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SOME INEQUALITIES FOR ISOTONIC LINEAR FUNCTIONALS

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ABSTRACT. In this paper is given a new variant of Minkowski-type inequality for isotonic linear functionals and then some variants of Qi's inequality for isotonic linear functionals using a new Young-type inequality. Also several applications are presented.

1. INTRODUCTION

In [1] are given new results which extend many generalizations of Young's inequality given before. We recall these results below in order to use them in the next sections.

Theorem 1. *Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

The following important definition is given in [3], [5] and we will recall it here.

Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbf{R}$ having the following properties:

(L1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $(af + bg) \in L$.

(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An *isotonic linear functional* is a functional $A : L \to \mathbf{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbf{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

The mapping A is said to be *normalised* if

(A3) $A(\mathbf{1}) = 1$.

The following Holder-type inequalities are obtained from Theorem 1.1 which is given in [1] and will be used in the next sections as an important tool in our demonstrations.

Theorem 2. *If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If f^p, g^q, fg, $f^{p\tau}$, $g^{q(1-\tau)} \in L$, $A(f^p) > 0$, $A(g^q) > 0$, $p\tau > 1$, $\tau < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f and g are positive functions then:*

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$$\frac{1}{p\tau}\left[1-\frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right] < 1-\frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)} <$$
$$< \frac{1}{q(1-\tau)}\left[1-\frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right].$$

2. A refinement of Minkowski's inequality for isotonic linear functional

Using inequalities from Theorem 2 we can obtain some extensions of the classical Minkowski's inequality for isotonic linear functionals.

Theorem 3. *Let* $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, *$L$ satisfying conditions L1, L2 and A satisfying A1, A2 on the set E. Considering the nonnegative functions f, h with* f^p, h^p, $(f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}$, $(f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}$, $(f+h)^{p-1}f$, $(f+h)^{p-1}h \in L$ *and* $A(f^p)>0$, $A(h^p)>0$, $A((f+h)^p)>0$ *we will have,*

$$A^{\frac{1}{p}}((f+h)^p) < A^{\frac{1}{p}}(f^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$
$$+A^{\frac{1}{p}}(h^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right], \tag{2.1}$$

and

$$A^{\frac{1}{p}}(f^p)\left[1-\frac{q_1}{q}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$
$$+A^{\frac{1}{p}}(h^p)\left[1-\frac{q_1}{q}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right] < A^{\frac{1}{p}}((f+h)^p). \tag{2.2}$$

Proof. We will check only inequality (2.1). Applying inequality from Theorem 2 first time for $f$and $\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}$ and then for h and $\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}$ we will have:

$$A^{\frac{1}{p}}((f+h)^p) = A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}(f+h)\right) =$$
$$= A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}f\right)+A\left(\frac{(f+h)^{p-1}}{A^{\frac{1}{q}}((f+h)^{q(p-1)})}h\right) \leq$$
$$< A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}\left(\frac{(f+h)^{q(p-1)}}{A((f+h)^{q(p-1)})}\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$
$$+A^{\frac{1}{p}}(h^p)A^{\frac{1}{q}}\left(\frac{(f+h)^{q(p-1)}}{A((f+h)^{q(p-1)})}\right)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right] =$$

$$= A^{\frac{1}{p}}(f^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(f^p)}\right)\right]+$$

$$+A^{\frac{1}{p}}(h^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left((f+h)^{\frac{p}{q_1}}h^{\frac{p}{p_1}}\right)}{A^{\frac{1}{q_1}}((f+h)^p)A^{\frac{1}{p_1}}(h^p)}\right)\right].$$

∎

This result allow us to give a refinement of Minkowski's inequality for the cases of the time scales Cauchy delta, Cauchy nabla and α-diamond integrals.

