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OPIAL TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish the Opial type inequalities for conformable fractional integral and give some results in special cases of α. The results presented here would provide generalizations of those given in earlier works.

1. Introduction

In the year 1960, Opial established the following interesting integral inequality[9]:

Theorem 1. *Let $x(t)\in C^{(1)}[0,h]$ be such that $x(0)=x(h)=0$, and $x(t)>0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt \tag{1.1}$$

The constant $h/4$ is best possible

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [3],[4], [10]-[15].

In [14], Traple gave the inequalities in the following theorem:

Theorem 2. *Let w be a nonnegative and continuous function on $[0,h]$. Let u be an absolutely continuous function on $[0,h]$ with $u(0)=u(h)=0$. Then the following inequalities hold*

$$\int_0^h w(t)\,|u(t)|^2\,dt \leq \frac{h}{4}\left(\int_0^h w(t)dt\right)\left(\int_0^h |u'(t)|^2\,dt\right) \tag{1.2}$$

$$\int_0^h w(t)\,|u(t)u'(t)|\,dt \leq \left(\frac{h}{4}\int_0^h w^2(t)dt\right)^{\frac{1}{2}}\left(\int_0^h |u'(t)|^2\,dt\right). \tag{1.3}$$

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2. Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in (see, [1], [2], [5]-[8]).

Definition 1 (Conformable fractional derivative)**.** *Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha(f)(t)=\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon} \tag{2.1}$$

for all $t>0,\ \alpha\in(0,1)$. If f is $\alpha-$differentiable in some $(0,a),\ \alpha>0,\ \lim_{t\to 0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 3. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*

i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{fD_\alpha(g)-gD_\alpha(f)}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.3}$$

Definition 2 (Conformable fractional integral)**.** *Let $\alpha\in(0,1]$ and $0\le a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_\alpha x:=\int_a^b f(x)\,x^{\alpha-1}dx \tag{2.4}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$.

Remark 1.

$$I_\alpha^a(f)(t)=I_1^a\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 4. *Let $f:(a,b)\to\mathbb{R}$ be differentiable and $0<\alpha\le 1$. Then, for all $t>a$ we have*

$$I_\alpha^a D_\alpha^a f(t)=f(t)-f(a). \tag{2.5}$$

Theorem 5 (Integration by parts)**.** *Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\,D_\alpha^a(g)(x)\,d_\alpha x=fg|_a^b-\int_a^b g(x)\,D_\alpha^a(f)(x)\,d_\alpha x. \tag{2.6}$$

Theorem 6. *Assume that* $f : [a, \infty) \to \mathbb{R}$ *such that* $f^{(n)}(t)$ *is continuous and* $\alpha \in (n, n+1]$. *Then, for all* $t > a$ *we have*

$$D_{\alpha}^{a} f(t) I_{\alpha}^{a} = f(t).$$

We can give the Hölder's inequality in conformable integral as follows:

Lemma 1. *Let* $f, g \in C[a, b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\int_{a}^{b} |f(x)g(x)| \, d_{\alpha}x \leq \left(\int_{a}^{b} |f(x)|^{p} \, d_{\alpha}x \right)^{\frac{1}{p}} \left(\int_{a}^{b} |g(x)|^{q} \, d_{\alpha}x \right)^{\frac{1}{q}}.$$

Remark 2. *If we take* $p = q = 2$ *in Lemma 1 , the we have the Cauchy-Schwartz inequality for conformable integral.*

In this paper, we establish the Opial type inequalities for conformable fractional integral and give some results in special cases of α. The results presented here would provide generalizations of those given in earlier works.

