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(p, q) –INTEGRAL INEQUALITIES

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Abstract. In this paper, we establish (p, q)-analogue of some of the most important integral equalities as trapezoid, Ostrowski, Cauchy-Bunyakowski-Schwarz, Grüss and Grüss-Chebyshev integral inequalities by using (p, q) –derivative and (p, q) –integral on finite intervals.

1. INTRODUCTION

Mathematical inequalities play an important role on many branches of mathematics as analysis, differential equations, geometry etc. In recent years $q-$integral inequalities and some of generalization forms of quantum type inequalities have been studied by many authors, see [3, 4, 5, 8, 14, 15, 16]. One of the generalization of $q-$calculus is (p, q) –calculus, see [6, 7, 13, 17, 18]. The aim of this paper is to establish some new integral inequalities on finite intervals via (p, q) –calculus.

Now, we give some definitions and results via (p, q) –calculus which will be used in the sequel, [6, 7, 13]. Let $0 < q < p \leq 1$. The (p, q) –integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

For each $k, n \in \mathbb{N}$, $n \geq k \geq 0$, the (p, q) –factorial and (p, q) –binomial are defined by

$$\begin{aligned} [n]_{p,q}! &= \prod_{k=1}^{n} [k]_{p,q}, \ n \geq 1, \ [0]_{p,q}! = 1 \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[n-k]_{p,q}! \, [k]_{p,q}!}. \end{aligned}$$

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$. The (p, q) –derivative of the function f is defined as*

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q) x}, \ x \neq 0 \tag{1.1}$$

provided that $D_{p,q} f(0) = f'(0)$.

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Definition 2. *Let $f : C[0,a] \to \mathbb{R}$ $(a>0)$ then the $(p,q)-$integration of f defined by*

$$\begin{aligned}\int_0^a f(t)\,d_{p,q}t &= (q-p)\,a\sum_{n=0}^{\infty}\frac{p^n}{q^{n+1}}f\left(\frac{p^n}{q^{n+1}}a\right) \ \textit{if} \left|\frac{p}{q}\right|<1 \\ \int_0^a f(t)\,d_{p,q}t &= (p-q)\,a\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}a\right) \ \textit{if} \left|\frac{p}{q}\right|>1.\end{aligned} \tag{1.2}$$

The formula of $(p,q)-$integration by parts is given by

$$\int_a^b f(px)\,D_{p,q}g(x)\,d_{p,q}t = f(x)\,g(x)|_a^b - \int_a^b g(qx)\,D_{p,q}f(x)\,d_{p,q}t. \tag{1.3}$$

All notions written above reduce to the $q-$analogs when $p=1$. For more details, see [6, 13].

2. PRELIMINARIES

Let $I := [a,b] \subset \mathbb{R}$ be an interval and $0 < q_k < p_k \le 1$ be constants. The $(p,q)-$derivative of a function f is defined on I at $t \in I$ on $[a,b]$ as:

Definition 3. [17]*Let $f : I \to \mathbb{R}$ be a continuous function and assume that $t \in I$. Then the following equality*

$$\begin{aligned}{}_aD_{p,q}f(t) &= \frac{f(pt+(1-p)a)-f(qt+(1-q)a)}{(p-q)(t-a)}, u \ne u_k \\ {}_aD_{p,q}f(t) &= \lim_{t\to a}{}_aD_{p,q}f(t)\end{aligned} \tag{2.1}$$

is called the (p,q)-derivative of a function f at t.

Obviously, f is (p,q)-differentiable on I provided ${}_aD_{p,q}f(t)$ exists for all $t \in I$. In Definition 3, if $p=1$, then $D_{p,q}f = D_qf$ which is the q-derivative of the function f and also if $q \to 1, a = 0$, (2.1) reduces to q-derivative of the function f, see [8, 16].

Example 1. [17]*For $t \in I$, if $f(t) = (t-a)^n$, then*

$$ {}_aD_{p,q}f(t) = [n]_p\,(t-a)^{n-1} \tag{2.2}$$

where $[n]_{p,q} = \frac{p^n-q^n}{p-q}$.

(p,q)-integral of f on a interval I is defined as follows:

Definition 4. [17]*Let $f : I \to \mathbb{R}$ is a continuous function. Then for $0<q<p\le 1$,*

$$\int_a^t f(s)\,{}_ad_{p,q}s = (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t+\left(1-\frac{q^n}{p^{n+1}}\right)a\right) \tag{2.3}$$

is called (p,q)-integral of f for $t \in I$.

