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INEQUALITIES FOR POSITIVE LINEAR MAPS OF SELFADJOINT OPERATORS VIA A KITTANEH-MANASRAH RESULT

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ABSTRACT. Some inequalities for positive linear maps of positive selfadjoint linear operators in Hilbert spaces via a Kittaneh-Manasrah result, are given. Operator and vector inequalities involving the weighted operator geometric mean are also obtained. Reverses of the celebrated Ando's inequality are provided.

1. INTRODUCTION

As is well-known, the famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.5) is also called as *ν-weighted arithmetic-geometric mean inequality.*

Kittaneh and Manasrah [9], [10] provided a refinement and a reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.2}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.2) to an identity and is of no interest.

We observe that, if a, $b \in [m, M] \subset (0, \infty)$, then $\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{M} - \sqrt{m}$ and by (1.2) we obtain the following simple reverse of Young inequality

$$(1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{M}-\sqrt{m}\right)^2. \tag{1.3}$$

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^{+}(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [16, p. 18]) we can introduce:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

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for any λ, $\mu \in \mathbb{C}$ *and* A, $B \in \mathcal{B}(H)$. *The linear map* $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ *is positive if it preserves the operator order, i.e. if* $A \in \mathcal{B}^+(H)$ *then* $\Phi(A) \in \mathcal{B}^+(K)$. *We write* $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. *The linear map* $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ *is normalised if it preserves the identity operator, i.e.* $\Phi(1_H) = 1_K$. *We write* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

The following refinement of Cauchy-Bunyakowsky-Schwarz inequality for n-tuples of nonnegative real numbers $(a_1, ..., a_n)$, $(b_1, ..., b_n)$ was established by Callebaut in 1965 [1] and can be stated as follows:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^{2(1-\lambda)} b_i^{2\lambda} \sum_{i=1}^{n} a_i^{2\lambda} b_i^{2(1-\lambda)} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2, \tag{1.4}$$

for any $\lambda \in [0, 1]$.

In [8], by the use of (1.2), we obtained the following refinement and reverse of the second part of Callebaut inequality (1.4) in the case of positive maps:

$$\begin{aligned} 0 &\leq 2r\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle - \left\langle \Phi\left(f(A)g(A)\right)x, x\right\rangle^2\right) \\ &\leq \left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle \\ &- \left\langle \Phi\left(f^{2\nu}(A)g^{2(1-\nu)}(A)\right)x, x\right\rangle \left\langle \Phi\left(f^{2(1-\nu)}(A)g^{2\nu}(A)\right)x, x\right\rangle \\ &\leq R\left(\left\langle \Phi\left(g^2(A)\right)x, x\right\rangle \left\langle \Phi\left(f^2(A)\right)x, x\right\rangle - \left\langle \Phi\left(f(A)g(A)\right)x, x\right\rangle^2\right), \end{aligned} \tag{1.5}$$

for any $x \in K$, where f, $g : I \to \mathbb{R}$ are continuous functions on the interval I, A is a selfadjoint operator with $\operatorname{Sp}(A) \subset I$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$, $R = \max\{1-\nu, \nu\}$ and $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$.

In [8] we have proved the following reverse of Hölder's inequality as well:

Let f, $g : I \to \mathbb{R}$ be continuous functions on the interval I such that

$$0 < m_1 \leq f \leq M_1 < \infty,\ 0 < m_2 \leq g \leq M_2 < \infty, \tag{1.6}$$

for some constants m_1, m_2, M_1 and M_2, A, B be two selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\begin{aligned} 0 &\leq 1 - \frac{\left\langle \Phi\left(f(A)g(A)\right)x, x\right\rangle}{\left\langle \Phi\left(f^p(A)\right)x, x\right\rangle^{1/p} \left\langle \Phi\left(g^q(A)\right)x, x\right\rangle^{1/q}} \\ &\leq S\left(\max\left\{\left(\frac{M_1}{m_1}\right)^{\frac{p}{2}}, \left(\frac{M_2}{m_2}\right)^{\frac{q}{2}}\right\} - \min\left\{\left(\frac{m_1}{M_1}\right)^{\frac{p}{2}}, \left(\frac{m_2}{M_2}\right)^{\frac{q}{2}}\right\}\right)^2 \end{aligned} \tag{1.7}$$

for any $x \in K$, $x \neq 0$, where $s = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $S = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

For some recent inequalities for positive maps of Hilbert space operators see [4]-[7], [11]-[15], [17]-[19] and the references therein.