3. Opial type inequalities for conformable fractional integral

Opial inequality can be represented in conformable fractional integral forms as follows:

Theorem 7. *Let* $\alpha \in (0, 1]$ *and* u *be an* α*-fractional differantiable function on* $(0, h)$ *with* $u(0) = u(h) = 0$. *Then, the following inequality for conformable fractional integral holds:*

$$\int_{0}^{h} |u(t) D_{\alpha}(u)(t)| \, d_{\alpha}t \leq \frac{h^{\alpha}}{4\alpha} \int_{0}^{h} |D_{\alpha}(u)(t)|^{2} \, d_{\alpha}t. \tag{3.1}$$

Proof. Let

$$y(t) = \int_{0}^{t} |D_{\alpha}(u)(s)| \, d_{\alpha}s$$

and

$$z(t) = \int_{t}^{h} |D_{\alpha}(u)(s)| \, d_{\alpha}s.$$

Then, we have

$$D_{\alpha}(y)(t) = |D_{\alpha}(u)(t)| = -D_{\alpha}(z)(t) \tag{3.2}$$

and for all $t \in [0, h]$,

$$u(t) \leq y(t), \quad u(t) \leq z(t). \tag{3.3}$$

Using (3.2) and (3.3), we get

$$\int_{0}^{\frac{h}{2^{1/\alpha}}} |u(t) D_{\alpha}(u)(t)| \, d_{\alpha}t \leq \int_{0}^{\frac{h}{2^{1/\alpha}}} y(t) D_{\alpha}(y)(t) d_{\alpha}t. \tag{3.4}$$

Using the integration by parts in conformable fractional integral, we have

$$\int_{0}^{\frac{h}{2^{1/\alpha}}} y(t) D_{\alpha}(y)(t) d_{\alpha}t = y^{2}\left(\frac{h}{2^{1/\alpha}}\right) - \int_{0}^{\frac{h}{2^{1/\alpha}}} y(t) D_{\alpha}(y)(t) d_{\alpha}t$$

i.e.

$$\int_{0}^{\frac{h}{2^{1/\alpha}}} y(t) D_{\alpha}(y)(t) d_{\alpha}t = \frac{1}{2} y^{2}\left(\frac{h}{2^{1/\alpha}}\right). \tag{3.5}$$

Combining (3.4) and (3.5), it follows that

$$\int_0^{\frac{h}{2^{1/\alpha}}} |u(t)D_\alpha(u)(t)|\, d_\alpha t \le \frac{1}{2} y^2\left(\frac{h}{2^{1/\alpha}}\right). \tag{3.6}$$

Similarly, using (3.2) and (3.3), we get

$$\int_{\frac{h}{2^{1/\alpha}}}^{h} |u(t)D_\alpha(u)(t)|\, d_\alpha t \le -\int_{\frac{h}{2^{1/\alpha}}}^{h} z(t)D_\alpha(z)(t) d_\alpha t = \frac{1}{2} z^2\left(\frac{h}{2^{1/\alpha}}\right). \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\int_0^{h} |u(t)D_\alpha(u)(t)|\, d_\alpha t \le \frac{1}{2}\left[y^2\left(\frac{h}{2^{1/\alpha}}\right) + z^2\left(\frac{h}{2^{1/\alpha}}\right)\right]. \tag{3.8}$$

On the other hand, using the Cauchy-Schwartz inequality for conformable integral, we get

$$\begin{aligned} y^2\left(\frac{h}{2^{1/\alpha}}\right) &= \left(\int_0^{\frac{h}{2^{1/\alpha}}} |D_\alpha(u)(s)|\, d_\alpha s\right)^2 \\ &\le \left(\int_0^{\frac{h}{2^{1/\alpha}}} d_\alpha s\right)\left(\int_0^{\frac{h}{2^{1/\alpha}}} |D_\alpha(u)(s)|^2 d_\alpha s\right) \\ &= \frac{h^\alpha}{2\alpha}\int_0^{\frac{h}{2^{1/\alpha}}} |D_\alpha(u)(s)|^2 d_\alpha s \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} z^2\left(\frac{h}{2^{1/\alpha}}\right) &= \left(\int_{\frac{h}{2^{1/\alpha}}}^{h} |D_\alpha(u)(s)|\, d_\alpha s\right)^2 \\ &\le \left(\int_{\frac{h}{2^{1/\alpha}}}^{h} d_\alpha s\right)\left(\int_{\frac{h}{2^{1/\alpha}}}^{h} |D_\alpha(u)(s)|^2 d_\alpha s\right) \\ &= \frac{h^\alpha}{2\alpha}\int_{\frac{h}{2^{1/\alpha}}}^{h} |D_\alpha(u)(s)|^2 d_\alpha s. \end{aligned} \tag{3.10}$$