Moreover, if $c\in(a,t)$, then $(p,q)-$integral is defined by

$$\begin{aligned}
&\int_c^t f(s)\ {}_ad_{p,q}s \\
&= \int_a^t f(s)\ {}_ad_{p,q}s-\int_a^c f(s)\ {}_ad_{p,q}s \\
&= (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}t+\left(1-\frac{q^n}{p^{n+1}}\right)a\right) \\
&\quad -(p-q)(c-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}c+\left(1-\frac{q^n}{p^{n+1}}\right)a\right).
\end{aligned}\tag{2.4}$$

Note that if $a=0$ and $p=1$, then (2.4) reduces to $q-$integral of the function. See, [16].

Example 2. *Let $f(t)=t$ for $t\in I$, then we have*

$$\begin{aligned}
\int_a^t f(s)\ {}_ad_{p,q}s &= \int_a^t s\ {}_ad_{p,q}s \\
&= (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}s+\left(1-\frac{q^n}{p^{n+1}}\right)a\right) \\
&= \frac{(t-a)(t-a(1-p-q))}{p+q}.
\end{aligned}\tag{2.5}$$

Example 3. *Let $c\in I$ be a constant. Then*

$$\begin{aligned}
\int_c^b (s-c)\ {}_ad_{p,q}s &= \int_a^b (s-c)\ {}_ad_{p,q}s-\int_a^c (s-c)\ {}_ad_{p,q}s \\
&= \left[\frac{(t-a)(t-a(1-p-q))}{p+q}\right]_a^b-\left[\frac{(t-a)(t-a(1-p-q))}{p+q}\right]_a^c \\
&= \frac{b^2-(p+q)bc+(p+q-1)c^2}{p+q} \\
&\quad -\frac{a(b-c)(2-p-q)}{p+q}.
\end{aligned}\tag{2.6}$$

It is easy to see that when $p=1$, (2.6) reduces $q-$integration of the function and also $q\to1$, it turns to the classical integration.

Theorem 1. [17] *The following formulas hold for $t\in I$:*

(a) ${}_aD_{p,q}\int_a^t f(s)\ {}_ad_{p,q}s=f(t)$

(b) $\int_a^t {}_aD_{p,q}f(s)\ {}_ad_{p,q}s=f(t)$

(c) $\int_c^t {}_aD_{p,q}f(s)\ {}_ad_{p,q}s=f(t)-f(c)$, *for* $c\in(a,t)$.

Theorem 2. [17] *Let $f,g:I\to\mathbb{R}$ are continuous functions. The following formulas hold:*

(a) $\int_a^t[f(s)+g(s)]\ {}_ad_{p,q}s=\int_a^t f(s)\ {}_ad_{p,q}s+\int_a^t g(s)\ {}_ad_{p,q}s$;

(b) $\int_a^t \lambda f(s)\ {}_ad_{p,q}s=\lambda\int_a^t f(s)\ {}_ad_{p,q}s$;

(c) $\int_a^t f(ps+(1-p)a)\ {}_aD_{p,q}g(s)\ {}_ad_{p,q}s=(fg)(s)|_a^t-\int_a^t g(qs+(1-q)a)\ {}_aD_{p,q}f(s)\ {}_ad_{p,q}s$
where $t\in I$, $\lambda\in\mathbb{R}$.

Lemma 1. *For $\lambda \in \mathbb{R}\backslash\{-1\}$, the following formula holds:*

$$\int_a^t (s-a)^{\lambda} \ {}_a d_{p,q} s = \frac{p-q}{p^{\lambda+1}-q^{\lambda+1}} (t-a)^{\lambda+1}.$$

Proof. Let $f(t) = (t-a)^{\lambda+1}$, $t \in I$ and $\lambda \in \mathbb{R}\backslash\{-1\}$. From Definition (), we have

$$\begin{aligned} {}_a D_{p,q} (t-a)^{\lambda+1} &= \frac{(pt+(1-p)a-a)^{\lambda+1} - (qt+(1-q)a-a)^{\lambda+1}}{(p-q)(t-a)} \\ &= [n+1]_p (t-a)^n . \end{aligned} \tag{2.7}$$

Taking $(p,q)-$integration of (2.7), we get the required inequality. □

The $(p,q)-$Hermite-Hadamard, Hölder and Minkowski integral inequalities are defined on $[a,b]$ as follows:

Theorem 3. [17]*Let $f : [a,b] \to \mathbb{R}$ be a convex function and $0 < q < p \leq 1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \ {}_a d_{p,q} x \leq \frac{(p+q-1) f(a) + f(b)}{p+q}. \tag{2.8}$$

Theorem 4. [17]*Let f and g be two functions defined on I, $0 < q < p \leq 1$ and $s_1, s_2 > 1$ with $\frac{1}{s_1} + \frac{1}{s_2} = 1$. Then*

$$\int_a^b |f(t) g(t)| \ {}_a d_{p,q} t \leq \left(\int_a^b |f(t)|^{s_1} \ {}_a d_{p,q} t\right)^{\frac{1}{s_1}} \left(\int_a^b |g(t)|^{s_2} \ {}_a d_{p,q} t\right)^{\frac{1}{s_2}}. \tag{2.9}$$

Theorem 5. [17]*Let f and g real-valued functions on $[a,b]$ such that $|f|^{s_1}$, $|g|^{s_1}$ and $|f+g|^{s_1}$ are (p,q)-integrable functions on $[a,b]$, $0 < q < p \leq 1$ and $s_1 > 1$. Then*

$$\left(\int_a^b |f(t)+g(t)|^{s_1} \ {}_a d_{p,q} t\right)^{\frac{1}{s_1}} \leq \left(\int_a^b |f(t)|^{s_1} \ {}_a d_{p,q} t\right)^{\frac{1}{s_1}} + \left(\int_a^b |g(t)|^{s_1} \ {}_a d_{p,q} t\right)^{\frac{1}{s_1}}. \tag{2.10}$$

Equality holds if and only if $f(t) = 0$ almost everwhere or $g(t) = \mu f(t)$ almost everywhere with a constant $\mu \geq 0$.

3. MAIN RESULTS

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a $(p,q)-$differentiable function and ${}_a D_{p,q} f$ be continuous with $0 < q < p \leq 1$. Then*

$$\begin{aligned} &\left| p \int_a^b f(qt+(1-q)a) \ {}_a d_{p,q} t - (b-a) \frac{f(a)+f(b)}{2} \right| \\ &\leq \|{}_a D_{p,q} f\| \left(\frac{(b-a)^2}{2p} \frac{p+q-pq+p^2-1}{p+q} \right). \end{aligned} \tag{3.1}$$

Proof. By (p,q) −integration by parts, we have

$$\begin{aligned}\int_a^b \left(pt+(1-p)a-\frac{a+b}{2}\right) {}_aD_{p,q}f(t)\ {}_ad_{p,q}t &= \left(t-\frac{a+b}{2}\right)f(t)\Big|_{x=a}^{x=b} - \int_a^b f(qt+(1-q)a)\,p\ {}_ad_{p,q}t \\ &= (b-a)\frac{f(a)+f(b)}{2} - p\int_a^b f(qt+(1-q)a)\ {}_ad_{p,q}t\ .\end{aligned}$$

Using the absolute value property, it is easy to see that

$$\begin{aligned}&\left|p\int_a^b f(qt+(1-q)a)\ {}_ad_{p,q}t-(b-a)\frac{f(a)+f(b)}{2}\right| \\ \leq\ & \int_a^b \left|pt+(1-p)a-\frac{a+b}{2}\right| \left|{}_aD_{p,q}f(t)\right|\ {}_ad_{p,q}t \\ \leq\ & \left\|{}_aD_{p,q}f\right\| \int_a^b \left|pt+(1-p)a-\frac{a+b}{2}\right|\ {}_ad_{p,q}t\end{aligned} \tag{3.2}$$

From Example 2 and Example 3, we obtain

$$\begin{aligned}&\int_a^b \left|pt+(1-p)a-\frac{a+b}{2}\right|\ {}_ad_{p,q}t \\ =\ & p\int_a^{a+\frac{b-a}{2p}}\left(a+\frac{b-a}{2p}-t\right){}_ad_{p,q}t + p\int_{a+\frac{b-a}{2p}}^b \left(t-a-\frac{b-a}{2p}\right){}_ad_{p,q}t \\ =\ & p\left(a+\frac{b-a}{2p}\right)\left(a+\frac{b-a}{2p}-a\right) \\ & -p\frac{\left(a+\frac{b-a}{2p}-a\right)\left(a+\frac{b-a}{2p}-a(1-p-q)\right)}{p+q} \\ & +p\frac{b^2-(p+q)b\left(a+\frac{b-a}{2p}\right)+(p+q-1)\left(a+\frac{b-a}{2p}\right)^2}{p+q} \\ & -p\frac{a\left(b-\left(a+\frac{b-a}{2p}\right)\right)(2-p-q)}{p+q}\end{aligned}$$

