Received 27/05/16

(p,q) –INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

MEVLÜT TUNÇ$^{\alpha}$ AND ESRA GÖV$^{\beta}$

ABSTRACT. In this paper, we establish (p,q)-analogue of an integral equality and also we derive some integral inequalities for convex functions.

1. INTRODUCTION

A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on J if the inequality

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) \tag{1.1}$$

holds for all $u, v \in J$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex. We say that $f : J \subset \mathbb{R} \to \mathbb{R}$ is a quasi-convex function, or that f belongs to the class $QC(J)$, if, for all $u, v \in J$ and $t \in [0,1]$, we have

$$f(tu + (1-t)v) \leq max\{f(x), f(y)\}.$$

Convex functions play an important role in mathematical inequalities. The most famous inequality have been used with convex functions is Hermite-Hadamard, which is stated as follows:

Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u, v \in J$ with $u < v$. The following double inequality:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(x)\,dx \leq \frac{f(u)+f(v)}{2}. \tag{1.2}$$

Mathematical inequalities play an important role on many branches of mathematics as analysis, differential equations, geometry etc. In recent years $q-$integral inequalities and some of generalization forms of quantum type inequalities have been studied by many authors, see [3, 4, 5, 8, 14, 15, 16]. One of the generalization of $q-$calculus is (p,q) –calculus, see [6, 7, 13]. The aim of this paper is to establish Hermite-Hadamard type integral inequalities for convex and quasi-convex functions via (p,q) –calculus. One can see that the results reduces to their $q-$analogues when $p = 1$, see [14].

Now, we give some definitions and results via (p,q) –calculus which will be used in the sequel, [6, 7, 13]. Let $0 < q < p \leq 1$. The (p,q) –integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

1991 *Mathematics Subject Classification.* 34A08, 26A33, 26D10, 26D20.

Key words and phrases. (p,q)-derivative; (p,q)-integral; convex functions; Hadamard inequality; Hölder inequality.

$^{\beta}$*Corresponding Author.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

For each $k, n \in \mathbb{N}$, $n \geq k \geq 0$, the (p, q) −factorial and (p, q) −binomial are defined by

$$\begin{aligned} [n]_{p,q}! &= \prod_{k=1}^{n} [k]_{p,q}, \ n \geq 1, \ [0]_{p,q}! = 1 \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[n-k]_{p,q}! \, [k]_{p,q}!}. \end{aligned}$$

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$. The (p, q) −derivative of the function f is defined as*

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q) x}, \ x \neq 0 \tag{1.3}$$

provided that $D_{p,q} f(0) = f'(0)$.

Definition 2. *Let $f : C[0, a] \to \mathbb{R}$ $(a > 0)$ then the (p, q) −integration of f defined by*

$$\begin{aligned} \int_0^a f(t) \, d_{p,q} t &= (q - p) \, a \sum_{n=0}^{\infty} \frac{p^n}{q^{n+1}} f\left(\frac{p^n}{q^{n+1}} a\right) \ \text{if} \ \left|\frac{p}{q}\right| < 1 \\ \int_0^a f(t) \, d_{p,q} t &= (p - q) \, a \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} a\right) \ \text{if} \ \left|\frac{p}{q}\right| > 1. \end{aligned} \tag{1.4}$$

The formula of (p, q) −integration by parts is given by

$$\int_a^b f(px) \, D_{p,q} g(x) \, d_{p,q} t = f(x) \, g(x)|_a^b - \int_a^b g(qx) \, D_{p,q} f(x) \, d_{p,q} t. \tag{1.5}$$

All notions written above reduce to the q−analogs when $p = 1$. For more details, see [6, 13].