Combining (3.8)-(3.10), we obtain the required result (3.1). □

Remark 3. *If we choose $\alpha = 1$ in Theorem 7, then the inequality (3.1) reduces the inequality (1.1).*

Theorem 8. *Let $\alpha \in (0,1]$ and p be a nonnegative and continuous function on $[0,h]$. Let u be an α-fractional differantiable function on $(0,h)$ with $u(0) = u(h) = 0$. Then the following inequalities for conformable fractional integral hold*

$$\int_0^h p(t)|u(t)|^2 d_\alpha t \le \frac{h^\alpha}{4\alpha}\left(\int_0^h p(t) d_\alpha t\right)\left(\int_0^h |D_\alpha(u)(t)|^2 d_\alpha t\right) \tag{3.11}$$

(3.12)

$$\int_0^h p(t)|u(t)D_\alpha(u)(t)|\, d_\alpha t \le \left(\frac{h^\alpha}{4\alpha}\int_0^h p^2(t) d_\alpha t\right)^{\frac{1}{2}}\left(\int_0^h |D_\alpha(u)(t)|^2 d_\alpha t\right).$$

Proof. From the inequalities in (3.3), we have

$$u(t) \leq \frac{y(t)+z(t)}{2} = \frac{1}{2}\int_0^h |D_\alpha(u)(s)|\, d_\alpha s. \tag{3.13}$$

Using the inequality (3.13) and Cauchy-Schwarz inequality for conformable integral, we obtain

$$\begin{aligned}
\int_0^h p(t)\,|u(t)|^2\, d_\alpha t &\leq \frac{1}{4}\int_0^h p(t)\left(\int_0^h |D_\alpha(u)(s)|\, d_\alpha s\right)^2 d_\alpha t \\
&\leq \frac{1}{4}\left(\int_0^h p(t) d_\alpha t\right)\left(\int_0^h d_\alpha s\right)\left(\int_0^h |D_\alpha(u)(s)|^2\, d_\alpha s\right) \\
&= \frac{h^\alpha}{4\alpha}\left(\int_0^h p(t) d_\alpha t\right)\left(\int_0^h |D_\alpha(u)(s)|^2\, d_\alpha s\right)
\end{aligned}$$

which completes the proof of the inequality (3.11).

For the proof of the inequality (3.12), by using the inequality (3.11) and Cauchy-Schwartz inequality for conformable integral, we have

$$\begin{aligned}
&\int_0^h p(t)\,|u(t)D_\alpha(u)(t)|\, d_\alpha t \\
\leq\ & \left(\int_0^h p^2(t)\,|u(t)|^2\, d_\alpha t\right)^{\frac{1}{2}}\left(\int_0^h |D_\alpha(u)(t)|^2\, d_\alpha t\right)^{\frac{1}{2}} \\
\leq\ & \left(\frac{h^\alpha}{4\alpha}\left(\int_0^h p^2(t) d_\alpha t\right)\left(\int_0^h |D_\alpha(u)(t)|^2\, d_\alpha t\right)\right)^{\frac{1}{2}}\left(\int_0^h |D_\alpha(u)(t)|^2\, d_\alpha t\right)^{\frac{1}{2}} \\
=\ & \left(\frac{h^\alpha}{4\alpha}\int_0^h p^2(t) d_\alpha t\right)^{\frac{1}{2}}\left(\int_0^h |D_\alpha(u)(t)|^2\, d_\alpha t\right).
\end{aligned}$$

This completes the proof. □

Remark 4. *If we choose $\alpha = 1$ in Theorem 8, then the inequalities (3.11) and (3.12) reduce the inequalities (1.2) and (1.3).*