In this paper, by the use of Kittaneh-Manasrah inequality (1.2) we establish some other inequalities for positive linear maps of positive selfadjoint linear operators in Hilbert spaces. Operator and vector inequalities involving the weighted operator geometric mean are also provided. Reverses of the celebrated Ando's inequality are given.

2. Inequalities for Operator Geometric Mean

Throughout this section A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^\nu A^{1/2},\ \nu \in [0,1]$$

the *weighted operator geometric mean.* When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

Ando's inequality says that if A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\Phi(A \sharp_\nu B) \leq \Phi(A) \sharp_\nu \Phi(B) \tag{2.1}$$

for any $\nu \in [0,1]$.

We have the following reverse of Ando's inequality:

Theorem 1. *Let A, B be positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$, then we have the following reverse of (2.1)*

$$\begin{aligned} 0 &\leq \Phi(A) \sharp_\nu \Phi(B) - \Phi(A \sharp_\nu B) \\ &\leq 4\left|\nu - \frac{1}{2}\right| \Phi(A) \nabla \Phi(B) + 2r\Phi(A) \sharp \Phi(B) - 2R\Phi(A \sharp B) \\ &\leq 2R\left(\Phi(A) \nabla \Phi(B) - \Phi(A \sharp B)\right), \end{aligned} \tag{2.2}$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. From $a = 1$ and $b = x > 0$ in (1.2) we have

$$2r\left(\frac{x+1}{2} - \sqrt{x}\right) \leq (1-\nu) + \nu x - x^\nu \leq R\left(\frac{x+1}{2} - \sqrt{x}\right),$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$, $R = \max\{1-\nu, \nu\}$.

If we use the functional calculus for the positive operator X, then we obtain

$$2r\left(\frac{X + 1_H}{2} - X^{1/2}\right) \leq (1-\nu) + \nu X - X^\nu \leq R\left(\frac{X + 1_H}{2} - X^{1/2}\right), \tag{2.3}$$

which by taking $X = C^{-1/2}DC^{-1/2}$ produces

$$2r\left(\frac{C^{-1/2}DC^{-1/2}+1_H}{2}-\left(C^{-1/2}DC^{-1/2}\right)^{1/2}\right) \tag{2.4}$$
$$\leq (1-\nu)+\nu C^{-1/2}DC^{-1/2}-\left(C^{-1/2}DC^{-1/2}\right)^{\nu}$$
$$\leq R\left(\frac{C^{-1/2}DC^{-1/2}+1_H}{2}-\left(C^{-1/2}DC^{-1/2}\right)^{1/2}\right),$$

provided C, D are positive invertible operators.

Now, if we multiply both sides of (2.4) by $C^{1/2}$, then we get

$$2r\left(C\nabla D - C\sharp D\right) \leq C\nabla_{\nu} D - C\sharp_{\nu} D \leq 2R\left(C\nabla D - C\sharp D\right), \tag{2.5}$$

where C, D are positive invertible operators and $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$, $R = \max\{1-\nu, \nu\}$.

Further, if we take in (2.5) $C = A$ and $D = B$ and then apply the positive map Φ, then we get

$$2r\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right) + \Phi(A\sharp_{\nu} B) \tag{2.6}$$
$$\leq \Phi(A)\nabla_{\nu}\Phi(B)$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right) + \Phi(A\sharp_{\nu} B),$$

where A, B are positive invertible operators and $\nu \in [0,1]$.

If we write the inequality (2.5) for $C = \Phi(A)$ and $D = \Phi(B)$, then we also have

$$2r\left(\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B)\right) + \Phi(A)\sharp_{\nu}\Phi(B) \tag{2.7}$$
$$\leq \Phi(A)\nabla_{\nu}\Phi(B)$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B)\right) + \Phi(A)\sharp_{\nu}\Phi(B),$$

where A, B are positive invertible operators and $\nu \in [0,1]$.