Corollary 1. *(i) Let* $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, *and the positive functions* $f,h \in C_{rd}([a,b),\mathbf{R})$. *The following inequality takes place:*

$$\left(\int_a^b (f(x)+h(x))^p \Delta x\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b (f(x)+h(x))^p\Delta x\right)^{\frac{1}{q_1}}\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p_1}}}\right)\right]+$$

$$+\left(\int_a^b h^p(x)\Delta x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} h^{\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b (f(x)+h(x))^p\Delta x\right)^{\frac{1}{q_1}}\left(\int_a^b h^p(x)\Delta x\right)^{\frac{1}{p_1}}}\right)\right].$$

(ii) Let $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, *and the positive functions* $f,h \in C_{ld}((a,b],\mathbf{R})$. *The following inequality takes place:*

$$\left(\int_a^b (f(x)+h(x))^p \nabla x\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b f^p(x)\nabla x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x)\nabla x}{\left(\int_a^b (f(x)+h(x))^p\nabla x\right)^{\frac{1}{q_1}}\left(\int_a^b f^p(x)\nabla x\right)^{\frac{1}{p_1}}}\right)\right]+$$

$$+\left(\int_a^b h^p(x)\nabla x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} h^{\frac{p}{p_1}}(x)\nabla x}{\left(\int_a^b (f(x)+h(x))^p\nabla x\right)^{\frac{1}{q_1}}\left(\int_a^b h^p(x)\nabla x\right)^{\frac{1}{p_1}}}\right)\right].$$

(iii) Let $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$ *and the positive functions* $f,h:[a,b]\to\mathbf{R}$ *be* $\diamond_\alpha$*-integrable functions. The following inequality takes place:*

$$\left(\int_a^b (f(x)+h(x))^p \diamond_\alpha x\right)^{\frac{1}{p}} <$$

$$< \left(\int_a^b f^p(x)\diamond_\alpha x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}} f^{\frac{p}{p_1}}(x)\diamond_\alpha x}{\left(\int_a^b (f(x)+h(x))^p\diamond_\alpha x\right)^{\frac{1}{q_1}}\left(\int_a^b f^p(x)\diamond_\alpha x\right)^{\frac{1}{p_1}}}\right)\right]+$$

$$+\left(\int_a^b h^p(x)\diamond_\alpha x\right)^{\frac{1}{p}}\left[1-\frac{p_1}{p}\left(1-\frac{\int_a^b (f(x)+h(x))^{\frac{p}{q_1}}h^{\frac{p}{p_1}}(x)\diamond_\alpha x}{\left(\int_a^b (f(x)+h(x))^p\diamond_\alpha x\right)^{\frac{1}{q_1}}\left(\int_a^b h^p(x)\diamond_\alpha x\right)^{\frac{1}{p_1}}}\right)\right].$$

3. Some variants of Qi's inequality for isotonic linear functionals

In this section we give several variants of some inequalities from [9] in the case of isotonic linear functionals for $p>1$ using the corresponding Holder's inequalities from Theorem 2.

Lemma 1. *Let E, L and A be such that L1, L2, A1, A2 are satisfied.*

If $f,\ g,\ \frac{f^p}{g^{p-1}},\ f^{\frac{p}{p_1}}g^{1-\frac{p}{p_1}}\in L$ are positive functions with $A(g)>0,\ A\left(\frac{f^p}{g^{p-1}}\right)>0$ then

$$\left[1-\frac{p_1}{p}\left(1-\frac{A\left(f^{\frac{p}{p_1}}g^{1-\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{p-1}}\right)A^{\frac{1}{q_1}}(g)}\right)\right]^p A\left(\frac{f^p}{g^{p-1}}\right)>\frac{A^p(f)}{A^{p-1}(g)},$$

where $p>p_1>1$ with $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We apply Holder' s inequality from Theorem 2 when $p>1$ and $f,\ g,\ \frac{f^p}{g^{p-1}},\ f^{\frac{p}{p_1}}g^{1-\frac{p}{p_1}}\in L$ are positive functions, obtaining:

$$A(f)=A\left(\frac{f}{g^{\frac{1}{q}}}g^{\frac{1}{q}}\right)<A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{1}{q}}(g)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(\frac{f^{\frac{p}{p_1}}}{g^{\frac{p}{p_1q}}}g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{1}{q_1}}(g)}\right)\right].$$

Then we take the p-th power on both sides of the inequalities and have:

$$A^p(f)<A\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{p}{q}}(g)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(\frac{f^{\frac{p}{p_1}}}{g^{\frac{p}{p_1q}}}g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{1}{q_1}}(g)}\right)\right]^p.$$

∎

Theorem 4. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If $f,\ f^{p+2},\ f^{\frac{p+2}{p_1}}\in L$, f is positive function and $A(f)\geq A^2(\mathbf{1})$ then*

$$A^{p-1}(\mathbf{1})A(f^{p+2})\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2}>A^{p+1}(f),$$

takes place for $p+2>p_1>1$.

