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(p,q) –INTEGRAL INEQUALITIES ON FINITE INTERVALS

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ABSTRACT. In this paper, we obtain (p,q)-analogues of some of the well known basic inequalities in restricted forms. Hölder, Minkowski, Cauchy, Hermite-Hadamard, Trapezoid and Ostrowski (p,q)-integral inequalities are proved.

1. INTRODUCTION

Mathematical inequalities play an important role on many branches of mathematics as analysis, differential equations, geometry etc. In recent years $q-$integral inequalities and some of generalization forms of quantum type inequalities have been studied by many authors, see [4, 5, 6, 9, 15, 16, 14]. One of the generalization of $q-$calculus is $(p,q)-$calculus, see [7, 8, 13].

The aim of this paper is to establish $(p,q)-$analogues of some well known integral inequalities. Hölder, Minkowski, Hermite-Hadamard, Trapezoid, Ostrowski integral inequalities are considered. The results are compared with the $q-$analogs of these inequalities and also the with the classical forms.

Now, we give some definitions and results via $(p,q)-$calculus which will be used in the sequel [7, 8, 13]. Let $0<q<p\leq 1$. The $(p,q)-$integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}. \tag{1.1}$$

For each $k,n\in\mathbb{N}$, $n\geq k\geq 0$, the $(p,q)-$factorial and $(p,q)-$binomial are defined by

$$\begin{aligned}[n]_{p,q}! &= \prod_{k=1}^{n}[k]_{p,q}\,,\ n\geq 1,\ \ [0]_{p,q}!=1\\ \begin{bmatrix} n\\ k\end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.\end{aligned}$$

Definition 1. *Let $f:\mathbb{R}\to\mathbb{R}$. The $(p,q)-$derivative of the function f is defined as*

$$D_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x},\ x\neq 0 \tag{1.2}$$

provided that $D_{p,q}f(0)=f'(0)$.

$(p,q)-$derivative of a function is a linear operator. For any constants a and b,

$$D_{p,q}[af(x)+bg(x)]=aD_{p,q}f(x)+bD_{p,q}f(x).$$

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The (p,q) −derivative of a product is given as

$$D_{p,q}\left[f(x)g(x)\right] = f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x), \quad = g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x). \tag{1.3}$$

The (p,q) −derivative fulfills the following product rules

$$D_{p,q}\left[\frac{f(x)}{g(x)}\right] = \frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)} = \frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

The (p,q) −power basis is defined by

$$(x\oplus a)_{p,q}^{n}=(x\oplus a)(px\oplus qa)\left(p^{2}x\oplus q^{2}a\right)\cdots\left(p^{n-1}x\oplus q^{n-1}a\right)$$

and

$$(x\ominus a)_{p,q}^{n}=(x\ominus a)(px\ominus qa)\left(p^{2}x\ominus q^{2}a\right)\cdots\left(p^{n-1}x\ominus q^{n-1}a\right)$$

The following statements hold true:

$$D_{p,q}(x\ominus a)_{p,q}^{n} = [n]_{p,q}(px\ominus a)_{p,q}^{n-1}$$

$$D_{p,q}(\alpha x\ominus a)_{p,q}^{n} = \alpha[n]_{p,q}(\alpha px\ominus a)_{p,q}^{n-1}, \alpha\in\mathbb{C}$$

$$D_{p,q}(a\ominus x)_{p,q}^{n} = -[n]_{p,q}(a\ominus qx)_{p,q}^{n-1}.$$

Definition 2. *Let $f:C[0,a]\to\mathbb{R}$ $(a>0)$ then the (p,q) −integration of f defined by*

$$\int_{0}^{a}f(t)\,d_{p,q}t = (q-p)\,a\sum_{n=0}^{\infty}\frac{p^{n}}{q^{n+1}}f\left(\frac{p^{n}}{q^{n+1}}a\right) \text{ if } \left|\frac{p}{q}\right|<1 \tag{1.4}$$

$$\int_{0}^{a}f(t)\,d_{p,q}t = (p-q)\,a\sum_{n=0}^{\infty}\frac{q^{n}}{p^{n+1}}f\left(\frac{q^{n}}{p^{n+1}}a\right) \text{ if } \left|\frac{p}{q}\right|>1.$$