Now, if we use the first inequality in (2.7) and the second inequality in (2.6), then we conclude that

$$2r\left(\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B)\right) + \Phi(A)\sharp_{\nu}\Phi(B)$$
$$\leq \Phi(A)\nabla_{\nu}\Phi(B)$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right) + \Phi(A\sharp_{\nu} B),$$

which implies that

$$2r\left(\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B)\right) + \Phi(A)\sharp_{\nu}\Phi(B)$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right) + \Phi(A\sharp_{\nu} B),$$

namely

$$\Phi(A)\sharp_{\nu}\Phi(B) - \Phi(A\sharp_{\nu} B) \tag{2.8}$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right) - 2r\left(\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B)\right)$$
$$\leq 2R\left(\Phi(A)\nabla\Phi(B) - \Phi(A\sharp B)\right)$$

since

$$\Phi(A)\nabla\Phi(B) - \Phi(A)\sharp\Phi(B) \geq 0.$$

Observe that

$$2(R-r) = 4\left|\nu - \frac{1}{2}\right|,\ \nu \in [0,1],$$

then by (2.8) we get the desired inequality (2.2). □

Corollary 1. *Let A, B be positive operators such that there exist the positive numbers $0 < m < M$ with the property*

$$mA \leq B \leq MA.$$

Then for any $\Phi \in \mathfrak{P}\left[\mathcal{B}\left(H\right), \mathcal{B}\left(K\right)\right]$, we have

$$0 \leq \Phi\left(A\right) \sharp_{\nu} \Phi\left(B\right) - \Phi\left(A \sharp_{\nu} B\right) \leq RC\left(m, M\right) \Phi\left(A\right), \tag{2.9}$$

where $\nu \in [0,1]$, $R = \max\{1-\nu, \nu\}$ and

$$C\left(m, M\right) := \begin{cases} \left(1-\sqrt{m}\right)^2 & \text{if } M < 1 \\ \max\left\{\left(1-\sqrt{m}\right)^2, \left(\sqrt{M}-1\right)^2\right\} & \text{if } m \leq 1 \leq M \\ \left(\sqrt{M}-1\right)^2 & \text{if } 1 < m. \end{cases}$$

Proof. It is clear that if $x \in [m, M] \subset (0, \infty)$, then

$$\max_{x \in [m,M]} \left(\sqrt{x}-1\right)^2 = C\left(m, M\right).$$

This implies that

$$\frac{x+1}{2} - \sqrt{x} \leq \frac{1}{2} C\left(m, M\right) \tag{2.10}$$

if $x \in [m, M]$.

Now, since $mA \leq B \leq MA$, then by multiplying both sides with $A^{-1/2}$ we get $m1_H \leq A^{-1/2}BA^{-1/2} \leq M1_H$, which implies, by (2.10), that

$$\frac{A^{-1/2}BA^{-1/2} + 1_H}{2} - \left(A^{-1/2}BA^{-1/2}\right)^{1/2} \leq \frac{1}{2} C\left(m, M\right) 1_H.$$

If we multiply this inequality both sides by $A^{1/2}$ we get

$$A \nabla B - A \sharp B \leq \frac{1}{2} C\left(m, M\right) A$$

and by taking the positive map Φ, we deduce

$$\Phi\left(A\right) \nabla \Phi\left(B\right) - \Phi\left(A \sharp B\right) \leq \frac{1}{2} C\left(m, M\right) \Phi\left(A\right),$$

which, by (2.2), produces the desired result (2.9). □

3. Vector Inequalities

In this section we establish some vector inequalities for positive invertible self-adjoint operators as follows:

Theorem 2. *Let A, B be positive invertible operators on a complex Hilbert space $\left(H, \langle \cdot, \cdot \rangle\right)$. If Φ, $\Psi \in \mathfrak{P}_N\left[\mathcal{B}\left(H\right), \mathcal{B}\left(K\right)\right]$, then we have*

$$\begin{aligned} & 2r\left(\langle \left(\Phi\left(A\right) \nabla \Psi\left(B\right)\right) x, x \rangle - \left\langle \Phi\left(A^{1/2}\right) x, x \right\rangle \left\langle \Psi\left(B^{1/2}\right) x, x \right\rangle\right) \\ & \leq \langle \left(\Phi\left(A\right) \nabla_{\nu} \Psi\left(B\right)\right) x, x \rangle - \left\langle \Phi\left(A^{1-\nu}\right) x, x \right\rangle \langle \Psi\left(B^{\nu}\right) x, x \rangle \\ & \leq 2R\left(\langle \left(\Phi\left(A\right) \nabla \Psi\left(B\right)\right) x, x \rangle - \left\langle \Phi\left(A^{1/2}\right) x, x \right\rangle \left\langle \Psi\left(B^{1/2}\right) x, x \right\rangle\right) \end{aligned} \tag{3.1}$$

for any $x \in K$, $\|x\| = 1$, where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Proof. From (1.2) we have for any $t,\ s\in\mathbb{R}_+$ that

$$r\left(t+s-2\sqrt{ts}\right)\le(1-\nu)\,t+\nu s-t^{1-\nu}s^{\nu}\le R\left(t+s-2\sqrt{ts}\right),\tag{3.2}$$

where $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

Fix $s\in\mathbb{R}_+$, then by the functional calculus for the operator A we have

$$\begin{aligned}r\left(A+s1_H-2\sqrt{s}A^{1/2}\right)&\le(1-\nu)\,A+\nu s1_H-s^{\nu}A^{1-\nu}\\&\le R\left(A+s1_H-2\sqrt{s}A^{1/2}\right)\end{aligned}$$

and by taking the normalised positive map Φ and the inner product for $x\in K$, $\|x\|=1$, we have

$$\begin{aligned}&r\left(\langle\Phi(A)\,x,x\rangle+s-2\sqrt{s}\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\right)\\&\le(1-\nu)\,\langle\Phi(A)\,x,x\rangle+\nu s-s^{\nu}\left\langle\Phi\left(A^{1-\nu}\right)x,x\right\rangle\\&\le R\left(\langle\Phi(A)\,x,x\rangle+s-2\sqrt{s}\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\right)\end{aligned}$$

for any $s\in\mathbb{R}_+$.

Using the functional calculus for the operator B we have

$$\begin{aligned}&r\left(\langle\Phi(A)\,x,x\rangle\,1_H+B-2\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle B^{1/2}\right)\\&\le(1-\nu)\,\langle\Phi(A)\,x,x\rangle\,1_H+\nu B-\left\langle\Phi\left(A^{1-\nu}\right)x,x\right\rangle B^{\nu}\\&\le R\left(\langle\Phi(A)\,x,x\rangle\,1_H+B-2\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle B^{1/2}\right)\end{aligned}$$

for any $x\in K$, $\|x\|=1$.

If we take the normalised positive map Ψ and the inner product for $y\in K$, $\|y\|=1$, then we have

$$\begin{aligned}&r\left(\langle\Phi(A)\,x,x\rangle+\langle\Psi(B)\,y,y\rangle-2\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Psi\left(B^{1/2}\right)y,y\right\rangle\right)\\&\le(1-\nu)\,\langle\Phi(A)\,x,x\rangle+\nu\,\langle\Psi(B)\,y,y\rangle-\langle\Psi(B^{\nu})\,y,y\rangle\left\langle\Phi\left(A^{1-\nu}\right)x,x\right\rangle\\&\le R\left(\langle\Phi(A)\,x,x\rangle+\langle\Psi(B)\,y,y\rangle-2\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Psi\left(B^{1/2}\right)y,y\right\rangle\right)\end{aligned}$$

for any $x,\ y\in K$, $\|x\|=\|y\|=1$.

