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INEQUALITIES FOR POSITIVE LINEAR MAPS OF SELFADJOINT OPERATORS VIA SOME MULTIPLICATIVE REFINEMENTS AND REVERSES OF YOUNG'S INEQUALITY

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ABSTRACT. Inequalities for positive linear maps of positive selfadjoint operators in Hilbert spaces via some results of Tominaga, Zou et al. and Liao et al. are given. Operator and vector inequalities involving the weighted operator geometric mean are also obtained. Reverses of the celebrated Ando's inequality are provided.

1. INTRODUCTION

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators and $\nu\in[0,1]$

$$A\nabla_\nu B:=(1-\nu)A+\nu B,$$

the *weighted operator arithmetic mean,* and

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},$$

the *weighted operator geometric mean,* [12]. When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if a, $b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [20]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality:

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

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where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [22] while the first one is due to Furuichi [11].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds:

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu, \tag{1.5}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.5) was obtained by Zou et al. in [25] while the second by Liao et al. [13].

In [25] the authors also showed that

$$K^r(h) \geq S(h^r) \text{ for } h > 0 \text{ and } r \in \left[0, \frac{1}{2}\right] \tag{1.6}$$

implying that the lower bound in (1.5) is better than the lower bound from (1.3).

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [18, p. 18]) we can introduce:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H) \Psi \Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

The celebrated *Ando's inequality* [1] (see also [19]) says that if A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$, then

$$\Phi(A \sharp_\nu B) \leq \Phi(A) \sharp_\nu \Phi(B) \tag{1.7}$$

for any $\nu \in [0,1]$.

For some recent inequalities for positive maps of Hilbert space operators see [5]-[8], [9]-[17], [21]-[24] and the references therein.

Motivated by the above results, several inequalities for positive linear maps of positive selfadjoint linear operators in Hilbert spaces via some results of Tominaga, Zou et al. and Liao et al. are given. Operator and vector inequalities involving the weighted operator geometric mean are also obtained. Multiplicative reverses of the celebrated Ando's inequality are provided.

2. Multiplicative Reverses of Ando's Inequality

The following lemma is of interest in itself:

Lemma 1. *Assume that C, D are positive invertible operators and the constants $M > m > 0$ are such that*

$$mC \leq D \leq MC \tag{2.1}$$

in the operator order. Let $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. Then we have the inequalities

$$\varphi_r(m,M)\, C\sharp_\nu D \leq C\nabla_\nu D \leq \Gamma(m,M)\, C\sharp_\nu D, \tag{2.2}$$

where

$$\Gamma(m,M) := \begin{cases} S(m) \text{ if } M < 1, \\ \max\{S(m), S(M)\} \text{ if } m \leq 1 \leq M, \\ S(M) \text{ if } 1 < m, \end{cases} \tag{2.3}$$

$$\varphi_r(m,M) := \begin{cases} S(M^r) \text{ if } M < 1, \\ 1 \text{ if } m \leq 1 \leq M, \\ S(m^r) \text{ if } 1 < m, \end{cases}$$

and

$$\psi_r(m,M)\, C\sharp_\nu D \leq C\nabla_\nu D \leq \Psi_R(m,M)\, C\sharp_\nu D, \tag{2.4}$$

where

$$\Psi_R(m,M) := \begin{cases} K^R(m) \text{ if } M < 1, \\ \max\left\{K^R(m), K^R(M)\right\} \text{ if } m \leq 1 \leq M, \\ K^R(M) \text{ if } 1 < m, \end{cases} \tag{2.5}$$

$$\psi_r(m,M) := \begin{cases} K^r(M) \text{ if } M < 1, \\ 1 \text{ if } m \leq 1 \leq M, \\ K^r(m) \text{ if } 1 < m. \end{cases}$$

Proof. From the inequality (1.3) we have

$$x^\nu \min_{x\in[m,M]} S(x^r) \leq S(x^r)\, x^\nu \leq (1-\nu) + \nu x \leq S(x)\, x^\nu \leq x^\nu \max_{x\in[m,M]} S(x) \tag{2.6}$$

where $x \in [m, M]$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$.

