda)$$

$$=(1-\lambda)f\bigg(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\bigg)+\lambda\bigg[\frac{f(a)+f(a+\eta(mb,a))}{2}\bigg]-\frac{2a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+m\eta(b,a)}\frac{f(x)}{x^2}dx$$

Proof. Integrating by parts, we have

$$I_1=\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\int_0^{\frac{1}{2}}\frac{\lambda-2t}{(a+t\eta(mb,a))^2}f'\bigg(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\bigg)dt$$
$$=\frac{1-\lambda}{2}f\bigg(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\bigg)+\frac{\lambda}{2}f(a+\eta(mb,a))-\int_0^{\frac{1}{2}}f\bigg(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\bigg)dt,$$

and

$$I_2=\frac{a(a,a+\eta(mb,a))\eta(b,a)}{2}\int_{\frac{1}{2}}^{1}\frac{2t-2+\lambda}{(a+t\eta(mb,a))^2}f'\bigg(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\bigg)dt$$
$$=\frac{1-\lambda}{2}f\bigg(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\bigg)+\frac{\lambda}{2}f(a)-\int_{\frac{1}{2}}^{1}f\bigg(\frac{a(a+\eta(mb,a))}{x+t\eta(mb,a)}\bigg)dt$$

Thus

$$I_1+I_2$$
$$=(1-\lambda)f\bigg(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\bigg)+\lambda\bigg[\frac{f(a)+f(a+\eta(mb,a))}{2}\bigg]-\frac{2a(a+\eta(mb,a))}{\eta(mb,a)}\int_a^{a+m\eta(b,a)}\frac{f(x)}{x^2}dx$$

which is the required result. □

Theorem 2.3. *Let $f:[a,a+\eta(mb,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f'\in[a,a+\eta(mb,a)]$ and $|f'|^q$ is harmonic (s,m)-preinvex function on I for $q\geq1$ and $\lambda\in[0,1]$, then*

$$\Big|\Upsilon_f(a,a+\eta(mb,a);\lambda)\Big|$$
$$\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\sigma_1(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_2(a,b;\lambda,s)|f'(a)|^q+m\sigma_3(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}}$$
$$+\sigma_4(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_5(a,b;\lambda,s)|f'(a)|^q+m\sigma_6(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}}\Bigg],$$

where one can evaluate these integrals using any mathematical software (i.e maple).

$$\sigma_1(a,b;\lambda)=\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}dt,$$

$$\sigma_2(a,b;\lambda,s)=\int_0^{\frac{1}{2}}\frac{|\lambda-2t|(1-t)^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_3(a,b;\lambda,s)=\int_0^{\frac{1}{2}}\frac{|\lambda-2t|t^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_4(a,b;\lambda) = \int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}dt$$

$$\sigma_5(a,b;\lambda,s) = \int_0^{\frac{1}{2}} \frac{|2-2t-\lambda|(1-t)^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_6(a,b;\lambda,s) = \int_0^{\frac{1}{2}} \frac{|2-2t-\lambda|t^s}{(a+t\eta(mb,a))^2}.$$

Proof. Using Lemma 2.2 and the power mean inequality, we have

$$\begin{aligned}
&\left|\Upsilon_f(a,a+\eta(mb,a);\lambda)\right| \\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt \\
&\quad+\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt\Bigg] \\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg] \\
&\leq \frac{a(a+\eta(mb,a))\eta(b,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}\Bigg] \\
&= \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\sigma_1(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_2(a,b;\lambda,s)|f'(a)|^q+m\sigma_3(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}} \\
&\quad+\sigma_4(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_5(a,b;\lambda,s)|f'(a)|^q+m\sigma_6(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}}\Bigg],
\end{aligned}$$

which is the required result. □

If $q=1$, then Theorem .. reduces to the following result, which appears to be a better mew one than already exists.

Corollary 2.4. *Let $f:[a,a+\eta(mb,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be a differentiable function on the interior of I° of I. If $f'\in[a,a+\eta(mb,a)]$ and $|f'|$ is harmonic (s,m)-preinvex function on I and $\lambda\in[0,1]$, then*

$$\begin{aligned}&\left|\Upsilon_f(a,a+\eta(mb,a);\lambda)\right|\\ &\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Big[\{\sigma_2(a,b;\lambda,s)+m\sigma_3(a,b;\lambda,s)\}|f'(a)|\\ &\quad+\{\sigma_5(a,b;\lambda,s)+m\sigma_6(a,b;\lambda,s)\}|f'(b)|\Big],\end{aligned}$$

where $\sigma_2(a,b;\lambda,s)$, $\sigma_3(a,b;\lambda,s)$, $\sigma_5(a,b;\lambda,s)$, $\sigma_6(a,b;\lambda,s)$ are given as in Theorem...