The (p,q) −integral on an interval defined as

$$\int_{a}^{b}f(t)\,d_{p,q}t=\int_{0}^{b}f(t)\,d_{p,q}t-\int_{0}^{a}f(t)\,d_{p,q}t.$$

If F is an antiderivative of the function f and f is continuous at $t=0$

$$\int_{a}^{b}f(t)\,d_{p,q}t=F(b)-F(a)$$

and for any function of f, we have

$$D_{p,q}\int_{a}^{x}f(t)\,d_{p,q}t=f(x).$$

We denote the following set by I

$$I=[a,b]_{p,q}=\left\{b\frac{q^{k}}{p^{k}}:0\leq k\leq n\right\}$$

where $b>0$, $a=b\frac{q^{n}}{p^{n}}$ and $n\in\mathbb{N}$ and we display the integral on I as $\int_{I}f(t)\,d_{p,q}t$. So it easy to show that

$$\int_{a}^{b}f(t)\,d_{p,q}t=(p-q)\,b\sum_{k=0}^{n-1}\frac{q^{k}}{p^{k+1}}f\left(\frac{q^{k}}{p^{k+1}}b\right). \tag{1.5}$$

The formula of (p,q) −integration by parts is given by

$$\int_a^b f(px)\, D_{p,q}g(x)\, d_{p,q}t = f(x)\, g(x)|_a^b - \int_a^b g(qx)\, D_{p,q}f(x)\, d_{p,q}t. \tag{1.6}$$

All notions written above reduce to the q−analogs when $p=1$. For more details, see [7, 13].

2. PRELIMINARIES

Lemma 1. *The following formula holds:*

$$\begin{aligned} \int_0^a (m-nt)\, d_{p,q}t &= ma - \frac{a^2 n}{p+q} \\ \int_0^a (nt-m)\, d_{p,q}t &= \frac{a^2 n}{p+q} - ma. \end{aligned} \tag{2.1}$$

Proof. From Definition 2, we have

$$\begin{aligned} \int_0^a (m-nt)\, d_{p,q}t &= (p-q)\, a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(m - \frac{q^k}{p^{k+1}} na \right) \\ &= (p-q)\, a \left(\frac{m}{p} \frac{1}{1-\frac{q}{p}} - \frac{na}{p^2} \frac{1}{1-\frac{q^2}{p^2}} \right) \\ &= ma - \frac{a^2 n}{p+q} \end{aligned}$$

and similarly it is easy to see that,

$$\begin{aligned} \int_0^a (nt-m)\, d_{p,q}t &= (p-q)\, a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{q^k}{p^{k+1}} na - m \right) \\ &= (p-q)\, a \left(\frac{na}{p^2} \frac{1}{1-\frac{q^2}{p^2}} - \frac{m}{p} \frac{1}{1-\frac{q}{p}} \right) \\ &= \frac{a^2 n}{p+q} - ma. \end{aligned}$$

□

Lemma 2. *The following formula holds:*

$$\int_0^a t(nt-m)\, d_{p,q}t = \frac{na^3}{p^2+pq+q^2} - \frac{ma^2}{p+q}. \tag{2.2}$$

Proof. From Definition 2, we have

$$\begin{aligned} \int_0^a t(nt-m)\, d_{p,q}t &= n \int_0^a t^2 d_{p,q}t - m \int_0^a t d_{p,q}t \\ &= n(p-q)\, a \sum_{K=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{q^k}{p^{k+1}} a \right)^2 - m(p-q)\, a \sum_{K=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{q^k}{p^{k+1}} a \right) \\ &= \frac{na^3}{p^2+pq+q^2} - \frac{ma^2}{p+q}. \end{aligned}$$

□

3. MAIN RESULTS

Lets start with $(p,q)-$Hölder integral ineqality:

Theorem 1. *Let f and g be two functions defined on I, $0<q<p\leq 1$ and $s_1,s_2>1$ with $\frac{1}{s_1}+\frac{1}{s_2}=1$. Then*

$$\int_I |f(t)\,g(t)|\,d_{p,q}t \leq \left(\int_I |f(t)|^{s_1}\,d_{p,q}t\right)^{\frac{1}{s_1}}\left(\int_I |g(t)|^{s_2}\,d_{p,q}t\right)^{\frac{1}{s_2}}. \tag{3.1}$$

Proof. From Definition 2 and discrete Hölder inequality, we get

$$\begin{aligned}
\int_I |f(t)\,g(t)|\,d_{p,q}t &= (p-q)\,b\sum_{n=0}^{k-1}\frac{q^n}{p^{n+1}}\left|f\left(\frac{q^n}{p^{n+1}}b\right)g\left(\frac{q^n}{p^{n+1}}b\right)\right| \\
&= (p-q)\,b\sum_{n=0}^{k-1}\left|f\left(\frac{q^n}{p^{n+1}}b\right)\left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{s_1}}\right|\left|g\left(\frac{q^n}{p^{n+1}}b\right)\left(\frac{q^n}{p^{n+1}}\right)^{\frac{1}{s_2}}\right| \\
&\leq \left((p-q)\,b\sum_{n=0}^{k-1}\left|f\left(\frac{q^n}{p^{n+1}}b\right)\right|^{s_1}\left(\frac{q^n}{p^{n+1}}\right)\right)^{\frac{1}{s_1}} \\
&\quad\times\left((p-q)\,b\sum_{n=0}^{k-1}\left|g\left(\frac{q^n}{p^{n+1}}b\right)\right|^{s_2}\left(\frac{q^n}{p^{n+1}}\right)\right)^{\frac{1}{s_2}} \\
&= \left(\int_I |f(t)|^{s_1}\,d_{p,q}t\right)^{\frac{1}{s_1}}\left(\int_I |g(t)|^{s_2}\,d_{p,q}t\right)^{\frac{1}{s_2}}.
\end{aligned}$$

Thus, the proof is complete. □

It easy to show that we obtain the same result in the statement $p<q$.

Corollary 1. *Under the assumptions of Theorem 1, if we take $s_1=s_2=2$, then we have the following formula,*

$$\int_I |f(t)\,g(t)|\,d_{p,q}t \leq \left(\int_I |f(t)|^{2}\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_I |g(t)|^{2}\,d_{p,q}t\right)^{\frac{1}{2}} \tag{3.2}$$

which we call $(p,q)-$Cauchy-Schwarz integral inequality.

Corollary 2. *Let f and g be two functions defined on $[0,b]$, $0<q<p\leq 1$ and $s_1,s_2>1$ with $\frac{1}{s_1}+\frac{1}{s_2}=1$. Then*

$$\int_0^b |f(t)\,g(t)|\,d_{p,q}t \leq \left(\int_0^b |f(t)|^{s_1}\,d_{p,q}t\right)^{\frac{1}{s_1}}\left(\int_0^b |g(t)|^{s_2}\,d_{p,q}t\right)^{\frac{1}{s_2}}$$

and

$$\int_0^b |f(t)\,g(t)|\,d_{p,q}t \leq \left(\int_0^b |f(t)|^{2}\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^b |g(t)|^{2}\,d_{p,q}t\right)^{\frac{1}{2}}.$$

Remark 1. *If $p=1$, (3.1) and (3.2) reduces to q-Hölder integral inequality and q-Cauchy-Schwarz integral inequality respectively.*

Theorem 2. *Let f and g real-valued functions on I such that $|f|^{s_1}$, $|g|^{s_1}$ and $|f+g|^{s_1}$ are (p,q) −integrable functions on $[a,b]$, $0<q<p\leq 1$ and $s_1>1$. Then*

$$\left(\int_I |f(t)+g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} \leq \left(\int_I |f(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} + \left(\int_I |g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}}. \tag{3.3}$$

Equality holds if and only if $f(t)=0$ almost everwhere or $g(t)=\mu f(t)$ almost everywhere with a constant $\mu\geq 0$.

