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CALLEBAUT AND HÖLDER TYPE INEQUALITIES FOR POSITIVE LINEAR MAPS OF SELFADJOINT OPERATORS VIA A KITTANEH-MANASRAH RESULT

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Abstract. Some inequalities of Callebaut and Hölder type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces, are given. Applications for power function are provided as well.

1. Introduction

As is well-known, the famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.4) is also called as *ν-weighted arithmetic-geometric mean inequality.*

Kittaneh and Manasrah [8], [9] provided a refinement and a reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.2}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We observe that, if $a,\ b\in[m,M]\subset(0,\infty)$, then $\left|\sqrt{a}-\sqrt{b}\right|\leq\sqrt{M}-\sqrt{m}$ and by (1.2) we obtain the following simple reverse of Young inequality

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{1.3}$$

We can give a direct proof for (1.2) as follows.

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Recall the following result obtained by Dragomir in 2006 [3] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \\ &\leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right) \\ &\leq n \max_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \tag{1.4}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\ldots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (1.4) that

$$\begin{aligned} &2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y] \\ &\leq 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.5}$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$.

If we take $\Phi(x) = \exp(x)$, then we get from (1.5) that

$$\begin{aligned} &2\min\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu\exp(x) + (1-\nu)\exp(y) - \exp[\nu x + (1-\nu)y] \\ &\leq 2\max\{\nu, 1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2} - \exp\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.6}$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$. Further, denote $\exp(x) = a$, $\exp(y) = b$ with a, $b > 0$, then from (1.6) we obtain the inequality (1.2).

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^{+}(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [15, p. 18]) we can introduce:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^{+}(H)$ then $\Phi(A) \in \mathcal{B}^{+}(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

For some recent inequalities for positive maps of Hilbert space operators see [4]-[7], [10]-[14], [16]-[18] and the references therein.

In this paper we establish some inequalities of Callebaut and Hölder type for positive linear maps of continuous functions of selfadjoint linear operators in Hilbert spaces. Applications for power function are provided as well.

2. Callebaut's Type Inequalities

The following refinement of Cauchy-Bunyakowsky-Schwarz inequality for n-tuples of nonnegative real numbers $(a_1, ..., a_n)$, $(b_1, ..., b_n)$ was established by Callebaut in 1965 [1] and can be stated as follows:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^{2(1-\lambda)} b_i^{2\lambda} \sum_{i=1}^{n} a_i^{2\lambda} b_i^{2(1-\lambda)} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{2.1}$$

for any $\lambda \in [0, 1]$.

In this section we obtain, as a main result, the following refinement and reverse of the second part of Callebaut inequality 2.1 in the case of positive maps:

$$\begin{aligned} 0 &\leq 2r\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle - \left\langle \Phi\left(f(A)g(A)\right)x, x\right\rangle^2\right) \\ &\leq \left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle \\ &- \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x, x\right\rangle \left\langle \Phi\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)x, x\right\rangle \\ &\leq R\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle - \left\langle \Phi\left(f(A)g(A)\right)x, x\right\rangle^2\right), \end{aligned} \tag{2.2}$$

for any $x \in K$, where f, $g : I \to \mathbb{R}$ are continuous functions on the interval I, A is a selfadjoint operator with $\operatorname{Sp}(A) \subset I$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$, $R = \max\{1-\nu, \nu\}$ and $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$.

This result can be obtained as a particular case of the more general result incorporated in:

Theorem 1. *Let f, $g : I \to \mathbb{R}$ be continuous functions on the interval I and A, B be two selfadjoint operators with* $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$. *If $\nu \in [0,1]$ and Φ,*

$\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$, *then*

$$
\begin{aligned}
(2.3)\qquad & 0 \le r\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle\left\langle \Psi\left(f^2(B)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle\right. \\
& \left.\times\left\langle \Psi\left(f(B)g(B)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle\left\langle \Psi\left(g^2(B)\right)x,x\right\rangle\right) \\
& \le (1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle\left\langle \Psi\left(f^2(B)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle\left\langle \Psi\left(g^2(B)\right)x,x\right\rangle \\
& - \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x,x\right\rangle\left\langle \Psi\left(f^{2(1-\nu)}(B)g^{2\nu}(B)\right)x,x\right\rangle \\
& \le R\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle\left\langle \Psi\left(f^2(B)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle\right. \\
& \left.\times\left\langle \Psi\left(f(B)g(B)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle\left\langle \Psi\left(g^2(B)\right)x,x\right\rangle\right)
\end{aligned}
$$

for any $x \in K$, *where* $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Proof. Let $t, s \in I$ such that $g(t), g(s) \neq 0$. If we use the inequalities (1.2) for