2. PRELIMINARIES

Let $I_k := [u_k, u_{k+1}]$ be an interval and $0 < q_k < p_k \leq 1$ be constants. The (p_k, q_k) −derivative of a function f is defined on I_k as:

Definition 3. [17]*Let $f : I_k \to \mathbb{R}$ be a continuous function and assume that $u \in I_k$. Then the following equality*

$$\begin{aligned} D_{p_k,q_k} f(u) &= \frac{f(p_k u + (1 - p_k) u_k) - f(q_k u + (1 - q_k) u_k)}{(p_k - q_k)(u - u_k)}, u \neq u_k \\ D_{p_k,q_k} f(u_k) &= \lim_{u \to u_k} D_{p_k,q_k} f(u) \end{aligned} \tag{2.1}$$

is called the (p_k, q_k)-derivative of a function f at u.

Obviously, f is (p_k, q_k)-differentiable on I_k provided $D_{p_k,q_k} f(u)$ exists for all $u \in I_k$. In Definition 3, if $p_k = 1$, then $D_{p_k,q_k} f = D_{q_k} f$ which is the q_k-derivative of the function f and also if $q_k \to 1, u_k = 0$, (2.1) reduces to q-derivative of the function f, see [8, 16].

Example 1. [17]*For $u \in I_k$, if $f(u) = (u - u_k)^n$, then*

$$D_{p_k,q_k} f(u) = [n]_{p,k} (u - u_k)^{n-1} \tag{2.2}$$

where $[n]_{p_k,q_k} = \frac{p_k^n - q_k^n}{p_k - q_k}$. *If* $p_k = 1$ *in (2.2), then (2.2) reduces*

$$D_{q_k} f(u) = [n]_{q_k} (u - u_k)^{n-1}$$

which is given in [16]. *Also if* $q_k \to 1, u_k = 0$, *it reduces* q*-derivative of the given function, see* [8].

(p_k, q_k)-integral of f on a finite interval is given as follows:

Definition 4. [17]*Let* $f : I_k \to \mathbb{R}$ *is a continuous function. Then for* $0 < q_k < p_k \leq 1$,

(2.3)
$$\int_{u_k}^{u} f(s)\, d_{p_k,q_k} s = (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k\right)$$

is called (p_k, q_k)*-integral of* f *for* $u \in I_k$.

Moreover, if $a \in (u_k, u)$, then (p_k, q_k)-integral is defined by

(2.4)
$$\begin{aligned} & \int_{a}^{u} f(s)\, d_{p_k,q_k} s \\ = & \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s \int_{u_k}^{a} f(s)\, d_{p_k,q_k} s \\ = & (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k\right) \\ & - (p_k - q_k)(a - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} a + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k\right). \end{aligned}$$

Note that if $u_k = 0$ and $p = 1$, then (2.4) reduces to q_k-integral of the function. See, [16].

Remark 1. [17]*We assume* $0 < q_k < p_k \leq 1$ *for all of the above results. We shall mention that* $0 < q_k < 1$, $0 < p_k \leq 1$ *for interchanging* p_k *and* q_k *in the formulas. So, we have*

(2.5)
$$\begin{aligned} \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s &= (p_k - q_k)(u - u_k) \sum_{n=0}^{\infty} \frac{q_k^n}{p_k^{n+1}} f\left(\frac{q_k^n}{p_k^{n+1}} u + \left(1 - \frac{q_k^n}{p_k^{n+1}}\right) u_k\right), \left|\frac{p}{q}\right| > 1 \\ \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s &= (q_k - p_k)(u - u_k) \sum_{n=0}^{\infty} \frac{p_k^n}{q_k^{n+1}} f\left(\frac{p_k^n}{q_k^{n+1}} u + \left(1 - \frac{p_k^n}{q_k^{n+1}}\right) u_k\right), \left|\frac{p}{q}\right| < 1. \end{aligned}$$

where $0 < q_k < 1$, $0 < p_k \leq 1$.

Remark 2. [17]*Note that, if we take* $u_k = 0$ *in (2.5), then (2.5) reduces to (1.4),* [13, Definition 5]. *Also, if* $p_k = 1$ *in (2.3), then (2.3) reduces to* q_k-*integral of a function* f *defined by*

$$\int_{u_k}^{u} f(s)\, d_{q_k} s = (1 - q_k)(u - u_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n u + (1 - q_k^n) u_k).$$

For more details, see [16].