Since, by the properties of Specht's ratio S, we have

$$\max_{x\in[m,M]} S(x) = \begin{cases} S(m) \text{ if } M < 1, \\ \\ \max\{S(m), S(M)\} \text{ if } m \le 1 \le M, \\ \\ S(M) \text{ if } 1 < m, \end{cases} = \Gamma(m, M)$$

and

$$\min_{x\in[m,M]} S(x^r) = \begin{cases} S(M^r) \text{ if } M < 1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ S(m^r) \text{ if } 1 < m, \end{cases} = \varphi_r(m, M),$$

then by (2.6) we have

$$x^\nu \varphi_r(m, M) \le (1-\nu) + \nu x \le x^\nu \Gamma(m, M) \tag{2.7}$$

for any $x \in [m, M]$ and $\nu \in [0, 1]$.

Using the functional calculus for the operator X with $mI \le X \le MI$ we have from (2.7) that

$$X^\nu \varphi_r(m, M) \le (1-\nu) I + \nu X \le X^\nu \Gamma(m, M) \tag{2.8}$$

for any $\nu \in [0, 1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $C^{-1/2}$ we get $mI \le C^{-1/2} D C^{-1/2} \le MI$ and by taking $X = C^{-1/2} D C^{-1/2}$ in (2.8) we get

$$\begin{aligned} \left(C^{-1/2} D C^{-1/2}\right)^\nu \varphi_r(m, M) &\le (1-\nu) I + \nu C^{-1/2} D C^{-1/2} \\ &\le \left(C^{-1/2} D C^{-1/2}\right)^\nu \Gamma(m, M). \end{aligned} \tag{2.9}$$

Now, if we multiply (2.9) in both sides with $C^{1/2}$ we get the desired result (2.2).

The second part follows in a similar way by utilizing the inequality

$$x^\nu \min_{x\in[m,M]} K^r(x) \le K^r(x) x^\nu \le (1-\nu) + \nu x \le K^R(x) x^\nu \le x^\nu \max_{x\in[m,M]} K^R(x),$$

where $x \in [m, M]$, which follows from (1.5). The details are omitted. □

Remark 1. *By the inequality (1.6) we observe that the lower bound in (2.4) is better than the lower bound in (2.2). It has been shown in* [4] *that in general neither of the upper bounds in (1.3) and (1.5) is always best, then it makes sense to state the results in Lemma 1 as a single inequality, namely*

$$\psi_r(m, M) C \sharp_\nu D \le C \nabla_\nu D \le \min\{\Gamma(m, M), \Psi_R(m, M)\} C \sharp_\nu D, \tag{2.10}$$

provided that C, D are positive invertible operators and the constants $M > m > 0$ are such that the condition (2.1) is valid. Here $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

We have the following multiplicative reverse of Ando's inequality (1.7):

Theorem 1. *Let A, B be positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and the constants $M > m > 0$ are such that*

$$mA \leq B \leq MA. \tag{2.11}$$

If $\Phi \in \mathfrak{P}\left[\mathcal{B}(H), \mathcal{B}(K)\right]$, then we have

$$\left(\Phi\left(A\sharp_\nu B\right) \leq\right) \Phi(A)\sharp_\nu \Phi(B) \leq \frac{\min\left\{\Gamma(m,M), \Psi_R(m,M)\right\}}{\psi_r(m,M)} \Phi\left(A\sharp_\nu B\right), \tag{2.12}$$

where $\psi_r(m,M)$, $\Phi(m,M)$ and $\Psi_R(m,M)$ are defined in Lemma 1 and $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu,\nu\}$.

Proof. If we write the inequality (2.10) for $C = A$ and $D = B$, then we get

$$\psi_r(m,M) A\sharp_\nu B \leq A\nabla_\nu B \leq \min\left\{\Gamma(m,M), \Psi_R(m,M)\right\} A\sharp_\nu B, \tag{2.13}$$

for $\nu \in [0,1]$.

By taking the positive map Φ in (2.13) we get

$$\begin{aligned}\psi_r(m,M)\Phi\left(A\sharp_\nu B\right) &\leq \Phi(A)\nabla_\nu \Phi(B) \\ &\leq \min\left\{\Gamma(m,M), \Psi_R(m,M)\right\}\Phi\left(A\sharp_\nu B\right),\end{aligned} \tag{2.14}$$

for $\nu \in [0,1]$.

From (2.11) we have that $m\Phi(A) \leq \Phi(B) \leq M\Phi(A)$ and by the inequality (2.10) for $C = \Phi(A)$ and $D = \Phi(B)$ we have

$$\begin{aligned}\psi_r(m,M)\Phi(A)\sharp_\nu \Phi(B) &\leq \Phi(A)\nabla_\nu \Phi(B) \\ &\leq \min\left\{\Gamma(m,M), \Psi_R(m,M)\right\}\Phi(A)\sharp_\nu \Phi(B),\end{aligned} \tag{2.15}$$

for $\nu \in [0,1]$.