Theorem 9. *Let $\alpha \in (0,1]$ and $p \geq 0$, $q \geq 1$, $m \geq 1$ be real numbers. If u is an α-fractional differantiable function on $(0,h)$ with $u(0) = u(h) = 0$, then the following inequalities for conformable fractional integral hold*

$$\int_0^h |u(t)|^{m(p+q)}\, d_\alpha t \leq [(p+q)^m I(m)]^q \int_0^h |u(t)|^{mp}\, |D_\alpha(u)(t)|^{mq}\, d_\alpha t \tag{3.14}$$

and

$$\int_0^h |u(t)|^{m(p+q)}\, d_\alpha t \leq [(p+q)^m I(m)]^{p+q} \int_0^h |D_\alpha(u)(t)|^{m(p+q)}\, d_\alpha t \tag{3.15}$$

where

$$I(m) = \frac{1}{\alpha^{m-1}}\int_0^h \left[t^{(1-m)\alpha} + (h^\alpha - t^\alpha)^{1-m}\right]^{-1} d_\alpha t. \tag{3.16}$$

Proof. Using the integration by parts in conformable fractional integral, we have

$$u^{p+q}(t)=(p+q)\int_0^t u^{p+q-1}(s)D_\alpha(u)(s)d_\alpha s \tag{3.17}$$

and

$$u^{p+q}(t)=-(p+q)\int_t^h u^{p+q-1}(s)D_\alpha(u)(s)d_\alpha s \tag{3.18}$$

for $t\in[0,h]$. Using the Hölder's inequality for conformable integral with indices m, $\frac{m}{m-1}$ in (3.17) and (3.18), we get

$$\begin{aligned} &|u(t)|^{m(p+q)} \\ \leq\ & (p+q)^m\left(\int_0^t \left|u^{p+q-1}(s)D_\alpha(u)(s)\right| d_\alpha s\right)^m \\ \leq\ & (p+q)^m\left(\int_0^t d_\alpha s\right)^{m-1}\int_0^t |u(s)|^{m(p+q-1)}|D_\alpha(u)(s)|^m d_\alpha s \\ =\ & \frac{(p+q)^m}{\alpha^{m-1}}t^{(m-1)\alpha}\int_0^t |u(s)|^{m(p+q-1)}|D_\alpha(u)(s)|^m d_\alpha s \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &|u(t)|^{m(p+q)} \\ =\ & (p+q)\int_t^h \left|u^{p+q-1}(s)D_\alpha(u)(s\right|)d_\alpha s \\ \leq\ & \frac{(p+q)^m}{\alpha^{m-1}}(h^\alpha-t^\alpha)^{m-1}\int_t^h |u(s)|^{m(p+q-1)}|D_\alpha(u)(s)|^m d_\alpha s. \end{aligned} \tag{3.20}$$

for $t\in[0,h]$. Multiplyig the (3.19) by $t^{(1-m)\alpha}$ and (3.20) by $(h^\alpha-t^\alpha)^{1-m}$ and summing these inequalities, we obtain

$$\left[t^{(1-m)\alpha}+(h^\alpha-t^\alpha)^{1-m}\right]|u(t)|^{m(p+q)}\leq\frac{(p+q)^m}{\alpha^{m-1}}\int_0^h |u(s)|^{m(p+q-1)}|D_\alpha(u)(s)|^m d_\alpha s,$$

and thus,

$$\begin{aligned} &|u(t)|^{m(p+q)} \\ \leq\ & \frac{(p+q)^m}{\alpha^{m-1}}\left[t^{(1-m)\alpha}+(h^\alpha-t^\alpha)^{1-m}\right]^{-1}\int_0^h |u(s)|^{m(p+q-1)}|D_\alpha(u)(s)|^m d_\alpha s \\ =\ & \frac{(p+q)^m}{\alpha^{m-1}}\left[t^{(1-m)\alpha}+(h^\alpha-t^\alpha)^{1-m}\right]^{-1} \\ & \times\int_0^h |u(s)|^{mp/q}|D_\alpha(u)(s)|^m|u(s)|^{m(p+q-1)-mp/q}d_\alpha s. \end{aligned} \tag{3.21}$$