Finally, if we put $y=x$ above, then we get the desired result (3.1). □

Remark 1. *If we take in (3.1) $\Phi=\Psi$, then we get*

$$\begin{aligned}&2r\left(\langle(\Phi(A)\,\nabla\Phi(B))\,x,x\rangle-\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Phi\left(B^{1/2}\right)x,x\right\rangle\right)\\&\le\langle(\Phi(A)\,\nabla_{\nu}\Phi(B))\,x,x\rangle-\left\langle\Phi\left(A^{1-\nu}\right)x,x\right\rangle\langle\Phi(B^{\nu})\,x,x\rangle\\&\le 2R\left(\langle(\Phi(A)\,\nabla\Phi(B))\,x,x\rangle-\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Phi\left(B^{1/2}\right)x,x\right\rangle\right)\end{aligned}\tag{3.3}$$

for any $x\in K$, $\|x\|=1$.

If we choose in (3.1) $B=A$, then we get

$$\begin{aligned}&2r\left(\langle(\Phi(A)\,\nabla\Psi(A))\,x,x\rangle-\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Psi\left(A^{1/2}\right)x,x\right\rangle\right)\\&\le\langle(\Phi(A)\,\nabla_{\nu}\Psi(A))\,x,x\rangle-\left\langle\Phi\left(A^{1-\nu}\right)x,x\right\rangle\langle\Psi(A^{\nu})\,x,x\rangle\\&\le 2R\left(\langle(\Phi(A)\,\nabla\Psi(A))\,x,x\rangle-\left\langle\Phi\left(A^{1/2}\right)x,x\right\rangle\left\langle\Psi\left(A^{1/2}\right)x,x\right\rangle\right)\end{aligned}\tag{3.4}$$

for any $x \in K$, $\|x\| = 1$.

Moreover, if we put in (3.4) $\Phi = \Psi$, *then we have the interesting result*

$$\begin{aligned} &2r\left(\langle \Phi(A)x,x\rangle - \left\langle \Phi\left(A^{1/2}\right)x,x\right\rangle^2\right) \\ &\le \langle \Phi(A)x,x\rangle - \left\langle \Phi\left(A^{1-\nu}\right)x,x\right\rangle \langle \Phi(A^\nu)x,x\rangle \\ &\le 2R\left(\langle \Phi(A)x,x\rangle - \left\langle \Phi\left(A^{1/2}\right)x,x\right\rangle^2\right) \end{aligned} \tag{3.5}$$

for any $x \in K$, $\|x\| = 1$.

If we take in (3.5) $A = |T|^2$ *where* $T \in \mathcal{B}(H)$, *then we have*

$$\begin{aligned} &2r\left(\left\langle \Phi\left(|T|^2\right)x,x\right\rangle - \langle \Phi(|T|)x,x\rangle^2\right) \\ &\le \left\langle \Phi\left(|T|^2\right)x,x\right\rangle - \left\langle \Phi\left(|T|^{2(1-\nu)}\right)x,x\right\rangle \left\langle \Phi\left(|T|^{2\nu}\right)x,x\right\rangle \\ &\le 2R\left(\left\langle \Phi\left(|T|^2\right)x,x\right\rangle - \langle \Phi(|T|)x,x\rangle^2\right) \end{aligned} \tag{3.6}$$

for any $x \in K$, $\|x\| = 1$, *where* $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$.

We also have the complementary result:

Theorem 3. *Let* A, B *be positive invertible operators on a complex Hilbert space* $(H, \langle \cdot,\cdot \rangle)$. *If* Φ, $\Psi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, *then we have*

$$\begin{aligned} &2r\left(\langle (\Phi(A)\nabla\Psi(B))x,x\rangle - \langle \Psi(B)x,x\rangle^{1/2}\left\langle \Phi\left(A^{1/2}\right)x,x\right\rangle\right) \\ &\le \langle (\Phi(A)\nabla_\nu\Psi(g(A)))x,x\rangle - \langle \Psi(B)x,x\rangle^\nu \left\langle \Phi\left(A^{1-\nu}\right)x,x\right\rangle \\ &\le 2R\left(\langle (\Phi(A)\nabla\Psi(B))x,x\rangle - \langle \Psi(B)x,x\rangle^{1/2}\left\langle \Phi\left(A^{1/2}\right)x,x\right\rangle\right), \end{aligned} \tag{3.7}$$

for any $x \in K$, $\|x\| = 1$, *where* $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$.