$$= \frac{(b-a)^2}{2p}\frac{p+q-pq+p^2-1}{p+q} \tag{3.3}$$

Combining (3.2) with (3.3), we have

$$\begin{aligned}&\left|p\int_a^b f(qt+(1-q)a)\ {}_ad_{p,q}t-(b-a)\frac{f(a)+f(b)}{2}\right| \\ \leq\ & \left\|{}_aD_{p,q}f\right\|\left(\frac{(b-a)^2}{2p}\frac{p+q-pq+p^2-1}{p+q}\right).\end{aligned}$$

From which we obtain the required inequality. □

Remark 1. *If $p=1$, then (3.1) reduces to $q-$trapezoid inequality:*

$$\begin{aligned}&\left|\int_a^b f\left(qt+(1-q)a\right)\ {}_a d_q t-(b-a)\frac{f(a)+f(b)}{2}\right|\\ \leq\ & \left\|{}_a D_q f\right\|\frac{(b-a)^2}{2(1+q)}.\end{aligned}$$

See [15, Theorem 3.3]. *Also if $q\rightarrow 1$, it turns the classical form of trapezoid inequality on $[a,b]$:*

$$\left|\int_a^b f(t)\,dt-(b-a)\frac{f(a)+f(b)}{2}\right|\leq\left\|f'\right\|\frac{(b-a)^2}{4}.$$

See [2, 11].

Theorem 7. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a twice $(p,q)-$differentiable function and ${}_aD_{p,q}^2 f$ be continuous with $0<q<p\leq 1$. Then*

$$\begin{aligned}(3.4)\quad &\left|\int_a^b f\left(q^2t+\left(1-q^2\right)a\right)\ {}_a d_{p,q} t-\frac{(b-a)p}{p+q}\left(\frac{q}{p}f\left(qb+(1-q)a\right)+f(a)\right)\right|\\ \leq\ & \left\|{}_a D_{p,q}^2 f\right\|\frac{(b-a)^3p^2q^2}{(p+q)\left(p^3+2p^2q+2pq^2+q^3\right)}\end{aligned}$$

Proof. Applying $(p,q)-$integration by parts, we have

$$\begin{aligned}&\int_a^b (pt+(1-p)a-a)(b-pt-(1-p)a)\ {}_a D_{p,q}^2 f(t)\ {}_a d_{p,q} t\\ =\ & -\int_a^b {}_a D_{p,q} f\left(qt+(1-q)a\right)\ {}_a D_{p,q}\left((t-a)(b-t)\right)\ {}_a d_{p,q} t\ .\end{aligned}$$

From the $(p,q)-$derivative of the product, we obtain

$$\begin{aligned}(3.5)\quad & {}_a D_{p,q}\left((t-a)(b-t)\right)\\ =\ & (pt+(1-p)a-a)\ {}_a D_{p,q}(b-t)+(b-qt-(1-q)a)\ {}_a D_{p,q}(t-a)\\ =\ & -p(t-a)+(b-a-q(t-a))=(b-a)-(t-a)(p+q)\end{aligned}$$

Applying $(p,q)-$integration by parts again and by using (3.5), we see

$$\begin{aligned}&-\int_a^b {}_a D_{p,q} f\left(qt+(1-q)a\right)\ {}_a D_{p,q}\left((t-a)(b-t)\right)\ {}_a d_{p,q} t\ .\\ =\ & -\int_a^b \left((b-a)-(t-a)(p+q)\right)\ {}_a D_{p,q} f\left(qt+(1-q)a\right)\ {}_a d_{p,q} t\\ =\ & -\left[\left((b-a)-\left(1+\frac{q}{p}\right)(t-a)\right)f\left(qt+(1-q)a\right)\right]_{t=a}^{t=b}\\ & +\int_a^b f\left(q^2t+\left(1-q^2\right)a\right)\ {}_a D_{p,q}\left((b-a)-\left(1+\frac{q}{p}\right)(t-a)\right)\ {}_a d_{p,q} t\\ =\ & -(b-a)\left(-\frac{q}{p}f\left(qb+(1-q)a\right)-f(a)\right)\\ & -\left(1+\frac{q}{p}\right)\int_a^b f\left(q^2t+\left(1-q^2\right)a\right)\ {}_a d_{p,q} t\ .\end{aligned}$$