Theorem 1. [17] *The following formulas hold for $u \in I_k$:*
(a) $D_{p_k,q_k} \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s = f(u)$
(b) $\int_{u_k}^{u} D_{p_k,q_k} f(s)\, d_{p_k,q_k} s = f(u)$
(c) $\int_{a}^{u} D_{p_k,q_k} f(s)\, d_{p_k,q_k} s = f(u) - f(a)$, *for* $a \in (u_k, u)$.

Theorem 2. [17] *Let $f, g : I_k \to \mathbb{R}$ are continuous functions. The following formulas hold:*
(a) $\int_{u_k}^{u} [f(s) + g(s)]\, d_{p_k,q_k} s = \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s + \int_{u_k}^{u} g(s)\, d_{p_k,q_k}$;
(b) $\int_{u_k}^{u} \lambda f(s)\, d_{p_k,q_k} s = \lambda \int_{u_k}^{u} f(s)\, d_{p_k,q_k} s$;
(c) $\int_{u_k}^{u} f(q_k s + (1-q_k) u_k)\, D_{p_k,q_k} g(p_k s)\, d_{p_k,q_k} s = (fg)(s)\,_{u_k}^{u} - \int_{u_k}^{u} g(p_k s + (1-p_k) u_k)\, D_{p_k,q_k} f(s)\, d_{p_k,q_k} s$
or
$\int_{u_k}^{u} f(p_k s + (1-p_k) u_k)\, D_{p_k,q_k} g(q_k s)\, d_{p_k,q_k} s = (fg)(s)\,_{u_k}^{u} - \int_{u_k}^{u} g(q_k s + (1-q_k) u_k)\, D_{p_k,q_k} f(s)\, d_{p_k,q_k} s$
where $u \in I_k, \lambda \in \mathbb{R}$.

The (p,q) −Hermite-Hadamard integral inequality is defined on $[a,b]$ as follows:

Theorem 3. [17] *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $0 < q < p \leq 1$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\; {}_a d_{p,q} x \leq \frac{(p+q-1) f(a) + f(b)}{p+q}. \tag{2.6}$$

3. AUXILIARY RESULTS

Lemma 1. [18] *Let $0 < q < 1$ and $0 < t < p \leq 1$. The following formula holds:*

$$\begin{aligned} \int_0^a (m - nt)\, d_{p,q} t &= ma - \frac{a^2 n}{p+q} \\ \int_0^a (nt - m)\, d_{p,q} t &= \frac{a^2 n}{p+q} - ma. \end{aligned} \tag{3.1}$$

Lemma 2. [18] *Let $0 < q < 1$ and $0 < t < p \leq 1$. The following formula holds:*

$$\int_0^a t(nt - m)\, d_{p,q} t = \frac{na^3}{p^2 + pq + q^2} - \frac{ma^2}{p+q}. \tag{3.2}$$

Now, we give some auxiliary results which we use in the sequel:

Lemma 3. *Let $0 < q < 1$ and $0 < t < p \leq 1$. The following formula holds:*

$$\begin{aligned} M_1 &= \int_0^1 \left(1 - \frac{t}{p}\right) \left|1 - \frac{p+q}{p} t\right| {}_0 d_{p,q} t = \frac{1}{p}\left(\frac{1}{p+q} - \frac{1}{p}\frac{p+q}{p^2+pq+q^2}\right) \\ &\quad - \frac{2}{p}\left(\frac{p^2}{(p+q)^3} - \frac{p^2}{(p+q)^2 (p^2+pq+q^2)}\right) \\ &\quad + 2\frac{p}{p+q} - 2\frac{p}{(p+q)^2} + \frac{1}{p} - 1. \end{aligned} \tag{3.3}$$

Proof. By computing the integral, we have

$$\begin{aligned}
&\int_0^1 \left(1-\frac{t}{p}\right)\left|1-\frac{p+q}{p}t\right| {}_0d_{p,q}t \\
= &\int_0^{\frac{p}{p+q}} \left(1-\frac{t}{p}\right)\left(1-\frac{p+q}{p}t\right) {}_0d_{p,q}t \\
&+\int_{\frac{p}{p+q}}^1 \left(1-\frac{t}{p}\right)\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t\ .
\end{aligned}$$