$$a = \frac{f^2(t)}{g^2(t)},\ b = \frac{f^2(s)}{g^2(s)},$$

then we get

$$
\begin{aligned}
(2.4)\qquad & r\left(\frac{f(t)}{g(t)} - \frac{f(s)}{g(s)}\right)^2 \\
& \le (1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \\
& \le R\left(\frac{f(t)}{g(t)} - \frac{f(s)}{g(s)}\right)^2,
\end{aligned}
$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Therefore

$$
\begin{aligned}
(2.5)\qquad & r\left(\frac{f^2(t)}{g^2(t)} - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \frac{f^2(s)}{g^2(s)}\right) \\
& \le (1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \\
& \le R\left(\frac{f^2(t)}{g^2(t)} - 2\frac{f(t)}{g(t)}\frac{f(s)}{g(s)} + \frac{f^2(s)}{g^2(s)}\right).
\end{aligned}
$$

If we multiply (2.5) by $g^2(t)g^2(s)$, then we get

$$
\begin{aligned}
(2.6)\qquad & 0 \le r\left(f^2(t)g^2(s) - 2f(t)g(t)f(s)g(s) + f^2(s)g^2(t)\right) \\
& \le (1-\nu)f^2(t)g^2(s) + \nu g^2(t)f^2(s) \\
& - f^{2(1-\nu)}(t)g^{2\nu}(t)f^{2\nu}(s)g^{2(1-\nu)}(s) \\
& \le R\left(f^2(t)g^2(s) - 2f(t)g(t)f(s)g(s) + f^2(s)g^2(t)\right),
\end{aligned}
$$

which holds for any $t, s \in I$.

Fix $t \in I$ and use the continuous functional calculus for A to get

$$\begin{aligned}
(2.7)\qquad 0 &\le r\left(f^2(t)\,g^2(A) - 2f(t)\,g(t)\,f(A)\,g(A) + g^2(t)\,f^2(A)\right) \\
&\le (1-\nu)\,f^2(t)\,g^2(A) + \nu g^2(t)\,f^2(A) \\
&- f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(A)\,g^{2(1-\nu)}(A) \\
&\le R\left(f^2(t)\,g^2(A) - 2f(t)\,g(t)\,f(A)\,g(A) + g^2(t)\,f^2(A)\right).
\end{aligned}$$

If we take in the inequality (2.7) the map Φ, then we get

$$\begin{aligned}
(2.8)\qquad 0 &\le r\left(f^2(t)\,\Phi\left(g^2(A)\right) - 2f(t)\,g(t)\,\Phi\left(f(A)\,g(A)\right) + g^2(t)\,\Phi\left(f^2(A)\right)\right) \\
&\le (1-\nu)\,f^2(t)\,\Phi\left(g^2(A)\right) + \nu g^2(t)\,\Phi\left(f^2(A)\right) \\
&- f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,\Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right) \\
&\le R\left(f^2(t)\,\Phi\left(g^2(A)\right) - 2f(t)\,g(t)\,\Phi\left(f(A)\,g(A)\right) + g^2(t)\,\Phi\left(f^2(A)\right)\right)
\end{aligned}$$

for any $t \in I$.

If we take the inner product in (2.8), then we get

$$\begin{aligned}
(2.9)\qquad 0 &\le r\left(f^2(t)\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle - 2f(t)\,g(t)\left\langle \Phi\left(f(A)\,g(A)\right)x, x\right\rangle \right.\\
&\left. + g^2(t)\left\langle \Phi\left(f^2(A)\right)x, x\right\rangle\right) \\
&\le (1-\nu)\,f^2(t)\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle + \nu g^2(t)\left\langle \Phi\left(f^2(A)\right)x, x\right\rangle \\
&- f^{2(1-\nu)}(t)\,g^{2\nu}(t)\left\langle \Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)x, x\right\rangle \\
&\le R\left(f^2(t)\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle - 2f(t)\,g(t)\left\langle \Phi\left(f(A)\,g(A)\right)x, x\right\rangle \right.\\
&\left. + g^2(t)\left\langle \Phi\left(f^2(A)\right)x, x\right\rangle\right)
\end{aligned}$$

for any $t \in I$ and $x \in K$.

If we use the functional calculus for the operator B we have by (2.9) that

$$\begin{aligned}
(2.10)\qquad 0 &\le r\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle f^2(B) - 2\left\langle \Phi\left(f(A)\,g(A)\right)x, x\right\rangle f(B)\,g(B) \right.\\
&\left. + \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle g^2(B)\right) \\
&\le (1-\nu)\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle f^2(B) + \nu\left\langle \Phi\left(f^2(A)\right)x, x\right\rangle g^2(B) \\
&- \left\langle \Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)x, x\right\rangle f^{2(1-\nu)}(B)\,g^{2\nu}(B) \\
&\le R\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle f^2(B) - 2\left\langle \Phi\left(f(A)\,g(A)\right)x, x\right\rangle f(B)\,g(B) \right.\\
&\left. + \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle g^2(B)\right)
\end{aligned}$$

for any $x \in K$.