Therefore,

(3.6)
$$\begin{aligned}
&\left|\int_a^b f\left(q^2t+\left(1-q^2\right)a\right)\ {}_ad_{p,q}t-\frac{(b-a)\,p}{p+q}\left(\frac{q}{p}f\left(qb+(1-q)\,a\right)+f\left(a\right)\right)\right| \\
\leq\ & \int_a^b \left(pt+(1-p)\,a-a\right)\left(b-pt-(1-p)\,a\right)\ {}_aD^2_{p,q}f\left(t\right)\ {}_ad_{p,q}t \\
\leq\ & \left\|{}_aD^2_{p,q}f\right\|\int_a^b p\left(t-a\right)\left(b-a-p\left(t-a\right)\right)\ {}_ad_{p,q}t
\end{aligned}$$

From Lemma 1, we obtain

$$\begin{aligned}
&\int_a^b p\left(t-a\right)\left(b-a-p\left(t-a\right)\right)\ {}_ad_{p,q}t\ \ {}_ad_{p,q}t \\
=\ & (b-a)\,p\int_a^b \left(t-a\right)\ {}_ad_{p,q}t-p^2\int_a^b \left(t-a\right)^2\ {}_ad_{p,q}t \\
=\ & (b-a)\,\frac{p}{p+q}\left(b-a\right)^2-\frac{p^2}{p^2+pq+q^2}\left(b-a\right)^3 \\
=\ & (b-a)^3\left(\frac{p}{p+q}-\frac{p^2}{p^2+pq+q^2}\right) \\
=\ & (b-a)^3\,\frac{pq^2}{p^3+2p^2q+2pq^2+q^3}. \qquad (3.7)
\end{aligned}$$

Combining (3.6) with (3.7), we get the desired inequality. □

Remark 2. *If $p=1$, then (3.4) reduces to $q-$trapezoid inequality with the second order as*

$$\begin{aligned}
&\left|\int_a^b f\left(q^2t+\left(1-q^2\right)a\right)d_qt-\frac{(b-a)}{1+q}\left(qf\left(qb+(1-q)\,a\right)+f\left(a\right)\right)\right| \\
\leq\ & \left\|{}_aD^2_qf\right\|\frac{(b-a)^3\,q^2}{(1+q)\left(1+2q+2q^2+q^3\right)}.
\end{aligned}$$

See [15, Theorem 3.4]. *Also if $q\to 1$, then (3.4) reduces to the well-known trapezoid inequality with the second order as*

$$\left|\int_a^b f\left(t\right)dt-(b-a)\,\frac{f\left(a\right)+f\left(b\right)}{2}\right|\leq\left\|f''\right\|\frac{(b-a)^3}{12}.$$

See, [2, 11].

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a (p,q) differentiable function and ${}_aD_{p,q}f$ be continuous with $0<q<p\leq 1$. Then*

(3.8)
$$\begin{aligned}
&\left|f\left(x\right)-\frac{1}{b-a}\int_a^b f\left(t\right)\ {}_ad_{p,q}t\right| \\
\leq\ & \left\|aD_{p,q}f\right\|(b-a)\left[\frac{2\left(p+q-1\right)}{p+q}\left(x-\frac{\frac{(3p+3q-4)a+(p+q)b}{4(p+q-1)}}{b-a}\right)^2-\frac{\left(-8p-8q+2pq+p^2+q^2+8\right)}{8\left(p+q-1\right)\left(p+q\right)}\right]
\end{aligned}$$

Proof. Using the Lagrange Mean Value Theorem, we obtain

$$
\begin{aligned}
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\ {}_ad_{p,q}t\right| &= \frac{1}{b-a}\left|\int_a^b (f(x)-f(t))\ {}_ad_{p,q}t\right| \\
&\le \frac{1}{b-a}\int_a^b |f(x)-f(t)|\ {}_ad_{p,q}t \\
&\le \frac{\|{}_aD_{p,q}f\|}{b-a}\int_a^b |x-t|\ {}_ad_{p,q}t\ . \qquad (3.9)
\end{aligned}
$$