Dividing the integral into two parts and using Lemma 1, Lemma 2, we get

$$\begin{aligned}
I_1 &= \int_0^{\frac{p}{p+q}} \left(1-\frac{p+q}{p}t\right) {}_0d_{p,q}t - \int_0^{\frac{p}{p+q}} \frac{t}{p}\left(1-\frac{p+q}{p}t\right) {}_0d_{p,q}t \\
&= \frac{p}{p+q} - \frac{p}{(p+q)^2} - \frac{1}{p}\left(\frac{p^2}{(p+q)^3} - \frac{p^2}{(p+q)^2(p^2+pq+q^2)}\right)
\end{aligned}$$

and similarly it is easy to see that,

$$\begin{aligned}
I_2 &= \int_{\frac{p}{p+q}}^1 \left(1-\frac{t}{p}\right)\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t \\
&= \int_0^1 \left(1-\frac{t}{p}\right)\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t - \int_0^{\frac{p}{p+q}} \left(1-\frac{t}{p}\right)\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t \\
&= \int_0^1 \left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t - \int_0^1 \frac{t}{p}\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t \\
&\quad - \int_0^{\frac{p}{p+q}} \left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t + \int_0^{\frac{p}{p+q}} \frac{t}{p}\left(\frac{p+q}{p}t-1\right) {}_0d_{p,q}t \\
&= \left(\frac{1}{p}-1\right) - \left(\frac{p}{(p+q)^2} - \frac{p}{p+q}\right) - \frac{1}{p}\left(\frac{1}{p}\frac{p+q}{p^2+pq+q^2} - \frac{1}{p+q}\right) \\
&\quad \frac{1}{p} + \left(\frac{p^2}{(p+q)^2(p^2+pq+q^2)} - \frac{p^2}{(p+q)^3}\right).
\end{aligned}$$

Adding I_1 and I_2, we get the required inequality. □

Remark 3. *In Lemma 3, if we take $p=1$, (3.3) reduces to Lemma 3.3 in* [14].

Lemma 4. *Let $0<q<1$ and $0<t<p\leq 1$. The following formula holds:*

$$\begin{aligned}
M_2 &= \int_0^1 \frac{t}{p}\left|1-\frac{p+q}{p}t\right| d_{p,q}t = \frac{2}{p}\left(\frac{p^2}{(p+q)^3} - \frac{p^2}{(p+q)^2(p^2+pq+q^2)}\right) \\
&\quad -\frac{1}{p}\left(\frac{1}{p+q} - \frac{1}{p}\frac{p+q}{p^2+pq+q^2}\right).
\end{aligned} \tag{3.4}$$

Proof. Following from the proof of Lemma 3, it easy to obtain the desired equality given as M_2. □

Remark 4. *In Lemma 4, if we take $p=1$, (3.4) reduces to Lemma 3.2 in* [14].

4. MAIN RESULTS

We begin with the following lemma.

Lemma 5. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $0 < q < 1$ and $0 < t < p \leq 1$. If ${}_aD_{p,q}f$ is integrable on (a,b), then the following equality holds:*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) \; {}_ad_{p,q}t - \frac{pqf(a) + (pq-(p+q)) f\left(\left(1-\frac{1}{p}\right)a + \frac{b}{p}\right)}{p+q} \right| \\ = & \frac{q(b-a)}{p+q} \int_0^1 \left(1-(p+q)\frac{t}{p}\right) {}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a + \frac{t}{p}b\right) \; {}_0d_{p,q}t \; . \end{aligned}$$