If we use the first inequality in (2.15) and the second inequality in (2.14), then we get the desired result (2.12). □

Remark 2. *For $\nu = \frac{1}{2}$, denote*

$$\Psi(m,M) := \Psi_{1/2}(m,M) = \begin{cases} \frac{m+1}{2\sqrt{m}} \text{ if } M < 1, \\ \max\left\{\frac{m+1}{2\sqrt{m}}, \frac{M+1}{2\sqrt{M}}\right\} \text{ if } m \leq 1 \leq M, \\ \frac{M+1}{2\sqrt{M}} \text{ if } 1 < m, \end{cases} \tag{2.16}$$

and

$$\psi(m,M) := \psi_{1/2}(m,M) = \begin{cases} \frac{M+1}{2\sqrt{M}} \text{ if } M < 1, \\ 1 \text{ if } m \leq 1 \leq M, \\ \frac{m+1}{2\sqrt{m}} \text{ if } 1 < m. \end{cases} \tag{2.17}$$

By taking $\nu = \frac{1}{2}$ in (2.12) we then get the simple inequality

$$\left(\Phi\left(A\sharp B\right) \leq\right) \Phi(A)\sharp\Phi(B) \leq \frac{\min\left\{\Gamma(m,M), \Psi(m,M)\right\}}{\psi(m,M)} \Phi\left(A\sharp B\right), \tag{2.18}$$

where A, B are positive invertible operators that satisfy condition (2.11).

3. Vector Inequalities

We have the following vector inequality:

Theorem 2. *Let A, B be positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and the constants $M_1>m_1>0$ and $M_2>m_2>0$ satisfying the conditions*

$$M_1 1_H \geq A \geq m_1 1_H \text{ and } M_2 1_H \geq B \geq m_2 1_H.$$

If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$, then we have

$$\begin{aligned}(3.1)\quad &\psi_r\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)\langle\Phi\left(A^{1-\nu}\right)x,x\rangle\langle\Psi\left(B^{\nu}\right)y,y\rangle\\ &\leq(1-\nu)\langle\Phi(A)x,x\rangle+\nu\langle\Psi(B)y,y\rangle\\ &\leq\min\left\{\Gamma\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right),\Psi_R\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)\right\}\langle\Phi\left(A^{1-\nu}\right)x,x\rangle\langle\Psi\left(B^{\nu}\right)y,y\rangle,\end{aligned}$$

for any x, $y\in K$, $\|x\|=\|y\|=1$, where $\psi_r(\cdot,\cdot)$, $\Phi(\cdot,\cdot)$ and $\Psi_R(\cdot,\cdot)$ are defined in Lemma 1 and $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, we have

$$\begin{aligned}(3.2)\quad \psi_r\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)&\leq\frac{\langle(\Phi(A)\nabla_\nu\Psi(B))x,x\rangle}{\langle\Phi\left(A^{1-\nu}\right)x,x\rangle\langle\Psi\left(B^{\nu}\right)x,x\rangle}\\ &\leq\min\left\{\Gamma\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right),\Psi_R\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)\right\},\end{aligned}$$

for any $x\in K$, $\|x\|=1$.

Proof. From the proof of Lemma 1 we have the scalar inequality

$$(3.3)\quad \psi_r(m,M)\,t^{\nu}\leq(1-\nu)+\nu t\leq\min\{\Gamma(m,M),\Psi_R(m,M)\}\,t^{\nu}$$

for any $t\in[m,M]$ and $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

If we take in (3.3) $t=\frac{b}{a}\in[m,M]$, then we get

$$(3.4)\quad \psi_r(m,M)\,a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq\min\{\Gamma(m,M),\Psi_R(m,M)\}\,a^{1-\nu}b^{\nu}$$

for $\nu\in[0,1]$.

If $a\in[m_1,M_1]$ and $b\in[m_2,M_2]$ then $\frac{b}{a}\in\left[\frac{m_2}{M_1},\frac{M_2}{m_1}\right]$ and by (3.4) we have

$$\begin{aligned}(3.5)\quad \psi_r\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)a^{1-\nu}b^{\nu}&\leq(1-\nu)a+\nu b\\ &\leq\min\left\{\Gamma\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right),\Psi_R\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)\right\}a^{1-\nu}b^{\nu}\end{aligned}$$

for $\nu\in[0,1]$.