From Example 2 and Example 3, we have

$$
\begin{aligned}
\int_a^b |x-t|\ {}_ad_{p,q}t &= \int_a^x (x-t)\ {}_ad_{p,q}t+\int_x^b (t-x)\ {}_ad_{p,q}t \\
&= x(x-a)-\frac{(x-a)(x-a(1-p-q))}{p+q} \\
&\quad +\frac{b^2-(p+q)bx+(p+q-1)x^2}{p+q} \\
&\quad -\frac{a(b-x)(2-p-q)}{p+q}. \qquad (3.10) \\
&= \frac{2(p+q-1)}{p+q}\left(x-\frac{(3p+3q-4)a+(p+q)b}{4(p+q-1)}\right)^2 \\
&\quad -\frac{(b-a)^2\left(-8p-8q+2pq+p^2+q^2+8\right)}{8(p+q-1)(p+q)}
\end{aligned}
$$

Combining (3.9) with (3.10), we get the required inequality. □

Remark 3. *If $p=1$, then (3.8) reduces to $q-$Ostrowski integral inequality as*

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\ {}_ad_qt\right| \\
\le\ & \|aD_qf\|(b-a)\left[\frac{2q}{1+q}\left(\frac{x-\frac{(3q-1)a+(1+q)b}{4q}}{b-a}\right)^2+\frac{-q^2+6q-1}{8q(1+q)}\right].
\end{aligned}
$$

See [15, Theorem 3.5]. *Also if $q\to 1$, then (3.8) reduces to the well known Ostrowski integral inequality as*

$$
\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|.
$$

See, [2, 11].

Lemma 2. *Let $f,g:I\to\mathbb{R}$ be continuous functions on I and $0<q<p\le 1$. Then*

$$
\begin{aligned}
(3.11)\qquad & \frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\ {}_ad_{p,q}x\ \ {}_ad_{p,q}y \\
=\ & (b-a)\int_a^b f(x)g(x)\ {}_ad_{p,q}x-\left(\int_a^b f(x)\ {}_ad_{p,q}x\right)\left(\int_a^b g(x)\ {}_ad_{p,q}x\right).
\end{aligned}
$$

Proof. From Definition 4, we have

$$\begin{aligned}
&\int_a^b\int_a^b \left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\ {}_ad_{p,q}x\ {}_ad_{p,q}y\\
&\int_a^b\int_a^b \left[f(x)g(x)-f(x)g(y)-f(y)g(x)+f(y)g(y)\right]\ {}_ad_{p,q}x\ {}_ad_{p,q}y\\
=\ & (p-q)(b-a)^2\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\\
&-(p-q)^2(b-a)^2\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right)\\
&\times\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right)\\
&-(p-q)^2(b-a)^2\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}g\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right)\\
&\times\left(\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right)\\
&+(p-q)(b-a)^2\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a\right)\\
=\ & 2(b-a)\int_a^b f(x)g(x)\ {}_ad_{p,q}x-2\left(\int_a^b f(x)\ {}_ad_{p,q}x\right)\left(\int_a^b g(x)\ {}_ad_{p,q}x\right).
\end{aligned}$$

Thus, the proof is complete. □

Remark 4. *Note that, if $p=1$ and both $p=1$ and $q\to 1$, then (3.11) reduces to $q-$Korkine and the usual Korkine identity respectively, see* [2, 15].

$(p,q)-$analogue of Cauchy-Bunyakovsky-Schwarz integral inequality for double integrals on I is proved as follows:

Lemma 3. *Let $f,g:I\to\mathbb{R}$ be continuous functions on I and $0<q<p\le 1$. Then*

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b f(x,y)\,g(x,y)\ {}_ad_{p,q}x\ {}_ad_{p,q}y \qquad (3.12)\\
\le\ & \left(\int_a^b\int_a^b f^2(x,y)\ {}_ad_{p,q}x\ {}_ad_{p,q}y\right)^{\frac{1}{2}}\left(\int_a^b\int_a^b g^2(x,y)\ {}_ad_{p,q}x\ {}_ad_{p,q}y\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof. From Definition 4, we have

$$\begin{aligned}
&\int_a^b\int_a^b f(x,y)\ {}_ad_{p,q}x\ {}_ad_{p,q}y\\
=\ & \int_a^b\left((p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a,y\right)\right)\ {}_ad_{p,q}y\\
=\ & (p-q)^2(b-a)^2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{n+k}}{p^{n+k+1}}f\left(\frac{q^n}{p^{n+1}}b+\left(1-\frac{q^n}{p^{n+1}}\right)a,\frac{q^k}{p^{k+1}}b+\left(1-\frac{q^k}{p^{k+1}}\right)a\right).
\end{aligned}$$