If we take in the inequality (2.10) the map Ψ and the inner product for $y \in K$, then we get

$$0 \le r\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(B)\right)y,y\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Psi\left(f(B)g(B)\right)y,y\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(B)\right)y,y\right\rangle\right) \tag{2.11}$$
$$\le (1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(B)\right)y,y\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(B)\right)y,y\right\rangle - \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Psi\left(f^{2(1-\nu)}(B)g^{2\nu}(B)\right)y,y\right\rangle$$
$$\le R\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(B)\right)y,y\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Psi\left(f(B)g(B)\right)y,y\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(B)\right)y,y\right\rangle\right)$$

for any x, $y \in K$.

If we take in (2.11) $y = x$, then we get the desired result (2.3). □

Remark 1. *In particular, under the assumptions of Theorem 1 we have*

$$0 \le r\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Phi\left(f^2(B)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Phi\left(f(B)g(B)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Phi\left(g^2(B)\right)x,x\right\rangle\right) \tag{2.12}$$
$$\le (1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Phi\left(f^2(B)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Phi\left(g^2(B)\right)x,x\right\rangle - \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Phi\left(f^{2(1-\nu)}(B)g^{2\nu}(B)\right)x,x\right\rangle$$
$$\le R\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Phi\left(f^2(B)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Phi\left(f(B)g(B)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Phi\left(g^2(B)\right)x,x\right\rangle\right)$$

and

$$0 \le r\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(A)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Psi\left(f(A)g(A)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(A)\right)x,x\right\rangle\right) \tag{2.13}$$
$$\le (1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(A)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(A)\right)x,x\right\rangle - \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Psi\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)x,x\right\rangle$$
$$\le R\left(\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(A)\right)x,x\right\rangle - 2\left\langle \Phi\left(f(A)g(A)\right)x,x\right\rangle \times \left\langle \Psi\left(f(A)g(A)\right)x,x\right\rangle + \left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(A)\right)x,x\right\rangle\right)$$

for any $x \in K$.

If we take in (2.12) $B=A$, then we get (2.2).

The inequality (2.2) has some particular inequalities of interest for power function.

Indeed if $p,\ q>0,\ A\in\mathcal{B}^{+}(H)$ and $p>0,\ q<0$ or $p<0,\ q>0$, or $p<0,\ q<0$ and $A\in\mathcal{B}^{++}$, then by (2.11) we have

$$0\le 2r\left(\left\langle \Phi\left(A^{2p}\right)x,x\right\rangle \left\langle \Phi\left(A^{2q}\right)x,x\right\rangle -\left\langle \Phi\left(A^{p+q}\right)x,x\right\rangle ^{2}\right) \tag{2.14}$$

$$\le \left\langle \Phi\left(A^{2p}\right)x,x\right\rangle \left\langle \Phi\left(A^{2q}\right)x,x\right\rangle -\left\langle \Phi\left(A^{2\nu p+2(1-\nu)q}\right)x,x\right\rangle \left\langle \Phi\left(A^{2(1-\nu)p+2\nu q}\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(A^{2p}\right)x,x\right\rangle \left\langle \Phi\left(A^{2q}\right)x,x\right\rangle -\left\langle \Phi\left(A^{p+q}\right)x,x\right\rangle ^{2}\right),$$

for any $x\in K$, where $\Phi\in\mathfrak{P}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, if we take $p=s,\ q=-s$ with $s>0$, in (2.14), then we have

$$0\le 2r\left(\left\langle \Phi\left(A^{2s}\right)x,x\right\rangle \left\langle \Phi\left(A^{-2s}\right)x,x\right\rangle -\left\langle \Phi\left(1_{H}\right)x,x\right\rangle ^{2}\right) \tag{2.15}$$

$$\le \left\langle \Phi\left(A^{2s}\right)x,x\right\rangle \left\langle \Phi\left(A^{-2s}\right)x,x\right\rangle -\left\langle \Phi\left(A^{2(2\nu-1)s}\right)x,x\right\rangle \left\langle \Phi\left(A^{-2(2\nu-1)s}\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(A^{2s}\right)x,x\right\rangle \left\langle \Phi\left(A^{-2s}\right)x,x\right\rangle -\left\langle \Phi\left(1_{H}\right)x,x\right\rangle ^{2}\right),$$

which by $s=\frac{1}{2}$ gives the inequality

$$0\le 2r\left(\left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -\left\langle \Phi\left(1_{H}\right)x,x\right\rangle ^{2}\right) \tag{2.16}$$

$$\le \left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -\left\langle \Phi\left(A^{2\nu-1}\right)x,x\right\rangle \left\langle \Phi\left(A^{-(2\nu-1)}\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -\left\langle \Phi\left(1_{H}\right)x,x\right\rangle ^{2}\right),$$

for any $x\in K$, where $A\in\mathcal{B}^{++}(H)$, $\Phi\in\mathfrak{P}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