for $t \in [0,h]$. Integrating (3.21) on $[0,h]$ and using the Hölder's inequality for conformable integral with indices q, $\frac{q}{q-1}$, find that

$$\begin{aligned}
(3.22)\qquad & \int_0^h |u(t)|^{m(p+q)}\, d_\alpha t \\
&\leq (p+q)^m I(m) \int_0^h |u(s)|^{mp/q} \left|D_\alpha(u)(s)\right|^m |u(s)|^{m(p+q-1)-mp/q}\, d_\alpha s \\
&\leq (p+q)^m I(m) \\
&\quad \times \left(\int_0^h |u(s)|^{mp} \left|D_\alpha(u)(s)\right|^{mq} d_\alpha s\right)^{\frac{1}{q}} \left(\int_0^h |u(s)|^{m(p+q)}\, d_\alpha s\right)^{\frac{q-1}{q}}.
\end{aligned}$$

If $\int_0^h |u(s)|^{m(p+q)}\, d_\alpha s = 0$, then (3.14) is trivally true, otherwise, dividing the both sides of (3.22) by $\left(\int_0^h |u(t)|^{m(p+q)}\, d_\alpha t\right)^{\frac{q-1}{q}}$ and taking the qth power on both sides of resulting inequality we get the desired inequality (3.14).

By using the Hölder's inequality for conformable integral with indices $(p+q)/p$ and $(p+q)/q$ to the integral on the right hand side of (3.14), we have

$$\begin{aligned}
(3.23)\qquad & \int_0^h |u(t)|^{m(p+q)}\, d_\alpha t \\
&\leq [(p+q)^m I(m)]^q \\
&\quad \times \left(\int_0^h |u(t)|^{m(p+q)}\, d_\alpha t\right)^{\frac{p}{p+q}} \left(\int_0^h \left|D_\alpha(u)(t)\right|^{m(p+q)} d_\alpha t\right)^{\frac{q}{p+q}}.
\end{aligned}$$

Dividing the both sides of (3.23) by $\left(\int_0^h |u(t)|^{m(p+q)}\, d_\alpha t\right)^{\frac{p}{p+q}}$ and taking the $\frac{p+q}{q}$th power on both sides of resulting inequality we get the desired inequality (3.15). □

Theorem 10. *Let $\alpha \in (0,1]$ and $p \geq 0$, $q \geq 1$, $r \geq 0$, $m \geq 1$ be real numbers. If u is an α-fractional differantiable function on $(0,h)$ satisfies $u(0) = u(h) = 0$, then the following inequalities for conformable fractional integral hold*

$$\begin{aligned}
(3.24)\qquad & \int_0^h |u(t)|^{m(p+q)} \left|D_\alpha(u)(t)\right|^{mr} d_\alpha t \\
&\leq [(p+q)^m I(m)]^q \int_0^h |u(t)|^{mp} \left|D_\alpha(u)(t)\right|^{m(q+r)} d_\alpha t.
\end{aligned}$$

and

$$\begin{aligned}
(3.25)\qquad & \int_0^h |u(t)|^{m(p+q)} \left|D_\alpha(u)(t)\right|^{mr} d_\alpha t \\
&\leq [(p+q)^m I(m)]^{p+q} \int_0^h \left|D_\alpha(u)(t)\right|^{m(p+q+r)} d_\alpha t.
\end{aligned}$$

where $I(m)$ defined by (3.16).