Proof. Since $|f|^{s_1}$, $|g|^{s_1}$ and $|f+g|^{s_1}$ are (p,q) −integrable on $[a,b]$, by using the triangle inequality, we can write

$$\begin{aligned}\int_I |f(t)+g(t)|^{s_1} d_{p,q}t &= \int_I |f(t)+g(t)|\,|f(t)+g(t)|^{s_1-1} d_{p,q}t\\ &\leq \int_I |f(t)|\,|f(t)+g(t)|^{s_1-1} d_{p,q}t + \int_I |g(t)|\,|f(t)+g(t)|^{s_1-1} d_{p,q}t.\end{aligned}$$

Taking $s_1,s_2>1$ with $\dfrac{1}{s_1}+\dfrac{1}{s_2}=1$ and using (p,q) −Hölder integral inequality, we have

$$\int_I |f(t)|\,|f(t)+g(t)|^{s_1-1} d_{p,q}t \leq \left(\int_I |f(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} \left(\int_I |f(t)+g(t)|^{(s_1-1)s_2} d_{p,q}t\right)^{\frac{1}{s_2}} \tag{3.4}$$

and

$$\int_I |g(t)|\,|f(t)+g(t)|^{s_1-1} d_{p,q}t \leq \left(\int_I |g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} \left(\int_I |f(t)+g(t)|^{(s_1-1)s_2} d_{p,q}t\right)^{\frac{1}{s_2}}. \tag{3.5}$$

Since $(s_1-1)s_2=s_1$, from (3.4) and (3.5), it easy to see that

$$\left(\int_I |f(t)+g(t)|^{s_1} d_{p,q}t\right)^{1-\frac{1}{s_2}} \leq \left(\int_I |f(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} + \left(\int_I |g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}}$$

from which we obtain the required inequality. □

Corollary 3. *Let f and g real-valued functions on $[0,b]$ such that $|f|^{s_1}$, $|g|^{s_1}$ and $|f+g|^{s_1}$ are integrable functions on $[0,b]$, $0<q<p\leq 1$ and $s_1>1$. Then*

$$\left(\int_0^b |f(t)+g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} \leq \left(\int_0^b |f(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}} + \left(\int_0^b |g(t)|^{s_1} d_{p,q}t\right)^{\frac{1}{s_1}}.$$

Equality holds if and only if $f(t)=0$ almost everwhere or $g(t)=\mu f(t)$ almost everywhere with a constant $\mu\geq 0$.

Remark 2. *If $p=1$, (3.3) reduces to*

$$\left(\int_0^x |f(t)+g(t)|^{s_1} d_q t\right)^{\frac{1}{s_1}} \leq \left(\int_0^x |f(t)|^{s_1} d_q t\right)^{\frac{1}{s_1}} + \left(\int_0^x |g(t)|^{s_1} d_q t\right)^{\frac{1}{s_1}}$$

which can be called q-Minkowski integral inequality.

Next, we present the (p,q) −Hermite-Hadamard integral inequality on $[a,b]$.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function with $a=b\frac{q^n}{p^n}$. Then*

(3.6)

$$\begin{aligned} f\left(\frac{a+b}{p+q}\right) &\leq \frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t \\ &\leq \frac{1}{p+q}\left(q\frac{b(p+q)-(a+b)}{bq-a}f\left(\frac{pa}{q}\right)+\frac{q(a+b)-a(p+q)}{bq-a}f(b)\right). \end{aligned}$$

Proof. From (1.5) we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t &= (p-q)\frac{b}{b-a}\sum_{k=0}^{n-1}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}b\right) \\ &= p\frac{1-\frac{q}{p}}{1-\frac{q^n}{p^n}}\sum_{k=0}^{n-1}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}b\right). \end{aligned}$$

If we consider

$$\begin{aligned} \lambda &= \left[\sum_{k=0}^{n-1}\frac{q^k}{p^{k+1}}\right]^{-1}\sum_{k=0}^{n-1}\frac{q^k}{p^{k+1}}\frac{q^k}{p^{k+1}}b \\ &= p\frac{1-\frac{q}{p}}{1-\frac{q^n}{p^n}}\frac{b}{p^2}\frac{1-\left(\frac{q^n}{p^n}\right)^2}{1-\left(\frac{q}{p}\right)^2} \\ &= \frac{b\left(1+\frac{q^n}{p^n}\right)}{p+q}=\frac{a+b}{p+q} \end{aligned}$$

and apply Jensen inequality for the convex functions, we have

$$f(\lambda)=f\left(\frac{a+b}{p+q}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t.$$