If $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and $x\in K$ with $\|x\|=1$, then by (2.16) we have

$$0\le 2r\left(\left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -1\right) \tag{2.17}$$

$$\le \left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -\left\langle \Phi\left(A^{2\nu-1}\right)x,x\right\rangle \left\langle \Phi\left(A^{-(2\nu-1)}\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(A\right)x,x\right\rangle \left\langle \Phi\left(A^{-1}\right)x,x\right\rangle -1\right).$$

If we take $p=\lambda\in[0,1]$ and $q=1-\lambda$ in (2.14), then we get

$$0\le 2r\left(\left\langle \Phi\left(A^{2\lambda}\right)x,x\right\rangle\left\langle \Phi\left(A^{2(1-\lambda)}\right)x,x\right\rangle-\left\langle \Phi\left(A\right)x,x\right\rangle^{2}\right) \tag{2.18}$$

$$\le \left\langle \Phi\left(A^{2\lambda}\right)x,x\right\rangle\left\langle \Phi\left(A^{2(1-\lambda)}\right)x,x\right\rangle-\left\langle \Phi\left(A^{2(1-\nu-\lambda+2\nu\lambda)}\right)x,x\right\rangle\left\langle \Phi\left(A^{2(\nu+\lambda-2\nu\lambda)}\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(A^{2\lambda}\right)x,x\right\rangle\left\langle \Phi\left(A^{2(1-\lambda)}\right)x,x\right\rangle-\left\langle \Phi\left(A\right)x,x\right\rangle^{2}\right),$$

for any $x\in K$, where $A\in\mathcal{B}^{+}(H)$, $\Phi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(K)]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

Remark 2. *If in (2.2) we take the function g to be constant 1, then we have by (2.2) that*

$$0\le 2r\left(\left\langle \Phi\left(1_{H}\right)x,x\right\rangle\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle-\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle^{2}\right) \tag{2.19}$$

$$\le \left\langle \Phi\left(1_{H}\right)x,x\right\rangle\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle-\left\langle \Phi\left(f^{2\nu}\left(A\right)\right)x,x\right\rangle\left\langle \Phi\left(f^{2(1-\nu)}\left(A\right)\right)x,x\right\rangle$$

$$\le R\left(\left\langle \Phi\left(1_{H}\right)x,x\right\rangle\left\langle \Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle-\left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle^{2}\right),$$

for any $x\in K$.

Let T_j, $j\in\{1,...,n\}$ be bounded linear operators in H and put

$$\Phi\left(A\right):=\sum_{i=}^{n}T_{j}^{*}AT_{j},\ \ A\in\mathcal{B}\left(H\right),$$

then $\Phi\in\mathfrak{P}[\mathcal{B}(H),\mathcal{B}(H)]$ and by (2.19) we have

$$0\le 2r\left(\left\langle \sum_{i=}^{n}T_{j}^{*}AT_{j}x,x\right\rangle\left\langle \sum_{i=}^{n}T_{j}^{*}f^{2}\left(A\right)T_{j}x,x\right\rangle-\left\langle \sum_{i=}^{n}T_{j}^{*}f\left(A\right)T_{j}x,x\right\rangle^{2}\right)$$

$$\le \left\langle \sum_{i=}^{n}T_{j}^{*}AT_{j}x,x\right\rangle\left\langle \sum_{i=}^{n}T_{j}^{*}f^{2}\left(A\right)T_{j}x,x\right\rangle$$

$$-\left\langle \sum_{i=}^{n}T_{j}^{*}f^{2\nu}\left(A\right)T_{j}x,x\right\rangle\left\langle \sum_{i=}^{n}T_{j}^{*}f^{2(1-\nu)}\left(A\right)T_{j}x,x\right\rangle$$

$$\le R\left(\left\langle \sum_{i=}^{n}T_{j}^{*}AT_{j}x,x\right\rangle\left\langle \sum_{i=}^{n}T_{j}^{*}f^{2}\left(A\right)T_{j}x,x\right\rangle-\left\langle \sum_{i=}^{n}T_{j}^{*}f\left(A\right)T_{j}x,x\right\rangle^{2}\right),$$

for any $x\in K$.