Proof. By using the Hölder's inequality for conformable integral with indices $(q+r)/r$ and $(q+r)/q$ and inequality (3.14), we find that

$$\begin{aligned}
& \int_0^h |u(t)|^{m(p+q)} |D_\alpha(u)(t)|^{mr} d_\alpha t \\
= & \int_0^h \left[|u(t)|^{m(pr/(q+r))} |D_\alpha(u)(t)|^{mr}\right] \left[|u(t)|^{m(p+q)-m(pr/(q+r))}\right] d_\alpha t \\
\leq & \left[\int_0^h \left[|u(t)|^{mp} |D_\alpha(u)(t)|^{m(q+r)}\right] d_\alpha t\right]^{\frac{r}{q+r}} \left[\int_0^h |u(t)|^{m(p+q+r)}\right]^{\frac{q}{q+r}} \\
\leq & \left[\int_0^h \left[|u(t)|^{mp} |D_\alpha(u)(t)|^{m(q+r)}\right] d_\alpha t\right]^{\frac{r}{q+r}} \\
& \times \left[[(p+q)^m I(m)]^{q+r} \int_0^h |u(t)|^{mp} |D_\alpha(u)(t)|^{m(q+r)} d_\alpha t\right]^{\frac{q}{q+r}} \\
= & [(p+q)^m I(m)]^q \int_0^h |u(t)|^{mp} |D_\alpha(u)(t)|^{m(q+r)} d_\alpha t.
\end{aligned}$$

This completes the proof of the inequality (3.24).

By using the Hölder's inequality for conformable integral with indices $(p+q)/p$ and $(p+q)/q$ to the integral on the right hand side of (3.24), we have

$$\begin{aligned}
(3.26) \quad & \int_0^h |u(t)|^{m(p+q)} |D_\alpha(u)(t)|^{mr} d_\alpha t \\
\leq & [(p+q)^m I(m)]^q \\
& \times \int_0^h \left[|u(t)|^{mp} |D_\alpha(u)(t)|^{m(rp/(p+q))}\right] \left[|D_\alpha(u)(t)|^{m(q+r)-rp/(p+q)}\right] d_\alpha t \\
\leq & [(p+q)^m I(m)]^q \left[\int_0^h |u(t)|^{m(p+q)} |D_\alpha(u)(t)|^{mr} d_\alpha t\right]^{\frac{p}{p+q}} \\
& \times \left[\int_0^h |D_\alpha(u)(t)|^{m(p+q+r)} d_\alpha t\right]^{\frac{q}{p+q}}.
\end{aligned}$$

Dividing the both sides of (3.26) by $\left[\int_0^h |u(t)|^{m(p+q)} |D_\alpha(u)(t)|^{mr} d_\alpha t\right]^{\frac{p}{p+q}}$ and taking the $\frac{p+q}{q}$th power on both sides of resulting inequality we get the required inequality (3.25). □

Remark 5. *If we choose $\alpha = 1$ in Theorem 9 and Theorem 10, then the inequalities (3.14)-(3.15) and (3.24)-(3.25) reduce the results given in* [12].

Theorem 11. *Let $\alpha \in (0,1]$ and $p \geq 0$, $q \geq 1$, $\beta \geq 0$ be real numbers and f be real valued α-fractional differentiable function on $(0,b)$ for fixed real number $b > 0$.*

Then the following inequalities for conformable fractional integral hold

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha}\left|f(t)\right|^{p+q} d_\alpha t \\
&\leq \left(\int_0^b t^{(\beta+1)}\left|f(t)\right|^{p}\left[t^{1-\alpha}\frac{|f(t)|}{b}+\left|D_\alpha(f)(t)\right|\right]^{q} d_\alpha t\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^b \left[t^{(\beta+1)}\left|f(t)\right|^{p+q}\right] d_\alpha t\right)^{\frac{q-1}{q}}
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha}\left|f(t)\right|^{p+q} d_\alpha t \\
&\leq M\left(\int_0^b t^{(\beta+1)q}\left|f(t)\right|^{p}\left[t^{1-\alpha}\frac{|f(t)|}{b}+\left|D_\alpha(f)(t)\right|\right]^{q} d_\alpha t\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^b \left|f(t)\right|^{p+q} d_\alpha t\right)^{\frac{q-1}{q}}
\end{aligned} \tag{3.28}$$

where $M=\max\left\{\frac{\beta+2}{\beta+1},\frac{2(p+q)}{\beta+1}\right\}$.