Fix $b\in[m_2,M_2]$, then by the functional calculus for the operator A we have

$$\begin{aligned}(3.6)\quad \psi_r\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)b^{\nu}A^{1-\nu}&\leq(1-\nu)A+\nu b1_H\\ &\leq\min\left\{\Gamma\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right),\Psi_R\left(\frac{m_2}{M_1},\frac{M_2}{m_1}\right)\right\}b^{\nu}A^{1-\nu}\end{aligned}$$

for $\nu\in[0,1]$.

Now, if we apply to (3.6) the normalised positive map Φ and then take the inner product for $x \in K$, $\|x\| = 1$ we get

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) b^\nu \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle \\ &\leq (1-\nu)\left\langle \Phi(A) x, x\right\rangle + \nu b \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} b^\nu \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle . \end{aligned} \tag{3.7}$$

If we use the functional calculus for the operator B then we have by (3.7) that

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle B^\nu \\ &\leq (1-\nu)\left\langle \Phi(A) x, x\right\rangle 1_H + \nu B \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle B^\nu \end{aligned} \tag{3.8}$$

for $x \in K$, $\|x\| = 1$.

If we apply to (3.8) the normalised positive map Ψ and then take the inner product for $y \in K$, $\|y\| = 1$ we get

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle \left\langle \Psi\left(B^\nu\right) y, y\right\rangle \\ &\leq (1-\nu)\left\langle \Phi(A) x, x\right\rangle + \nu \left\langle \Psi(B) y, y\right\rangle \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\langle \Phi\left(A^{1-\nu}\right) x, x\right\rangle \left\langle \Psi\left(B^\nu\right) y, y\right\rangle , \end{aligned} \tag{3.9}$$

for any $x, y \in K$, $\|x\| = \|y\| = 1$ and the inequality (3.1) is proved. □

Corollary 1. *With the assumptions of Theorem 2 we have the norm inequalities*

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\|\Phi\left(A^{1-\nu}\right)\right\| \left\|\Psi\left(B^\nu\right)\right\| \\ &\leq (1-\nu)\left\|\Phi(A)\right\| + \nu \left\|\Psi(B)\right\| \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\|\Phi\left(A^{1-\nu}\right)\right\| \left\|\Psi\left(B^\nu\right)\right\| , \end{aligned} \tag{3.10}$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Proof. Indeed, if we fix in (3.1) $y \in K$, $\|y\| = 1$ and take the supremum over $x \in K$, $\|x\| = 1$ we have

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\|\Phi\left(A^{1-\nu}\right)\right\| \left\langle \Psi\left(B^\nu\right) y, y\right\rangle \\ &\leq (1-\nu)\left\|\Phi(A)\right\| + \nu \left\langle \Psi(B) y, y\right\rangle \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\|\Phi\left(A^{1-\nu}\right)\right\| \left\langle \Psi\left(B^\nu\right) y, y\right\rangle . \end{aligned} \tag{3.11}$$

By taking the supremum over $y \in K$, $\|y\| = 1$ in (3.11) we get the desired result (3.10). □

Remark 3. *If we take in (3.1) $\Phi = \Psi$, then we have*

$$\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \leq \frac{\langle(\Phi(A)\nabla_\nu \Phi(B))x, x\rangle}{\langle \Phi(A^{1-\nu})x, x\rangle \langle \Phi(B^\nu)x, x\rangle} \leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\}, \tag{3.12}$$

for any $x \in K$, $\|x\| = 1$.

If $B = A$, $M_2 = M_1$, $m_2 = m_1$ and since $\psi_r\left(\frac{m_1}{M_1}, \frac{M_1}{m_1}\right) = 1$,

$$\Gamma\left(\frac{m_1}{M_1}, \frac{M_1}{m_1}\right) = S\left(\frac{M_1}{m_1}\right) \text{ and } \Psi_R\left(\frac{m_1}{M_1}, \frac{M_1}{m_1}\right) = K^R\left(\frac{M_1}{m_1}\right),$$

then we get by (3.12) that

$$1 \leq \frac{\langle \Phi(A)x, x\rangle}{\langle \Phi(A^{1-\nu})x, x\rangle \langle \Phi(A^\nu)x, x\rangle} \leq \min\left\{S\left(\frac{M_1}{m_1}\right), K^R\left(\frac{M_1}{m_1}\right)\right\}, \tag{3.13}$$

for any $x \in K$, $\|x\| = 1$.