Proof. From (1.2) we have for any $t \in \mathbb{R}_+$

$$\begin{aligned} &r\left(t + \langle \Psi(B)y,y\rangle - 2\sqrt{t}\langle \Psi(B)y,y\rangle^{1/2}\right) \\ &\le (1-\nu)t + \nu\langle \Psi(B)y,y\rangle - t^{1-\nu}\langle \Psi(B)y,y\rangle^\nu \\ &\le R\left(t + \langle \Psi(B)y,y\rangle - 2\sqrt{t}\langle \Psi(B)y,y\rangle^{1/2}\right) \end{aligned} \tag{3.8}$$

where $y \in K$, $\|y\| = 1$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

If we use the functional calculus for the operator A then we have by (3.8) that

$$\begin{aligned} &r\left(A + \langle \Psi(B)y,y\rangle 1_H - 2\langle \Psi(B)y,y\rangle^{1/2}A^{1/2}\right) \\ &\le (1-\nu)A + \nu\langle \Psi(B)y,y\rangle 1_H - \langle \Psi(B)y,y\rangle^\nu A^{1-\nu} \\ &\le R\left(A + \langle \Psi(B)y,y\rangle 1_H - 2\langle \Psi(B)y,y\rangle^{1/2}A^{1/2}\right) \end{aligned} \tag{3.9}$$

where $y \in K$, $\|y\| = 1$.

If we apply to the inequality (3.9) the normalised positive map Φ, then we get

$$\begin{aligned} & r\left(\Phi(A)+\langle\Psi(B) y, y\rangle 1_K-2\langle\Psi(B) y, y\rangle^{1/2} \Phi\left(A^{1/2}\right)\right) \\ & \leq(1-\nu) \Phi(A)+\nu\langle\Psi(B) y, y\rangle 1_K-\langle\Psi(B) y, y\rangle^\nu \Phi\left(A^{1-\nu}\right) \\ & \leq R\left(\Phi(A)+\langle\Psi(B) y, y\rangle 1_K-2\langle\Psi(B) y, y\rangle^{1/2} \Phi\left(A^{1/2}\right)\right) \end{aligned}$$

where $y \in K$, $\|y\|=1$.

If in this inequality we take the inner product for $x \in K$, $\|x\|=1$, we get

$$\begin{aligned} & r\left(\langle\Phi(A) x, x\rangle+\langle\Psi(B) y, y\rangle-2\langle\Psi(B) y, y\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right) \\ & \leq(1-\nu)\langle\Phi(A) x, x\rangle+\nu\langle\Psi(B) y, y\rangle-\langle\Psi(B) y, y\rangle^\nu\left\langle\Phi\left(A^{1-\nu}\right) x, x\right\rangle \\ & \leq R\left(\langle\Phi(A) x, x\rangle+\langle\Psi(B) y, y\rangle-2\langle\Psi(B) y, y\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right), \end{aligned}$$

which, for $y=x$, generates the desired inequality (3.7). □

Remark 2. *If we take in (3.7) $\Phi=\Psi$, then we get*

$$\begin{aligned} & 2r\left(\langle(\Phi(A) \nabla \Phi(B)) x, x\rangle-\langle\Phi(B) x, x\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right) \\ & \leq\left\langle\left(\Phi(A) \nabla_\nu \Phi(A)\right) x, x\right\rangle-\langle\Phi(B) x, x\rangle^\nu\left\langle\Phi\left(A^{1-\nu}\right) x, x\right\rangle \\ & \leq 2R\left(\langle(\Phi(A) \nabla \Phi(B)) x, x\rangle-\langle\Phi(B) x, x\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right), \end{aligned} \tag{3.10}$$

for any $x \in K$, $\|x\|=1$.