Proof. Using the integrating by parts in comformable integral, we have

$$\begin{aligned}
&\int_0^b \left[t^{\beta+1}-\frac{t^{\beta+2}}{b}\right]\left|f(t)\right|^{p+q-1}\left|D_\alpha(f)(t)\right| sgn\left(f(t)\right) d_\alpha t \\
&= -\frac{1}{p+q}\int_0^b \left[(\beta+1)\,t^{\beta+1-\alpha}-(\beta+2)\,\frac{t^{\beta+2-\alpha}}{b}\right]\left|f(t)\right|^{p+q} d_\alpha t.
\end{aligned} \tag{3.29}$$

From (3.29), we get

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha}\left|f(t)\right|^{p+q} d_\alpha t \\
&= \left(\frac{\beta+2}{\beta+1}\right)\int_0^b \frac{t^{\beta+2-\alpha}}{b}\left|f(t)\right|^{p+q} d_\alpha t \\
&\quad -\frac{p+q}{\beta+1}\int_0^b \left[t^{\beta+1}-\frac{t^{\beta+2}}{b}\right]\left|f(t)\right|^{p+q-1}\left|D_\alpha(f)(t)\right| sgn\left(f(t)\right) d_\alpha t \\
&\leq \left(\frac{\beta+2}{\beta+1}\right)\int_0^b \frac{t^{\beta+2-\alpha}}{b}\left|f(t)\right|^{p+q} d_\alpha t \\
&\quad +\frac{p+q}{\beta+1}\int_0^b t^{\beta+1}\left[1+\frac{t}{b}\right]\left|f(t)\right|^{p+q-1}\left|D_\alpha(f)(t)\right| d_\alpha t,
\end{aligned}$$

and then by Hölder's inegral inequality for comformable integral

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha} |f(t)|^{p+q} d_\alpha t \\
\leq &\left(\frac{\beta+2}{\beta+1}\right)\int_0^b \frac{t^{\beta+2-\alpha}}{b} |f(t)|^{p+q} d_\alpha t \\
&+\frac{2(p+q)}{\beta+1}\int_0^b t^{\beta+1} |f(t)|^{p+q-1} |D_\alpha(f)(t)| d_\alpha t \\
\leq &M\int_0^b t^{\beta+1} |f(t)|^{p+q-1}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right] d_\alpha t \\
= &M\int_0^b t^{(\beta+1)/q} |f(t)|^{p/q}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right]\left[t^{(\beta+1)\frac{(q-1)}{q}} |f(t)|^{p+q-1-\frac{p}{q}}\right] d_\alpha t \\
\leq &\left(\int_0^b t^{(\beta+1)} |f(t)|^{p}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right]^q d_\alpha t\right)^{\frac{1}{q}}\left(\int_0^b \left[t^{(\beta+1)} |f(t)|^{p+q}\right] d_\alpha t\right)^{\frac{q-1}{q}}.
\end{aligned}$$

In order to establish inequality (3.28), we use above second inequality, that is

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha} |f(t)|^{p+q} d_\alpha t \\
\leq &M\int_0^b t^{\beta+1} |f(t)|^{p+q-1}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right] d_\alpha t \\
= &M\int_0^b t^{\beta+1} |f(t)|^{p/q}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right] |f(t)|^{p+q-1-\frac{p}{q}} d_\alpha t
\end{aligned}$$

and by Hölder's inegral inequality for comformable integral

$$\begin{aligned}
&\int_0^b t^{\beta+1-\alpha} |f(t)|^{p+q} d_\alpha t \\
\leq &M\left(\int_0^b t^{(\beta+1)q} |f(t)|^{p}\left[t^{1-\alpha}\frac{|f(t)|}{b}+|D_\alpha(f)(t)|\right]^q d_\alpha t\right)^{\frac{1}{q}}\left(\int_0^b |f(t)|^{p+q} d_\alpha t\right)^{\frac{q-1}{q}}
\end{aligned}$$

which is completed the proof. □

Remark 6. *If we choose $\alpha = 1$ in Theorem 11, then the inequalities (3.27) and (3.28) reduce the results given in* [13].

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