From (3.10) we also have the norm inequality

$$\|\Phi(A)\| \leq \min\left\{S\left(\frac{M_1}{m_1}\right), K^R\left(\frac{M_1}{m_1}\right)\right\} \left\|\Phi\left(A^{1-\nu}\right)\right\| \|\Phi(A^\nu)\|$$

where $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Now, if we take in (3.13) $\nu = \frac{1}{2}$, then we get

$$1 \leq \frac{\langle \Phi(A)x, x\rangle}{\left\langle \Phi\left(A^{1/2}\right)x, x\right\rangle^2} \leq \min\left\{S\left(\frac{M_1}{m_1}\right), \frac{M_1 + m_1}{2\sqrt{m_1 M_1}}\right\}, \tag{3.14}$$

for any $x \in K$, $\|x\| = 1$.

Alternatively, we have:

Theorem 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\langle \Phi\left(A^{1-\nu}\right)x, x\right\rangle \langle \Psi(B)y, y\rangle^\nu \\ &\leq (1-\nu)\langle \Phi(A)x, x\rangle + \nu\langle \Psi(B)y, y\rangle \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\langle \Phi\left(A^{1-\nu}\right)x, x\right\rangle \langle \Psi(B)y, y\rangle^\nu, \end{aligned} \tag{3.15}$$

for any x, $y \in K$, $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\langle \Phi\left(A^{1-\nu}\right)x, x\right\rangle \langle \Psi(B)x, x\rangle^\nu \\ &\leq \langle(\Phi(A)\nabla_\nu \Psi(B))x, x\rangle \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\langle \Phi\left(A^{1-\nu}\right)x, x\right\rangle \langle \Psi(B)x, x\rangle^\nu, \end{aligned} \tag{3.16}$$

for any $x \in K$, $\|x\| = 1$.

Proof. By taking $b = \langle \Psi(B) y, y \rangle \in [m_2, M_2]$ for $y \in K$, $\|y\| = 1$ in (3.6) we get

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \langle \Psi(B) y, y \rangle^\nu A^{1-\nu} \\ &\leq (1-\nu) A + \nu \langle \Psi(B) y, y \rangle 1_H \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \langle \Psi(B) y, y \rangle^\nu A^{1-\nu} \end{aligned} \tag{3.17}$$

for $\nu \in [0, 1]$.

Now, by making use of a similar argument to the one outlined in the proof of Theorem 2 we obtain the desired result (3.15) and the details are omitted. □

We have:

Corollary 2. *With the assumptions of Theorem 2 we have the norm inequalities*

$$\begin{aligned} &\psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) \left\|\Phi\left(A^{1-\nu}\right)\right\| \|\Psi(B)\|^\nu \\ &\leq (1-\nu) \|\Phi(A)\| + \nu \|\Psi(B)\| \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\} \left\|\Phi\left(A^{1-\nu}\right)\right\| \|\Psi(B)\|^\nu, \end{aligned} \tag{3.18}$$

where $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Remark 4. *If we take in (3.16)* $\Phi = \Psi$, *then we have*

$$\begin{aligned} \psi_r\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right) &\leq \frac{\langle (\Phi(A) \nabla_\nu \Phi(B)) x, x \rangle}{\langle \Phi(A^{1-\nu}) x, x \rangle \langle \Phi(B) x, x \rangle^\nu} \\ &\leq \min\left\{\Gamma\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right), \Psi_R\left(\frac{m_2}{M_1}, \frac{M_2}{m_1}\right)\right\}, \end{aligned} \tag{3.19}$$

for any $x \in K$, $\|x\| = 1$, *and, for* $B = A$, *we get for replacing* ν *with* $1-\nu$

$$1 \leq \frac{\langle \Phi(A) x, x \rangle^\nu}{\langle \Phi(A^\nu) x, x \rangle} \leq \min\left\{S\left(\frac{M_1}{m_1}\right), K^R\left(\frac{M_1}{m_1}\right)\right\} \tag{3.20}$$

for any $x \in K$, $\|x\| = 1$.

The corresponding norm inequality is:

$$\|\Phi(A)\|^\nu \leq \min\left\{S\left(\frac{M_1}{m_1}\right), K^R\left(\frac{M_1}{m_1}\right)\right\} \|\Phi(A^\nu)\|. \tag{3.21}$$

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