If we put in (3.7) $B=A$, then we get

$$\begin{aligned} & 2r\left(\langle(\Phi(A) \nabla \Psi(A)) x, x\rangle-\langle\Psi(A) x, x\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right) \\ & \leq\left\langle\left(\Phi(A) \nabla_\nu \Psi(A)\right) x, x\right\rangle-\langle\Psi(A) x, x\rangle^\nu\left\langle\Phi\left(A^{1-\nu}\right) x, x\right\rangle \\ & \leq 2R\left(\langle(\Phi(A) \nabla \Psi(A)) x, x\rangle-\langle\Psi(A) x, x\rangle^{1/2}\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right), \end{aligned} \tag{3.11}$$

for any $x \in K$, $\|x\|=1$.

Moreover, if we choose in (3.11) $\Phi=\Psi$ and assume that $A \in \mathcal{B}^{++}(H)$, then by replacing ν with $1-\nu$ we get the interesting result

$$\begin{aligned} & 2r\langle\Phi(A) x, x\rangle^{\nu-1/2}\left(\langle\Phi(A) x, x\rangle^{1/2}-\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right) \\ & \leq\langle\Phi(A) x, x\rangle^\nu-\left\langle\Phi\left(A^\nu\right) x, x\right\rangle \\ & \leq 2R\langle\Phi(A) x, x\rangle^{\nu-1/2}\left(\langle\Phi(A) x, x\rangle^{1/2}-\left\langle\Phi\left(A^{1/2}\right) x, x\right\rangle\right), \end{aligned} \tag{3.12}$$

for any $x \in K$, $\|x\|=1$, where $\nu \in[0,1]$, $r=\min\{1-\nu, \nu\}$ and $R=\max\{1-\nu, \nu\}$.

References

[1] D. K. Callebaut, Generalization of Cauchy-Schwarz inequality, *J. Math. Anal. Appl.* **12** (1965), 491-494.

[2] M. D. Choi, Positive linear maps on C*-algebras. *Canad. J. Math.* **24** (1972), 520–529.

[3] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 417-478.

[4] S. S. Dragomir, Grüss' type inequalities for positive linear maps of selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 62. [Online http://rgmia.org/papers/v19/v19a62.pdf].

[5] S. S. Dragomir, Operator quasilinearity of some functionals associated to Davis-Choi-Jensen's inequality for positive maps, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 63. [Online http://rgmia.org/papers/v19/v19a63.pdf].

[6] S. S. Dragomir, Čebyšev type inequalities for positive linear maps of selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 65. [Online http://rgmia.org/papers/v19/v19a65.pdf].

[7] S. S. Dragomir, Some Hermite-Hadamard type inequalities for operator convex functions and positive maps, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 60. [Online http://rgmia.org/papers/v19/v19a80.pdf].

[8] S. S. Dragomir, Callebaut and Hölder type inequalities for positive linear maps of selfadjoint operators via a Kittaneh-Manasrah result, *RGMIA Res. Rep. Coll.* **19** (2016), Art. .

[9] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.,* **361** (2010), 262-269

[10] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra.,* **59** (2011), 1031-1037.

[11] X. Fu and C. He, Some operator inequalities for positive linear maps. *Linear Multilinear Algebra* **63** (2015), no. 3, 571–577.

[12] J. S. Matharu and M. S. Moslehian, Grüss inequality for some types of positive linear maps. *J. Operator Theory* **73** (2015), no. 1, 265–278.

[13] H. R. Moradi, M. E. Omidvar and S. S. Dragomir, An operator extension of Čebyšev inequality, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 81. [Online http://rgmia.org/papers/v19/v19a81.pdf].

[14] H. R. Moradi, M. E. Omidvar and S. S. Dragomir, More operator inequalities for positive linear maps, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 82. [Online http://rgmia.org/papers/v19/v19a82.pdf].

[15] M. Niezgoda, Shannon like inequalities for f-connections of positive linear maps and positive operators. *Linear Algebra Appl.* **481** (2015), 186–201.

[16] J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[17] R. Sharma and A. Thakur, More inequalities for positive linear maps. *J. Math. Inequal.* **7** (2013), no. 1, 1–9.

[18] J. Xue and X. Hu, Some generalizations of operator inequalities for positive linear maps. *J. Inequal. Appl.* **2016**, 2016:27, 6 pp.

[19] P. Zhang, More operator inequalities for positive linear maps. Banach J. Math. Anal. 9 (2015), no. 1, 166–172.

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