Theorem 2.5. *Let $f:[a,a+\eta(mb,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be a differentiable function on the interior of I° of I. If $f'\in[a,a+\eta(mb,a)]$ and $|f'|^q$ is harmonic (s,m)-preinvex function on I for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda\in[0,1]$, then*

$$\begin{aligned}&\left|\Upsilon_f(a,a+\eta(mb,a);\lambda)\right|\\ &\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[(\sigma_7(a,b;\lambda,p))^{\frac{1}{p}}\left(\{(1-\frac{1}{2^{s+1}})|f'(a)|^q+\frac{m}{2^{s+1}}|f'(b)|^q\}\right)^{\frac{1}{q}}\\ &\quad+(\sigma_8(a,b;\lambda,p))^{\frac{1}{p}}\left(\{m(1-\frac{1}{2^{s+1}})|f'(b)|^q+\frac{1}{2^{s+1}}|f'(a)|^q\}\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

where

$$\sigma_7(a,b;\lambda,p)=\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt,$$
$$\sigma_8(a,b;\lambda,p)=\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt.$$

Proof. Using Lemma 2.2 and Holder's integral inequality, we have

$$\begin{aligned}&\left|\Upsilon_f(a,a+\eta(mb,a);\lambda)\right|\\ &\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt\\ &\quad+\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt\Bigg]\\ &\leq\frac{a(a,a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\quad+\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\\ &\leq\frac{a(a+\eta(mb,a))\eta(b,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}\\ &\quad+\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}\Bigg]\\ &=\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\{(1-\frac{1}{2^{s+1}})|f'(a)|^q+m\frac{1}{2^{s+1}}|f'(b)|^q\}\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}\frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}\left(\{m(1-\frac{1}{2^{s+1}})|f'(b)|^q+\frac{1}{2^{s+1}}|f'(a)|^q\}\right)^{\frac{1}{q}}\Bigg]$$

The proof completes. □

Theorem 2.6. *Let $f:[a,a+\eta(mb,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f'\in[a,a+\eta(mb,a)]$ and $|f'|^q$ is harmonic (s,m)-preinvex function on I for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda\in[0,1]$, then*

$$\Big|\Upsilon_f(a,a+\eta(mb,a);\lambda)\Big|$$

$$\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\times\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\Big[(\sigma_9(a,b;\lambda,q)|f'(a)|^q+m\sigma_{10}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\sigma_{11}(a,b;\lambda,q)|f'(a)|^q+m\sigma_{12}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}\Big],$$

where

$$\sigma_9(a,b;\lambda,p)=\int_0^{\frac{1}{2}}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt$$

$$\sigma_{10}(a,b;\lambda,p)=\int_0^{\frac{1}{2}}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt$$

$$\sigma_{11}(a,b;\lambda,p)=\int_{\frac{1}{2}}^{1}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt$$

$$\sigma_{12}(a,b;\lambda,p)=\int_{\frac{1}{2}}^{1}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt$$

Proof. Using Lemma 2.2 and the Holder's integral inequality, we have

$$\Big|\Upsilon_f(a,a+\eta(mb,a);\lambda)\Big|$$

$$\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\int_0^{\frac{1}{2}}|\lambda-2t|\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|dt$$

$$+\int_{\frac{1}{2}}^{1}|2-2t-\lambda|\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|dt\Bigg]$$

$$\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}}|\lambda-2t|^pdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|^qdt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}|2-2t-\lambda|^pdt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t)\eta(mb,a))^2}\right|^qdt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\Bigg[\left(\int_0^{\frac{1}{2}}|\lambda-2t|^pdt\right)^{\frac{1}{p}}$$

$$\times\left(|f'(a)|^q\int_0^{\frac{1}{2}}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt+m|f'(b)|^q\int_0^{\frac{1}{2}}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{1}|2-2t-\lambda|^pdt\right)^{\frac{1}{p}}\left(|f'(a)|^q\int_{\frac{1}{2}}^{1}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt+m|f'(b)|^q\int_{\frac{1}{2}}^{1}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt\right)^{\frac{1}{q}}\Bigg]$$

$$= \frac{a(a+\eta(mb,a))\eta(mb,a)}{2} \times \left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}} \Big[(\sigma_9(a,b;\lambda,q)|f'(a)|^q + m\sigma_{10}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\sigma_{11}(a,b;\lambda,q)|f'(a)|^q + m\sigma_{12}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}\Big].$$

This completes the proof. □

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