Proof. From Definition 3, we have

$$\begin{aligned} & \int_0^1 \left(1-(p+q)\frac{t}{p}\right) {}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a + \frac{t}{p}b\right) \; {}_0d_{p,q}t \\ = & \int_0^1 \left(1-(p+q)\frac{t}{p}\right) \frac{f\left(p\left(1-\frac{t}{p}\right)a + \frac{t}{p}pb + (1-p)a\right) - f\left(q\left(1-\frac{t}{p}\right)a + \frac{t}{p}qb + (1-q)a\right)}{\frac{t}{p}(b-a)(p-q)} \; {}_0d_{p,q}t \\ = & \frac{p}{(b-a)(p-q)} \int_0^1 \frac{f((1-t)a+tb) - f\left(\left(1-\frac{qt}{p}\right)a + \frac{qt}{p}b\right)}{t} \; {}_0d_{p,q}t \\ & - \frac{(p+q)}{(b-a)(p-q)} \int_0^1 \left(f((1-t)a+tb) - f\left(\left(1-\frac{qt}{p}\right)a + \frac{qt}{p}b\right)\right) \; {}_0d_{p,q}t \; . \end{aligned}$$

By using Definition 4, we get

$$\begin{aligned}
LHS &= \frac{p}{(b-a)}\left[\sum_{n=0}^{\infty} f\left(\left(1-\tfrac{q^n}{p^{n+1}}\right)a+\tfrac{q^n}{p^{n+1}}b\right)-\sum_{n=0}^{\infty} f\left(\left(1-\tfrac{q^{n+1}}{p^{n+2}}\right)a+\tfrac{q^{n+1}}{p^{n+2}}b\right)\right] \\
&\quad -\frac{(p+q)}{(b-a)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\left(1-\tfrac{q^n}{p^{n+1}}\right)a+\tfrac{q^n}{p^{n+1}}b\right)-\frac{p}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}} f\left(\left(1-\tfrac{q^{n+1}}{p^{n+2}}\right)a+\tfrac{q^{n+1}}{p^{n+2}}b\right)\right] \\
&= \frac{p}{(b-a)}\left[f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)-f(a)\right] \\
&\quad -\frac{(p+q)}{(b-a)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\left(1-\frac{q^n}{p^{n+1}}\right)a+\frac{q^n}{p^{n+1}}b\right)-\frac{p}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}} f\left(\left(1-\frac{q^n}{p^{n+1}}\right)a+\frac{q^n}{p^{n+1}}b\right)\right] \\
&= \frac{p}{(b-a)}\left[f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)-f(a)\right] \\
&\quad -\frac{(p+q)}{q(b-a)} f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right) \\
&\quad -\frac{(p+q)}{(b-a)}\left(1-\frac{p}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\left(1-\frac{q^n}{p^{n+1}}\right)a+\frac{q^n}{p^{n+1}}b\right) \\
&= \frac{\left(p-\frac{p+q}{q}\right) f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)-pf(a)}{b-a} \\
&\quad +\frac{(p+q)}{(b-a)^2}\frac{(p-q)(b-a)}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\left(1-\frac{q^n}{p^{n+1}}\right)a+\frac{q^n}{p^{n+1}}b\right) \\
&= \frac{(p+q)}{q(b-a)^2}\int_a^b f(x)\ {}_a d_{p,q}t-\frac{pqf(a)+(p+q-pq) f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{q(b-a)}.
\end{aligned}$$

Multiplying both sides with $\frac{(b-a)q}{p+q}$, we get the required equation. □

Remark 5. *Note that, when $p=1$, Lemma 5 reduces Lemma 3.1 in* [14].

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $0<q<1$ and $0<t<p\le 1$. If $|{}_aD_{p,q}f|$ is convex and integrable on (a,b), then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ {}_a d_{p,q}t-\frac{pqf(a)+(pq-(p+q)) f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \qquad (4.1)\\
&\le \frac{q(b-a)}{p+q}\left(M_1\left|{}_aD_{p,q}f(a)\right|+M_2\left|{}_aD_{p,q}f(b)\right|\right)
\end{aligned}$$

where M_1 and M_2 are as given above.