Proof. By Lemma 1 and hypothesis we have,

$$A(f^{p+2})\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2}=$$

$$= A\left(\frac{f^{p+2}}{1^{p+1}}\right)\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^{p+2} >$$

$$> \frac{A^{p+2}(f)}{A^{p+1}(\mathbf{1})} = \frac{A^{p+1}(f)A(f)}{A^{p-1}(\mathbf{1})A^2(\mathbf{1})} \geq \frac{A^{p+1}(f).}{A^{p-1}(\mathbf{1})}.$$

∎

Consequence 1. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, f^{p+2}, $f^{\frac{p+2}{p_1}} \in L$, f is positive and in addition A is normalised and $A(f) \geq 1$ then*

$$A(f^{p+2})\left[1-\frac{p_1}{p+2}\left(1-\frac{A(f^{\frac{p+2}{p_1}})}{A^{\frac{1}{p_1}}(f^{p+2})}\right)\right]^{p+2} > A^{p+1}(f),$$

takes place for $p+2 > p_1 > 1$.

As applications, we will give some refinements of several inequalities given by Qi and Yin, [9], in the cases of delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals.

Remark 1. *Let $a, b \in \mathbf{R}$, $a < b$. If $f \in C([a,b])$ is strictly positive and*

$$\int_a^b f(x)dx \geq (b-a)^2$$

then

$$\int_a^b f^{p+2}(x)dx\left[1-\frac{p_1}{p+2}\left(1-\frac{\int_a^b f^{\frac{p+2}{p_1}}(x)dx}{(b-a)^{\frac{1}{q_1}}(\int_a^b f^{p+2}(x)dx)^{\frac{1}{p_1}}}\right)\right]^{p+2} >$$

$$> \frac{1}{(b-a)^{p-1}}\left[\int_a^b f(x)dx\right]^{p+1},$$

where $p+2 > p_1 > 1$.

Moreover, when we have delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals similary inequalities can be stated like above.

Lemma 2. *Let E, L and A be such that L1, L2, A1, A2 are satisfied on the set E. If $1 < p_1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$ and $f, g, f^{\frac{p}{p_1}}g^{1-\frac{p}{p_1}}, \frac{f^p}{g^{p-1}} \in L$ are positive functions and $A(\frac{f^p}{g^{p-1}}) > 0$, $A(g) > 0$ then*

$$\frac{A^p(f)}{A^{p-1}(g)} > A\left(\frac{f^p}{g^{p-1}}\right)\left[1-\frac{q_1}{q}\left(1-\frac{A(f^{\frac{p}{p_1}}g^{1-\frac{p}{p_1}})}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{p-1}}\right)A^{\frac{1}{q_1}}(g)}\right)\right]^p.$$

Proof. We apply Holder's inequality from Theorem 2 when $g, f^{\frac{p}{p_1}} g^{1-\frac{p}{p_1}}, \frac{f^p}{g^{p-1}} \in L$ are positive functions, obtaining:

$$A(f) = A\left(\frac{f}{g^{\frac{1}{q}}} g^{\frac{1}{q}}\right) > A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q}}(g)\left[1 - \frac{q_1}{q}\left(1 - \frac{A\left(\left(\frac{f}{g^{\frac{1}{q}}}\right)^{\frac{p}{p_1}} g^{\frac{1}{q_1}}\right)}{A^{\frac{1}{p_1}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q_1}}(g)}\right)\right].$$

Then we take the p-th power on both sides of the inequalities and we obtain by calculus the desired inequality. ∎

The inequality from Lemma 2 can be written again for particular isotonic linear functionals, see for example [3] like below:

Consequence 2. *Let $a, b \in \mathbf{T}$, and $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$. If $f, g \in C_{rd}(\mathbf{T}, \mathbf{R})$ are positive then*

$$\frac{(\int_a^b f(x)\Delta x)^p}{(\int_a^b g(x)\Delta x)^{p-1}} > \int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x\left[1 - \frac{q_1}{q}\left(1 - \frac{\int_a^b f^{\frac{p}{p_1}}(x) g^{1-\frac{p}{p_1}}(x)\Delta x}{\left(\int_a^b \frac{f^p(x)}{g^{p-1}(x)}\Delta x\right)^{\frac{1}{p_1}}\left(\int_a^b g(x)\Delta x\right)^{\frac{1}{q_1}}}\right)\right]^p.$$