By using discrete Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_a^b \int_a^b f(x,y)\, g(x,y)\ {}_a d_{p,q}x\ {}_a d_{p,q}y \\
=& \left((p-q)^2 (b-a)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{n+k}}{p^{n+k+1}} f\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a, \frac{q^n}{p^{k+1}} b + \left(1 - \frac{q^k}{p^{k+1}}\right) a \right) \right. \\
& \left. \times g\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a, \frac{q^n}{p^{k+1}} b + \left(1 - \frac{q^k}{p^{k+1}}\right) a \right) \right)^2 \\
\leq& \left((p-q)^2 (b-a)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{n+k}}{p^{n+k+1}} f^2\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a, \frac{q^n}{p^{k+1}} b + \left(1 - \frac{q^k}{p^{k+1}}\right) a \right) \right)^2 \\
& \times \left((p-q)^2 (b-a)^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{n+k}}{p^{n+k+1}} g^2\left(\frac{q^n}{p^{n+1}} b + \left(1 - \frac{q^n}{p^{n+1}}\right) a, \frac{q^n}{p^{k+1}} b + \left(1 - \frac{q^k}{p^{k+1}}\right) a \right) \right)^2 \\
=& \left(\int_a^b \int_a^b f^2(x,y)\ {}_a d_{p,q}x\ {}_a d_{p,q}y \right) \left(\int_a^b \int_a^b g^2(x,y)\ {}_a d_{p,q}x\ {}_a d_{p,q}y \right).
\end{aligned}$$

Thus, the proof is complete. □

Remark 5. *Note that, if $p = 1$ and both $p = 1$ and $q \to 1$, then (3) reduces to $q-$Cauchy-Bunyakovsky-Schwarz and the usual Cauchy-Bunyakovsky-Schwarz integral inequality for double integrals on I respectively, see* [2, 15].

$(p,q)-$ analogue of Chebyshev functional can defined on I as follows:

$$\begin{aligned}
T(f,g) &= \frac{1}{b-a} \int_a^b f(x)\, g(x)\ {}_a d_{p,q}x \\
&\quad - \left(\frac{1}{b-a} \int_a^b f(x)\ {}_a d_{p,q}x \right) \left(\frac{1}{b-a} \int_a^b g(x)\ {}_a d_{p,q}x \right).
\end{aligned} \tag{3.13}$$

Furthermore, we obtain $(p,q)-$ analogue of Grüss integral inequality on I from Lemma 2 and Lemma 3 with (3.13). The proof can be derived from the classical Grüss integral inequality, see [2].

Theorem 9. *Assume $f, g : I \to \mathbb{R}$ be continuous functions on I and $0 < q < p \leq 1$. Let $\phi \leq f(x) \leq \Phi$, $\gamma \leq g(x) \leq \Gamma$ for all $x \in I$, $\phi, \Phi, \gamma, \Gamma \in \mathbb{R}$. Then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x)\, g(x)\ {}_a d_{p,q}x - \left(\frac{1}{b-a} \int_a^b f(x)\ {}_a d_{p,q}x \right) \left(\frac{1}{b-a} \int_a^b g(x)\ {}_a d_{p,q}x \right) \right| \\
\leq& \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma).
\end{aligned} \tag{3.14}$$

Remark 6. *Note that, if $p = 1$ and both $p = 1$ and $q \to 1$, then (3.14) reduces to $q-$Grüss and the well-known Grüss integral inequality on I respectively, see* [2, 15].