The following upper bound may be stated as well:

Theorem 2. *Let* $f,\ g : I \to \mathbb{R}$ *be continuous functions on the interval* I *such that*

$$0 < m \le \frac{f}{g} \le M < \infty \tag{2.20}$$

for some constants $m,\ M,$ *and* $A,\ B$ *be two selfadjoint operators with* $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$. *If* $\nu \in [0,1]$ *and* $\Phi,\ \Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$, *then*

$$\begin{aligned}(1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(B)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(B)\right)x,x\right\rangle \\ -\left\langle \Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Psi\left(f^{2(1-\nu)}(B)\,g^{2\nu}(B)\right)x,x\right\rangle \\ \le R(M-m)^2\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(B)\right)x,x\right\rangle\end{aligned} \tag{2.21}$$

for any $x \in K$.

Proof. Let $t,\ s \in I$ such that $g(t),\ g(s) \neq 0$. If we use the inequalities (1.2) for

$$a = \frac{f^2(t)}{g^2(t)},\ b = \frac{f^2(s)}{g^2(s)},$$

then we get

$$(1-\nu)\frac{f^2(t)}{g^2(t)} + \nu\frac{f^2(s)}{g^2(s)} - \left(\frac{f^2(t)}{g^2(t)}\right)^{1-\nu}\left(\frac{f^2(s)}{g^2(s)}\right)^{\nu} \le R(M-m)^2, \tag{2.22}$$

where $\nu \in [0,1]$, $R = \max\{1-\nu, \nu\}$.

If we multiply (2.22) by $g^2(t)\,g^2(s)$, then we get

$$\begin{aligned}&(1-\nu)f^2(t)\,g^2(s) + \nu g^2(t)\,f^2(s) - f^{2(1-\nu)}(t)\,g^{2\nu}(t)\,f^{2\nu}(s)\,g^{2(1-\nu)}(s) \\ &\le R(M-m)^2 g^2(t)\,g^2(s)\end{aligned} \tag{2.23}$$

for any $t,\ s \in I$.

Now, by employing a similar argument to the one outlined in the proof of Theorem 1 we deduce the desired result (2.21). □

Remark 3. *In particular, we have the inequalities*

$$\begin{aligned}(1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Phi\left(f^2(B)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Phi\left(g^2(B)\right)x,x\right\rangle \\ -\left\langle \Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Phi\left(f^{2(1-\nu)}(B)\,g^{2\nu}(B)\right)x,x\right\rangle \\ \le R(M-m)^2\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Phi\left(g^2(B)\right)x,x\right\rangle,\end{aligned} \tag{2.24}$$

$$\begin{aligned}(1-\nu)\left\langle \Phi\left(g^2(A)\right)x,x\right\rangle \left\langle \Psi\left(f^2(A)\right)x,x\right\rangle + \nu\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(A)\right)x,x\right\rangle \\ -\left\langle \Phi\left(f^{2\nu}(A)\,g^{2(1-\nu)}(A)\right)x,x\right\rangle \left\langle \Psi\left(f^{2(1-\nu)}(A)\,g^{2\nu}(A)\right)x,x\right\rangle \\ \le R(M-m)^2\left\langle \Phi\left(f^2(A)\right)x,x\right\rangle \left\langle \Psi\left(g^2(A)\right)x,x\right\rangle\end{aligned} \tag{2.25}$$

and

$$0\le\left\langle\Phi\left(g^{2}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle -\left\langle\Phi\left(f^{2\nu}\left(A\right)g^{2(1-\nu)}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(f^{2(1-\nu)}\left(A\right)g^{2\nu}\left(A\right)\right)x,x\right\rangle \le R\left(M-m\right)^{2}\left\langle\Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(g^{2}\left(A\right)\right)x,x\right\rangle, \tag{2.26}$$

for any $x\in K$.

The inequality (2.26) can be also written as

$$0\le 1-\frac{\left\langle\Phi\left(f^{2\nu}\left(A\right)g^{2(1-\nu)}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(f^{2(1-\nu)}\left(A\right)g^{2\nu}\left(A\right)\right)x,x\right\rangle}{\left\langle\Phi\left(g^{2}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(f^{2}\left(A\right)\right)x,x\right\rangle} \le R\left(M-m\right)^{2}, \tag{2.27}$$

for any $x\in K$, $x\neq 0$.