On the other hand, by using the theorem via reverse Jensen inequality, cited in [11], we obtain

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t &\leq \frac{b-\lambda}{b-b\frac{q^{n-1}}{p^{n-1}}}f\left(b\frac{q^{n-1}}{p^{n-1}}\right)+\frac{\lambda-\frac{p}{q}a}{b-b\frac{q^{n-1}}{p^{n-1}}}f(b) \\ &= \frac{b-\frac{a+b}{p+q}}{b-\frac{p}{q}a}f\left(\frac{p}{q}a\right)+\frac{\frac{a+b}{p+q}-\frac{p}{q}a}{b-\frac{p}{q}a}f(b) \\ &= \frac{1}{p+q}\left(q\frac{b(p+q)-(a+b)}{bq-pa}f\left(\frac{pa}{q}\right)+\frac{q(a+b)-a(p+q)}{bq-pa}f(b)\right). \end{aligned}$$

□

Remark 3. *If $p=1$, (3.6) reduces to* [10, Theorem 5.1].

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a $(p,q)-$differentiable function and $D_{p,q}f$ be continuous with $0<q<p\leq 1$. Then*

(3.7)

$$\left|p\int_a^b f(qt)\,d_{p,q}t-(b-a)\frac{f(a)+f(b)}{2}\right|\leq\|D_{p,q}f\|\left(\frac{(1-p)(a+b)^2}{2p}-\frac{(a+b)^2}{2p(p+q)}+\frac{a^2+b^2}{p+q}\right)$$

Proof. By (p,q) −integration by parts, we have

$$\begin{aligned}\int_a^b \left(pt-\frac{a+b}{2}\right) D_{p,q}f(t)\,d_{p,q}t &= \left(t-\frac{a+b}{2}\right) f(t)\Big|_{x=a}^{x=b} - \int_a^b f(qt)\,p d_{p,q}t \\ &= \frac{b-a}{2} f(b) + \frac{b-a}{2} f(a) - p\int_a^b f(qt)\,d_{p,q}t\end{aligned}$$

Using the absolute value property, it is easy to see that

$$\begin{aligned}&\left|p\int_a^b f(qt)\,d_{p,q}t-(b-a)\frac{f(a)+f(b)}{2}\right| \\ &\leq \int_a^b \left|pt-\frac{a+b}{2}\right| |D_{p,q}f(t)|\,d_{p,q}t \\ &\leq \|D_{p,q}f\| \int_a^b \left|pt-\frac{a+b}{2}\right| d_{p,q}t\end{aligned} \tag{3.8}$$

From Lemma 1, we obtain

$$\begin{aligned}\int_a^b \left|pt-\frac{a+b}{2}\right| d_{p,q}t &= \int_a^{\frac{a+b}{2p}} \left(\frac{a+b}{2}-pt\right) d_{p,q}t + \int_{\frac{a+b}{2p}}^b \left(pt-\frac{a+b}{2}\right) d_{p,q}t \\ &= \int_0^{\frac{a+b}{2p}} \left(\frac{a+b}{2}-pt\right) d_{p,q}t - \int_0^a \left(\frac{a+b}{2}-pt\right) d_{p,q}t \\ &\quad + \int_0^b \left(pt-\frac{a+b}{2}\right) d_{p,q}t - \int_0^{\frac{a+b}{2p}} \left(pt-\frac{a+b}{2}\right) d_{p,q}t \\ &= 2\left(\frac{a+b}{2p}\right)^2 p - 2\left(\frac{a+b}{2p}\right)^2 \frac{p}{p+q} - \frac{(a+b)^2}{2} + \frac{\left(a^2+b^2\right)p}{p+q}\end{aligned}$$

$$= 2p\left(\frac{a+b}{2p}\right)^2 \left(1-\frac{1}{p+q}\right) - \frac{(a+b)^2}{2} + \frac{\left(a^2+b^2\right)p}{p+q}. \tag{3.9}$$