Theorem 10. *Assume $f, g : I \to \mathbb{R}$ be continuous functions on I and $0 < q < p \leq 1$. Let L_1, L_2 Lipschitzian continuous functions on I, such that*

$$|f(x) - f(y)| \leq L_1 |x - y|, \quad |g(x) - g(y)| \leq L_2 |x - y|$$

for all $x, y \in I$. Then the following inequality holds

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\ {}_a d_{p,q}x - \left(\frac{1}{b-a}\int_a^b f(x)\ {}_a d_{p,q}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\ {}_a d_{p,q}x\right)\right| \\
\leq\ & 2pq\frac{(b-a)^2}{(p+q)^2\,(p^2+pq+q^2)}L_1L_2.
\end{aligned} \tag{3.15}$$

Proof. $(p,q)-$ analogue of Korkine identity on I is obtained as follows in below:

$$\begin{aligned}
&(b-a)\int_a^b f(x)\,g(x)\ {}_a d_{p,q}x - \left(\int_a^b f(x)\ {}_a d_{p,q}x\right)\left(\int_a^b g(x)\ {}_a d_{p,q}x\right) \\
=\ & \frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))\,(g(x)-g(y))\ {}_a d_{p,q}x\ \ {}_a d_{p,q}y\ .
\end{aligned} \tag{3.16}$$

Under the assumptions of Theorem, we have

$$|f(x)-f(y)|\,|g(x)-g(y)| \leq L_1L_2\,(x-y)^2 \tag{3.17}$$

for all $x, y \in I$. The double $(p,q)-$ integration for (3.17) on $I \times I$ gives

$$\begin{aligned}
&\int_a^b\int_a^b |f(x)-f(y)|\,|g(x)-g(y)|\ {}_a d_{p,q}x\ \ {}_a d_{p,q}y \\
\leq\ & L_1L_2\int_a^b\int_a^b (x-y)^2\ {}_a d_{p,q}x\ \ {}_a d_{p,q}y \\
=\ & L_1L_2\int_a^b\int_a^b \left(x^2-2xy-y^2\right)\ {}_a d_{p,q}x\ \ {}_a d_{p,q}y \\
=\ & L_1L_2\left[2\,(b-a)\int_a^b x^2\ {}_a d_{p,q}x - 2\left(\int_a^b x\ {}_a d_{p,q}x\right)^2\right]. \qquad (3.18)
\end{aligned}$$

By direct computation, we get

$$\begin{aligned}
\int_a^b x^2\ {}_a d_{p,q}x =\ & \int_a^b (x-a+a)^2\ {}_a d_{p,q}x \\
=\ & \int_a^b (x-a)^2\ {}_a d_{p,q}x + 2a\int_a^b (x-a)\ {}_a d_{p,q}x \\
& + a^2\int_a^b {}_a d_{p,q}x \\
=\ & \frac{(b-a)^3}{p^2+pq+q^2} + 2a\frac{(b-a)^2}{p+q} + a^2\,(b-a) \qquad (3.19) \\
=\ & \frac{(b-a)}{(p^2+pq+q^2)\,(p+q)}\left((p+q)\,b^2 + \left(p^2+pq-p+q^2-q\right)2ab\right. \\
& \left. + \left(p^3+2p^2q-2p^2+2pq^2-2pq+p+q^3-2q^2+q\right)a^2\right).
\end{aligned}$$

Thus, from (3.19) and (2.5), we obtain

$$
\begin{aligned}
(3.20) \qquad & (b-a)\int_a^b x^2 \ {}_a d_{p,q}x - \left(\int_a^b x \ {}_a d_{p,q}x\right)^2 \\
= \ & pq\frac{(a-b)^4}{(p+q)^2(p^2+pq+q^2)}.
\end{aligned}
$$

Combining with (3.20) to (3.18), we have

$$
\begin{aligned}
& \int_a^b\int_a^b |f(x)-f(y)|\,|g(x)-g(y)| \ {}_a d_{p,q}x \ {}_a d_{p,q}y \\
\leq \ & 2pq\frac{(b-a)^4}{(p+q)^2(p^2+pq+q^2)}L_1L_2.
\end{aligned}
$$

Using (3.16), we get the required inequality. □

Remark 7. *If $p=1$, then (3.15) reduces to $q-$Grüss-Chebyshev integral inequality*

$$
\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(x)g(x) \ {}_a d_q x - \left(\frac{1}{b-a}\int_a^b f(x) \ {}_a d_q x\right)\left(\frac{1}{b-a}\int_a^b g(x) \ {}_a d_q x\right)\right| \\
\leq \ & \frac{qL_1L_2}{(1+q+q^2)(1+q)^2}(b-a)^2.
\end{aligned}
$$

See [15, Theorem 3.7]*. Also if $q\to 1$, then (3.15) reduces to the well known Grüss-Chebyshev integral inequality*

$$
\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \\
\leq \ & \frac{L_1L_2}{12}(b-a)^2.
\end{aligned}
$$

See [2]*.*

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