Proof. Using Lemma 5 and convexity of $|_aD_{p,q}f|$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ _ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \\
\leq &\left|\frac{q(b-a)}{p+q}\int_0^1\left(1-(p+q)\frac{t}{p}\right)\ _aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\ _0d_{p,q}t\right| \\
\leq &\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left|_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\right|\ _0d_{p,q}t \\
\leq &\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left(\left(1-\frac{t}{p}\right)|_aD_{p,q}f(a)|+\frac{t}{p}|_aD_{p,q}f(b)|\right)\ _0d_{p,q}t\ .
\end{aligned}$$

From Lemma 3 and Lemma 4, we obtain

$$\begin{aligned}
LHS \leq &\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left(1-\frac{t}{p}\right)|_aD_{p,q}f(a)| \\
&+\frac{q(b-a)}{p+q}\int_0^1\frac{t}{p}\left|1-(p+q)\frac{t}{p}\right||_aD_{p,q}f(b)|\ _0d_{p,q}t \\
\leq &\frac{q(b-a)}{p+q}|_aD_{p,q}f(a)|M_1+\frac{q(b-a)}{p+q}|_aD_{p,q}f(b)|M_2
\end{aligned}$$

which completes the proof. □

Remark 6. *If $p=1$, then the inequality (4.1) reduces to Theorem 4.1 in* [14], *see also $q\to 1$.*

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $0<q<1$ and $0<t<p\leq 1$. If $|_aD_{p,q}f|^r$ is convex and integrable on (a,b) and $r,s>1$, $\frac{1}{r}+\frac{1}{s}=1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ _ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \tag{4.2}$$

$$\leq \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)^{\frac{1}{s}}$$

$$\times\left(M_1|_aD_{p,q}f(a)|^r+M_2|_aD_{p,q}f(b)|^r\right)^{\frac{1}{r}}\ . \tag{4.3}$$

where M_1 and M_2 are as given above.

Proof. Using Lemma 5 and convexity of $\left|{}_aD_{p,q}f\right|^r$ and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \\
\leq\ & \left|\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|^{1-\frac{1}{r}}\left|1-(p+q)\frac{t}{p}\right|^{\frac{1}{r}}\ {}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\ {}_0d_{p,q}t\right| \\
\leq\ & \frac{q(b-a)}{p+q}\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t\right)^{\frac{1}{s}} \\
&\times\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left|{}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\right|^r\ {}_0d_{p,q}t\right)^{\frac{1}{r}} \\
\leq\ & \frac{q(b-a)}{p+q}\left(\frac{p}{p+q}-\frac{p}{(p+q)^2}+\frac{1}{p}-1-\frac{p}{(p+q)^2}+\frac{p}{p+q}\right)^{\frac{1}{s}} \\
&\times\left\{\left|{}_aD_{p,q}f(a)\right|^r\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left(1-\frac{t}{p}\right)\ {}_0d_{p,q}t\right)\right. \\
&\left.+\left|{}_aD_{p,q}f(b)\right|^r\left(\int_0^1\frac{t}{p}\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t\right)\right\}^{\frac{1}{r}}.
\end{aligned}$$

From Lemma 3 and Lemma 4, we obtain

$$\begin{aligned}
LHS\ \leq\ & \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)^{\frac{1}{s}} \\
& \left[M_1\left|{}_aD_{p,q}f(a)\right|^r+M_2\left|{}_aD_{p,q}f(b)\right|^r\right]^{\frac{1}{r}}.
\end{aligned}$$

Thus, the proof completes. □

Corollary 1. *Under the assumptions of Theorem 5, if we take $p=1$, then the following inequality holds:*

$$\begin{aligned}
(4.4)\ &\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qt-\frac{qf(a)+f(b)}{1+q}\right| \\
\leq\ & \frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{\frac{1}{s}} \\
&\times\left(\frac{q\left(2q^3+3q^2+1\right)}{(q+1)^3\left(q^2+q+1\right)}\left|{}_aD_{p,q}f(a)\right|^r+\frac{q\left(q^2+4q+1\right)}{(q+1)^3\left(q^2+q+1\right)}\left|{}_aD_{p,q}f(b)\right|^r\right)^{\frac{1}{r}}.
\end{aligned}$$