Combining (3.8) with (3.9), we have

$$\begin{aligned}&\left|p\int_a^b f(qt)\,d_{p,q}t-(b-a)\frac{f(a)+f(b)}{2}\right| \\ &\leq \|D_{p,q}f\| \left(2p\left(\frac{a+b}{2p}\right)^2 \left(1-\frac{1}{p+q}\right) - \frac{(a+b)^2}{2} + \frac{\left(a^2+b^2\right)p}{p+q}\right).\end{aligned}$$

□

Remark 4. *If $p=1$, then (3.7) reduces to*

$$\left|\int_a^b f(qt)\,d_qt-(b-a)\frac{f(a)+f(b)}{2}\right| \leq \|D_qf\| \frac{(b-a)^2}{2(1+q)}$$

and also if $q\to 1$, it turns the classical form

$$\left|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}\right| \leq \|f'\| \frac{(b-a)^2}{4}.$$

See [3, 12, 15].

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a twice $(p,q)-$differentiable function and $D_{p,q}^{2}f$ be continuous with $0<q<p\leq 1$. Then*

$$\left|\int_{a}^{b} f\left(q^{2}t\right)d_{p,q}t-\left(\frac{bq-ap}{p+q}f\left(qb\right)+\frac{bp-aq}{p+q}f\left(qa\right)\right)\right| \leq \left\|D_{p,q}^{2}f\right\|\frac{p}{p+q}\left(\frac{\left(b^{3}-a^{3}\right)pq^{2}}{\left(p^{2}+pq+q^{2}\right)\left(p+q\right)}-\frac{\left(b-a\right)qab}{p+q}\right). \tag{3.10}$$

Proof. Applying $(p,q)-$integration by parts, we have

$$\int_{a}^{b}\left(pt-a\right)\left(b-pt\right)D_{p,q}^{2}f\left(t\right)d_{p,q}t = -\int_{a}^{b}D_{p,q}f\left(qt\right)D_{p,q}\left(\left(t-a\right)\left(b-t\right)\right)d_{p,q}t.$$

From the $(p,q)-$derivative of the product, we obtain

$$\begin{aligned} D_{p,q}\left(\left(t-a\right)\left(b-t\right)\right) &= \left(pt-a\right)D_{p,q}\left(b-t\right)+\left(b-qt\right)D_{p,q}\left(t-a\right)\\ &= \left(b-qt\right)-\left(pt-a\right)\\ &= \left(a+b\right)-t\left(p+q\right). \end{aligned}\tag{3.11}$$

Applying $(p,q)-$integration by parts again and by using (3.11), we see

$$\begin{aligned} &-\int_{a}^{b}D_{p,q}f\left(qt\right)D_{p,q}\left(\left(t-a\right)\left(b-t\right)\right)d_{p,q}t\\ =& \int_{a}^{b}\left(t\left(p+q\right)-\left(a+b\right)\right)D_{p,q}f\left(qt\right)d_{p,q}t\\ =& \left[\left(t\left(1+\frac{q}{p}\right)-\left(a+b\right)\right)f\left(qt\right)\right]_{t=a}^{t=b}-\int_{a}^{b}f\left(q^{2}t\right)D_{p,q}\left(t\left(1+\frac{q}{p}\right)-\left(a+b\right)\right)d_{p,q}t\\ =& \left(bf\left(qb\right)-af\left(qa\right)\right)\left(1+\frac{q}{p}\right)-\left(a+b\right)\left(f\left(qb\right)-f\left(qa\right)\right)-\left(1+\frac{q}{p}\right)\int_{a}^{b}f\left(q^{2}t\right)d_{p,q}t. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\left(1+\frac{q}{p}\right)\int_{a}^{b}f\left(q^{2}t\right)d_{p,q}t-\left(bf\left(qb\right)-af\left(qa\right)\right)\left(1+\frac{q}{p}\right)-\left(a+b\right)\left(f\left(qb\right)-f\left(qa\right)\right)\right|\\ \leq& \int_{a}^{b}\left(pt-a\right)\left(b-pt\right)D_{p,q}^{2}f\left(t\right)d_{p,q}t\\ \leq& \left\|D_{p,q}^{2}f\right\|\int_{a}^{b}\left(pt-a\right)\left(b-pt\right)d_{p,q}t. \end{aligned}\tag{3.12}$$

From Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
\int_a^b (pt-a)(b-pt)\,d_{p,q}t &= b\int_a^b (pt-a)\,d_{p,q}t - p\int_a^b t(pt-a)\,d_{p,q}t \\
&= b\int_0^b (pt-a)\,d_{p,q}tb - b\int_0^a (pt-a)\,d_{p,q}t \\
&\quad -p\int_0^b t(pt-a)\,d_{p,q}t + p\int_0^a t(pt-a)\,d_{p,q}t \\
&= b\left(\frac{b^2p}{p+q} - ab - \left(\frac{a^2p}{p+q} - a^2\right)\right) \\
&\quad -p\left(\frac{pb^3}{p^2+pq+q^2} - \frac{ab^2}{p+q}\right) + p\left(\frac{pa^3}{p^2+pq+q^2} - \frac{a^3}{p+q}\right) \\
&= q(a-b)\frac{ab\left(p^2+q^2\right) - pq\left(a^2+b^2\right)}{(p+q)\left(p^2+pq+q^2\right)}. \qquad (3.13)
\end{aligned}$$

Combining (3.12) with (3.13), we get the desired inequality. □

Remark 5. *If $p=1$, $q\to 1$, then (3.12) reduces to*

$$\left|\int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2}\right| \le \|f''\|\frac{(b-a)^3}{12}.$$

See, [3, 12].

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a (p,q) differentiable function and $D_{p,q}f$ be continuous with $0<q<p\le 1$. Then*

$$\begin{aligned}
\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t\right| &\le \|D_{p,q}f\|(b-a)\left[\frac{2(p+q-1)}{1+q}\left(\frac{x-\frac{(a+b)(p+q)}{4(p+q-1)}}{b-a}\right)^2\right. \\
&\quad \left. -\frac{(p+q)^2(a+b)^2}{16(p+q-1)^2(b-a)^2} + \frac{a^2+b^2}{(p+q)(b-a)^2}\right]. \qquad (3.14)
\end{aligned}$$

Proof. Using the Lagrange Mean Value Theorem, we obtain

$$\begin{aligned}
\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,d_{p,q}t\right| &= \frac{1}{b-a}\left|\int_a^b (f(x)-f(t))\,d_{p,q}t\right| \\
&\le \frac{1}{b-a}\int_a^b |f(x)-f(t)|\,d_{p,q}t \\
&\le \frac{\|D_{p,q}f\|}{b-a}\int_a^b |x-t|\,d_{p,q}t. \qquad (3.15)
\end{aligned}$$

From Lemma 1, we have

$$
\begin{aligned}
\int_a^b |x-t|\, d_{p,q}t &= \int_a^x (x-t)\, d_{p,q}t + \int_x^b (t-x)\, d_{p,q}t \\
&= \int_0^x (x-t)\, d_{p,q}t - \int_0^a (x-t)\, d_{p,q}t \\
&\quad + \int_0^b (t-x)\, d_{p,q}t - \int_0^x (t-x)\, d_{p,q}t \\
&= \frac{a^2-x^2}{p+q} - x(a-x) + \frac{b^2-x^2}{p+q} - x(b-x) \qquad (3.16)\\
&= \frac{2(p+q-1)}{1+q}\left(x - \frac{(a+b)(p+q)}{4(p+q-1)}\right)^2 \\
&\quad - \frac{(p+q)^2(a+b)^2}{16(p+q-1)^2} + \frac{a^2+b^2}{p+q}.
\end{aligned}
$$

Combining (3.15) with (3.16), we get the required inequality. □

Remark 6. *If $p=1$, $q\to 1$, then (3.14) reduces to*

$$
\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\left\| f' \right\|.
$$

See, [3, 12].

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