3. Hölder's Type Inequalities

We have:

Theorem 3. *Let* $f,\ g:I\to(0,\infty)$ *be continuous functions on the interval* I, *and* A, B *be two selfadjoint operators with* $\operatorname{Sp}(A),\ \operatorname{Sp}(B)\subset I$ *and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *If* $\Phi,\ \Psi\in\mathfrak{P}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$, *then*

$$s\left(\frac{\left\langle\Psi\left(f^{p}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle}+\frac{\left\langle\Psi\left(g^{q}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}-\frac{2\left\langle\Psi\left(f^{\frac{p}{2}}\left(B\right)g^{\frac{q}{2}}\left(B\right)\right)x,x\right\rangle}{\sqrt{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}}\right)$$
$$\le\frac{1}{p}\frac{\left\langle\Psi\left(f^{p}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle}+\frac{1}{q}\frac{\left\langle\Psi\left(g^{q}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}-\frac{\left\langle\Phi\left(f\left(B\right)g\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle^{1/p}\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle^{1/q}} \tag{3.1}$$
$$\le S\left(\frac{\left\langle\Psi\left(f^{p}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle}+\frac{\left\langle\Psi\left(g^{q}\left(B\right)\right)x,x\right\rangle}{\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}-\frac{2\left\langle\Psi\left(f^{\frac{p}{2}}\left(B\right)g^{\frac{q}{2}}\left(B\right)\right)x,x\right\rangle}{\sqrt{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}}\right)$$

for any $x\in K$, $x\neq 0$, *where* $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and* $S=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

In particular, we have

$$2s\left(1-\frac{\left\langle\Phi\left(f^{\frac{p}{2}}\left(A\right)g^{\frac{q}{2}}\left(A\right)\right)x,x\right\rangle}{\sqrt{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}}\right) \le 1-\frac{\left\langle\Phi\left(f\left(A\right)g\left(A\right)\right)x,x\right\rangle}{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle^{1/p}\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle^{1/q}} \le 2S\left(1-\frac{\left\langle\Phi\left(f^{\frac{p}{2}}\left(A\right)g^{\frac{q}{2}}\left(A\right)\right)x,x\right\rangle}{\sqrt{\left\langle\Phi\left(f^{p}\left(A\right)\right)x,x\right\rangle\left\langle\Phi\left(g^{q}\left(A\right)\right)x,x\right\rangle}}\right), \tag{3.2}$$

for any $x\in K$, $x\neq 0$.

Proof. From (1.2) we have

$$s\left(a+b-2\sqrt{ab}\right)\leq\frac{1}{p}a+\frac{1}{q}b-a^{\frac{1}{p}}b^{\frac{1}{q}}\leq S\left(a+b-2\sqrt{ab}\right) \tag{3.3}$$

where $a,\ b\geq 0$.

If we choose in (3.3)

$$a=\frac{f^{p}(t)}{\langle\Phi(f^{p}(A))x,x\rangle},b=\frac{g^{q}(t)}{\langle\Phi(g^{q}(A))x,x\rangle},\ t\in I$$

then we get

$$s\left(\frac{f^{p}(t)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{g^{q}(t)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2f^{\frac{p}{2}}(t)\,g^{\frac{q}{2}}(t)}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right)$$

$$\leq\frac{1}{p}\frac{f^{p}(t)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{1}{q}\frac{g^{q}(t)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{f(t)\,g(t)}{\langle\Phi(f^{p}(A))x,x\rangle^{1/p}\langle\Phi(g^{q}(A))x,x\rangle^{1/q}}$$

$$\leq S\left(\frac{f^{p}(t)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{g^{q}(t)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2f^{\frac{p}{2}}(t)\,g^{\frac{q}{2}}(t)}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right)$$

for any $t\in I$ and $x\in K,\ x\neq 0$.

If we use the functional calculus for B we get

$$s\left(\frac{f^{p}(B)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{g^{q}(B)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2f^{\frac{p}{2}}(B)\,g^{\frac{q}{2}}(B)}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right)$$

$$\leq\frac{1}{p}\frac{f^{p}(B)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{1}{q}\frac{g^{q}(B)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{f(B)\,g(B)}{\langle\Phi(f^{p}(A))x,x\rangle^{1/p}\langle\Phi(g^{q}(A))x,x\rangle^{1/q}}$$

$$\leq S\left(\frac{f^{p}(B)}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{g^{q}(B)}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2f^{\frac{p}{2}}(B)\,g^{\frac{q}{2}}(B)}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right)$$

and by taking the map Ψ and the inner product over $y\in K,\ y\neq 0$ we get

$$s\left(\frac{\langle\Psi(f^{p}(B))y,y\rangle}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{\langle\Psi(g^{q}(B))y,y\rangle}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2\left\langle\Psi\left(f^{\frac{p}{2}}(B)\,g^{\frac{q}{2}}(B)\right)y,y\right\rangle}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right)$$