We shall note that, we obtain

$$\int_0^1\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t=\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1$$

by using the lemmas given in axuiliary results in above theorem. If $p=1$, we have

$$\int_0^1\left|1-(1+q)t\right|\ {}_0d_qt=\frac{2q}{(1+q)^2}. \tag{4.5}$$

However (4.5) is mentioned as $\int_0^1 |1-(1+q)t|\ {}_0d_qt = \frac{q(2+q+q^3)}{(1+q)^3}$ in [14, Theorem 4.2]. That is why we cannot reduce Theorem 5 into its $q-$analogue which is obtained in [14, Theorem 4.2], when $p=1$. We should also note that, one can easily prove (4.5) by using $q-$integral definition and some basic properties.

Note that, if $q\to 1$, (4.4) reduces Theorem 1 in [12].

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $0<q<1$ and $0<t<p\leq 1$. If $|{}_aD_{p,q}f|^r$ is convex and integrable on (a,b) and $r\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \tag{4.6}$$

$$\leq \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)^{1-\frac{1}{r}}$$

$$\times\left[M_1\left|{}_aD_{p,q}f(a)\right|^r+M_2\left|{}_aD_{p,q}f(b)\right|^r\right]^{\frac{1}{r}} \tag{4.7}$$

where M_1 and M_2 are as given above.

Proof. Using Lemma 5 convexity of $|{}_aD_{p,q}f|^r$ and power mean inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right|$$

$$\leq \left|\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|^{1-\frac{1}{r}}\left|1-(p+q)\frac{t}{p}\right|^{\frac{1}{r}}{}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\ {}_0d_{p,q}t\right|$$

$$\leq \frac{q(b-a)}{p+q}\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left|{}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\right|^r\ {}_0d_{p,q}t\right)^{\frac{1}{r}}$$

$$\leq \frac{q(b-a)}{p+q}\left(\frac{p}{p+q}-\frac{p}{(p+q)^2}+\frac{1}{p}-1-\frac{p}{(p+q)^2}+\frac{p}{p+q}\right)^{1-\frac{1}{r}}$$

$$\times\left\{\left|{}_aD_{p,q}f(a)\right|^r\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left(1-\frac{t}{p}\right)\ {}_0d_{p,q}t\right)\right.$$

$$\left.+\left|{}_aD_{p,q}f(b)\right|^r\left(\int_0^1\frac{t}{p}\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t\right)\right\}^{\frac{1}{r}}.$$

From Lemma 3 and Lemma 4, we obtain

$$LHS \leq \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)^{1-\frac{1}{r}}$$

$$\left[M_1\left|{}_aD_{p,q}f(a)\right|^r+M_2\left|{}_aD_{p,q}f(b)\right|^r\right]^{\frac{1}{r}}.$$

Thus, the proof completes. □

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function and $0<q<1$ and $0<t<p\leq 1$. If $\left|{}_aD_{p,q}f\right|^r$ is quasi-convex and integrable on (a,b) and $r\geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right| \leq \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right) \times\left(\max\left\{\left|{}_aD_{p,q}f(a)\right|,\left|{}_aD_{p,q}f(b)\right|\right\}\right)^r. \tag{4.8}$$

Proof. Since $\left|{}_aD_{p,q}f\right|^r$ is quasi-convex, using Lemma 5 and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ {}_ad_{p,q}t-\frac{pqf(a)+(pq-(p+q))f\left(\left(1-\frac{1}{p}\right)a+\frac{b}{p}\right)}{p+q}\right|\\
\leq&\left|\frac{q(b-a)}{p+q}\int_0^1\left|1-(p+q)\frac{t}{p}\right|^{1-\frac{1}{r}}\left|1-(p+q)\frac{t}{p}\right|^{\frac{1}{r}}{}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\ {}_0d_{p,q}t\right|\\
\leq&\frac{q(b-a)}{p+q}\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\ {}_0d_{p,q}t\right)^{1-\frac{1}{r}}\\
&\times\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left|{}_aD_{p,q}f\left(\left(1-\frac{t}{p}\right)a+\frac{t}{p}b\right)\right|^r\ {}_0d_{p,q}t\right)^{\frac{1}{r}}\\
\leq&\frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)^{1-\frac{1}{r}}\\
&\times\left(\max\left\{\left|{}_aD_{p,q}f(a)\right|,\left|{}_aD_{p,q}f(b)\right|\right\}\right)^r\left(\int_0^1\left|1-(p+q)\frac{t}{p}\right|\left(1-\frac{t}{p}\right)\ {}_0d_{p,q}t\right)^{\frac{1}{r}}
\end{aligned}$$