A new inequality for isotonic linear functional is the following:

Theorem 5. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If $f, f^p, f^{\frac{p_1}{p}} \in L$, f is positive, $A(f) > 0$ and $A(f) \geq A^{p-1}(\mathbf{1})$ then*

$$A^{p-1}(f) < A(f^p)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f^p) A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p$$

when $1 < p_1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Proof. By Lemma 1 and hypothesis we have,

$$A(f^p) = A\left(\frac{f^p}{\mathbf{1}^{p-1}}\right) = A\left(\frac{f^p}{1^{\frac{p}{q}}}\right)$$

and

$$\frac{A^p(f)}{A^{p-1}(\mathbf{1})} < A\left(\frac{f^p}{\mathbf{1}^{p-1}}\right)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f) A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p$$

or

$$A^{p-1}(f) \leq \frac{A(f)A^{p-1}(f)}{A^{p-1}(\mathbf{1})} < A(f^p)\left[1 - \frac{p_1}{p}\left(1 - \frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f) A^{\frac{1}{q_1}}(\mathbf{1})}\right)\right]^p.$$

∎

If, in addition, the functional is normalised then previous inequality becomes:

Consequence 3. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, f^p, $f^{\frac{p_1}{p}} \in L$, f is positive, $A(f)>0$, $A(f)\geq 1$ and in addition, A is normalised, then*

$$A^{p-1}(f) < A(f^p)\left[1-\frac{p_1}{p}\left(1-\frac{A\left(f^{\frac{p}{p_1}}\right)}{A^{\frac{1}{p_1}}(f^p)}\right)\right]^p$$

when $1<p_1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$.

As an application of Consequence 3 for Riemann integrals we obtain:

Consequence 4. *(i) Let $a, b \in \mathbf{R}$, $a<b$. If $f \in C([a,b])$ is positive, and*

$$\int_a^b f(x)dx \geq (b-a)^{p-1}$$

then

$$\left(\int_a^b f(x)dx\right)^{p-1} < \int_a^b f^p(x)dx\left[1-\frac{p_1}{p}\left(\frac{\int_a^b f^{\frac{p}{p_1}}(x)dx}{(b-a)^{\frac{1}{q_1}}(\int_a^b f(x)dx)^{\frac{1}{p_1}}}\right)\right]^p,$$

when $1<p_1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$.

(ii) In the case of delta time scale integrals, Cauchy nabla time-scales integrals and Cauchy α-diamond time scale integrals similary inequalities can be stated as above.

Now we take into account a particular case when A is a normalised functional and f,g are two applications such that $f,g : E \to \mathbf{R}$ like below, and we will obtain new variant of inequaliities from Lemma 2 and from [9] by using Theorem 2, see [7].

Lemma 3. *Let $A : L \to \mathbf{R}$ be an normalised isotonic linear functional and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,g : E \to \mathbf{R}$ are so that $f, g, \frac{f^p}{g^{p-1}} \in L$ and $0<m_1\leq f\leq M_1<\infty$, $0<m_2\leq g\leq M_2<\infty$ for some constants m_1, M_1, m_2, M_2 then we have:*

$$\frac{A^p(f)}{A^{p-1}(g)}K^{Up}\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^p\right) \geq A\left(\frac{f^p}{g^{p-1}}\right),$$

where $U=\max\{\frac{1}{p},\frac{1}{q}\}$ and K is the Kantorovich's ratio defined by $K(h)=\frac{(h+1)^2}{4h}$, $h>0$.

Proof. We use Theorem 2 from [7] where we replace f by $\frac{f}{g^{\frac{1}{q}}}$ and g by $g^{\frac{1}{q}}$ obtaining:

$$A(f)K^U\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^p\right) = A\left(\frac{f^p}{g^{\frac{1}{q}}}g^{\frac{1}{q}}\right)K^U\left(\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^p\right) \geq$$

$$\geq A^{\frac{1}{p}}\left(\frac{f^p}{g^{p-1}}\right)A^{\frac{1}{q}}(g).$$

Now we take the p-th power on both sides of the inequalities and we get the conclusion.

∎

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