$$\leq\frac{1}{p}\frac{\langle\Psi(f^{p}(B))y,y\rangle}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{1}{q}\frac{\langle\Psi(g^{q}(B))y,y\rangle}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{\langle\Phi(f(B)\,g(B))y,y\rangle}{\langle\Phi(f^{p}(A))x,x\rangle^{1/p}\langle\Phi(g^{q}(A))x,x\rangle^{1/q}}$$

$$\leq S\left(\frac{\langle\Psi(f^{p}(B))y,y\rangle}{\langle\Phi(f^{p}(A))x,x\rangle}+\frac{\langle\Psi(g^{q}(B))y,y\rangle}{\langle\Phi(g^{q}(A))x,x\rangle}-\frac{2\left\langle\Psi\left(f^{\frac{p}{2}}(B)\,g^{\frac{q}{2}}(B)\right)y,y\right\rangle}{\sqrt{\langle\Phi(f^{p}(A))x,x\rangle\langle\Phi(g^{q}(A))x,x\rangle}}\right).$$

Now, if in this inequality we put $y=x$, then we get (3). □

Remark 4. *If we take $f(t)=t^{\alpha}$, $g(t)=t^{\beta}$, then from (3) we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that*

$$2s\left(1-\frac{\left\langle \Phi\left(A^{\frac{\alpha p+\beta q}{2}}\right)x,x\right\rangle}{\sqrt{\left\langle \Phi\left(A^{\alpha p}\right)x,x\right\rangle \left\langle \Phi\left(A^{\beta q}\right)x,x\right\rangle}}\right) \tag{3.4}$$

$$\leq 1-\frac{\left\langle \Phi\left(A^{\alpha+\beta}\right)x,x\right\rangle}{\left\langle \Phi\left(A^{\alpha p}\right)x,x\right\rangle^{1/p}\left\langle \Phi\left(A^{\beta q}\right)x,x\right\rangle^{1/q}}$$

$$\leq 2S\left(1-\frac{\left\langle \Phi\left(A^{\frac{\alpha p+\beta q}{2}}\right)x,x\right\rangle}{\sqrt{\left\langle \Phi\left(A^{\alpha p}\right)x,x\right\rangle \left\langle \Phi\left(A^{\beta q}\right)x,x\right\rangle}}\right),$$

for any $x\in K$, $x\neq 0$, where α, $\beta\in\mathbb{R}\setminus\{0\}$ and $A\in\mathcal{B}^{++}$.

If we take in (3.4) $\alpha=\beta=1$, then we have

$$2s\left(1-\frac{\left\langle \Phi\left(A^{\frac{p+q}{2}}\right)x,x\right\rangle}{\sqrt{\left\langle \Phi\left(A^{p}\right)x,x\right\rangle \left\langle \Phi\left(A^{q}\right)x,x\right\rangle}}\right) \tag{3.5}$$

$$\leq 1-\frac{\left\langle \Phi\left(A^{2}\right)x,x\right\rangle}{\left\langle \Phi\left(A^{p}\right)x,x\right\rangle^{1/p}\left\langle \Phi\left(A^{q}\right)x,x\right\rangle^{1/q}}$$

$$\leq 2S\left(1-\frac{\left\langle \Phi\left(A^{\frac{p+q}{2}}\right)x,x\right\rangle}{\sqrt{\left\langle \Phi\left(A^{p}\right)x,x\right\rangle \left\langle \Phi\left(A^{q}\right)x,x\right\rangle}}\right),$$

for any $x\in K$, $x\neq 0$.

We have the following reverse of Hölder's inequality:

Theorem 4. *Let f, $g: I\to\mathbb{R}$ be continuous functions on the interval I such that*

$$0<m_1\leq f\leq M_1<\infty,\ 0<m_2\leq g\leq M_2<\infty, \tag{3.6}$$

for some constants m_1, m_2, M_1 and M_2, A, B be two selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then

$$0\leq 1-\frac{\left\langle \Phi\left(f(A)\,g(A)\right)x,x\right\rangle}{\left\langle \Phi\left(f^{p}(A)\right)x,x\right\rangle^{1/p}\left\langle \Phi\left(g^{q}(A)\right)x,x\right\rangle^{1/q}} \tag{3.7}$$

$$\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2$$

for any $x\in K$, $x\neq 0$, where $s=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$ and $S=\max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

Proof. It suffices to prove the inequality for $x\in K$, $\|x\|=1$.

By (3.6) we have

$$\langle \Phi (f^p (A)) x, x\rangle \le \langle M_1^p \Phi (1_H) x, x\rangle = M_1^p \langle x, x\rangle = M_1^p$$

and

$$m_1^p = m_1^p \langle x, x\rangle = \langle m_1^p \Phi (1_H) x, x\rangle \le \langle \Phi (f^p (A)) x, x\rangle$$

showing that

$$m_1^p \le \langle \Phi (f^p (A)) x, x\rangle \le M_1^p$$

for $x \in K$, $\|x\| = 1$.