From Lemma 3 and Lemma 4, we obtain

$$\begin{aligned}
LHS\ \leq&\ \frac{q(b-a)}{p+q}\left(\frac{2p(p+q-1)}{(p+q)^2}+\frac{1}{p}-1\right)\\
&\times\left(\max\left\{\left|{}_aD_{p,q}f(a)\right|,\left|{}_aD_{p,q}f(b)\right|\right\}\right)^r.
\end{aligned}$$

Thus, the proof completes. □

References

[1] Acar T, Aral A, Mohiuddine SA, On Kantorovich modification of $(p,q)-$Baskakov operators, J. Inequal. Appl. 2016 (2016) 98.

[2] Cerone P, Dragomir SS, Mathematical Inequalities. CRC Press, New York (2011)

[3] Ernst T, A Comprehensive Treatment of q-Calculus, Springer Basel, 2012.

[4] Ernst T, A method for q-calculus, J. Nonlinear Math. Phys. 10 (4) (2003) 487525.

[5] Gauchman H, Integral inequalities in q-calculus, Comput. Math. Appl. 47 (2004) 281–300.

[6] Hounkonnou MN, Désiré J, Kyemba B, $R(p,q)-$calculus: differentiation and integration, SUT Journal of Mathematics, Vol. 49, No. 2 (2013), 145-167.

[7] Jagannathan R, Rao KS,Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series,arXiv:math/0602613v1[math.NT], 27 Feb 2006

[8] Kac V, Cheung P, Quantum Calculus, Universitext, Springer-Verlag, New York, 2002.

[9] Marinković S, Rajković P, Stanković M, The inequalities for some types of $q-$integrals, Computers and Mathematics with Applications, 56 (2008), 2490–2498.
[10] Mitrinović DS, Pečarić JE, Fink AM, Classical and new inequalties in analysis, Kluwer Academic Publisher, 1993.
[11] Pachpatte BG, Analytic Inequalities: Recent Advances, Atlantis Press, Amsterdam-Paris (2012).
[12] Pearce CEM. and Pećcarić J., Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000) 51–55.
[13] Sadjang PN, On the fundamental theorem of $(p,q)-$calculus and some $(p,q)-$Taylor formulas, arXiv:1309.3934v1 [math.QA], 2013.
[14] Sudsutad W, Ntouyas SK, Tariboon J, Quantum integral inequalities for convex functions, J. Math. Inequal., 9 (2015), No. 3, 781–793.
[15] Tariboon J, Ntouyas SK, Quantum integral inequalities on finite intervals, J. Inequal. Appl. 2014 (2014) 121.
[16] Tariboon J, Ntouyas SK, Quantum calculus on finite intervals and applications to impulsive difference equations, Adv. Differ. Equ. 2013 (2013) 282.
[17] Tunç M., Göv E., Some Integral Inequalities via (p,q) $-$Calculus On Finite Intervals, Preprint, 2016.
[18] Tunç M., Göv E., (p,q) $-$Integral Inequalities on Finite Intervals, Preprint, 2016.

E-mail address: $^{\alpha}$`mevluttttunc@gmail.com`

E-mail address: $^{\beta}$`esordulu@gmail.com`

$^{\alpha,\beta}$Mustafa Kemal University, Faculty of Science and Arts, Department of Mathematics, 31000, Hatay, Turkey