Similarly,

$$m_2^q \le \langle \Phi (g^q (A)) x, x\rangle \le M_2^q$$

for $x \in K$, $\|x\| = 1$.

These imply

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{\langle \Phi (f^p (A)) x, x\rangle} \le \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{\langle \Phi (g^q (A)) x, x\rangle} \le \left(\frac{M_2}{m_2}\right)^q,$$

therefore

$$\begin{aligned}\min\left\{\left(\frac{m_1}{M_1}\right)^p, \left(\frac{m_2}{M_2}\right)^q\right\} &\le \frac{f^p}{\langle \Phi (f^p (A)) x, x\rangle}, \frac{g^q}{\langle \Phi (g^q (A)) x, x\rangle} \\ &\le \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}\end{aligned}$$

for $x \in K$, $\|x\| = 1$.

Let $t \in I$. By (1.3) we have for $\nu = \frac{1}{q}$, $a = \frac{f^p(t)}{\langle \Phi(f^p(A))x,x\rangle}$, $b = \frac{g^q(t)}{\langle \Phi(g^q(A))x,x\rangle}$, $m = \min\left\{\left(\frac{m_1}{M_1}\right)^p, \left(\frac{m_2}{M_2}\right)^q\right\}$ and $M = \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$ that

$$\begin{aligned}&\frac{1}{p}\frac{f^p (t)}{\langle \Phi (f^p (A)) x, x\rangle} + \frac{1}{q}\frac{g^q (t)}{\langle \Phi (g^q (A)) x, x\rangle} \\ &- \frac{f(t) g(t)}{[\langle \Phi (f^p (A)) x, x\rangle]^{1/p} [\langle \Phi (g^q (A)) x, x\rangle]^{1/q}} \\ &\le S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}, \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\} - \min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}, \left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2\end{aligned} \tag{3.8}$$

for $x \in K$, $\|x\| = 1$.

Using the functional calculus for A we have

$$\begin{aligned}&\frac{1}{p}\frac{f^p (A)}{\langle \Phi (f^p (A)) x, x\rangle} + \frac{1}{q}\frac{g^q (A)}{\langle \Phi (g^q (A)) x, x\rangle} \\ &- \frac{f(A) g(A)}{[\langle \Phi (f^p (A)) x, x\rangle]^{1/p} [\langle \Phi (g^q (A)) x, x\rangle]^{1/q}} \\ &\le S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}, \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\} - \min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}, \left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2\end{aligned} \tag{3.9}$$

for $x \in K$, $\|x\| = 1$.

Taking the normalized map Φ in (3.9) and then the inner product for $y \in K$, $\|y\| = 1$ we get

$$\frac{1}{p}\frac{\langle \Phi(f^p(A))y,y\rangle}{\langle \Phi(f^p(A))x,x\rangle}+\frac{1}{q}\frac{\langle \Phi g^q(A)y,y\rangle}{\langle \Phi(g^q(A))x,x\rangle}$$
$$-\frac{\langle \Phi(f(A)g(A))y,y\rangle}{[\langle \Phi(f^p(A))x,x\rangle]^{1/p}[\langle \Phi(g^q(A))x,x\rangle]^{1/q}}$$
$$\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}},\left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\}-\min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}},\left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2,$$

which for $y = x$ produces the desired result (3.7). □

Remark 5. *Assume that the operator A satisfy the inequality*

$$k1_H \leq A \leq K1_H$$

for some numbers k, K with $0 < k < K$. If α, $\beta > 0$ and we take $f(t) = t^\alpha$ and $g(t) = t^\beta$ in Theorem 4, then $m_1 = k^\alpha$, $M_1 = K^\alpha$, $m_2 = k^\beta$ and $M_2 = K^\beta$ and by the inequality (3.7) we have

$$0 \leq 1-\frac{\langle \Phi(A^{\alpha+\beta})x,x\rangle}{\langle \Phi(A^{\alpha p})x,x\rangle^{1/p}\langle \Phi(A^{\beta q})x,x\rangle^{1/q}} \tag{3.10}$$
$$\leq S\left(\max\left\{\left(\frac{K}{k}\right)^{\frac{\alpha p}{2}},\left(\frac{K}{k}\right)^{\frac{\beta q}{2}}\right\}-\min\left\{\left(\frac{k}{K}\right)^{\frac{\alpha p}{2}},\left(\frac{k}{K}\right)^{\frac{\beta q}{2}}\right\}\right)^2$$

for any $x \in K$, $x \neq 0$.

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