Received 01/06/16

ON JENSEN'S MULTIPLICATIVE INEQUALITY FOR POSITIVE CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR[1,2]

ABSTRACT. In this paper we obtain some multiplicative refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

1. Introduction

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [12]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality is due to Tominaga [13] and provides a multiplicative reverse for Young's inequality

$$(1-\nu)a + \nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a, b > 0$, $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10 .

Key words and phrases. Young's Inequality, Convex functions, Jensen's inequality, Selfadjoint operator, Functions of selfadjoint operators.

RGMIA Res. Rep. Coll. 19 (2016), Art. 99

The following multiplicative reverse of Young inequality in terms of Kantorovich's constant has been obtained by Liao et al. [8]

$$(1.5)\qquad (1-\nu)\,a+\nu b\le K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu},$$

where $a,\ b>0,\ \nu\in[0,1]$ and $R=\max\{1-\nu,\nu\}$.

The following result that provides a vector operator version for the Jensen inequality is well known, see for instance [10] or [11, p. 5]:

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars $m,\ M$ with $m<M$. If f is a convex function on $[m,M]$, then*

$$(1.6)\qquad f(\langle Ax,x\rangle)\le\langle f(A)x,x\rangle$$

for each $x\in H$ with $\|x\|=1$.

As a special case of Theorem 1 we have the *Hölder-McCarthy inequality* [9]: Let A be a selfadjoint positive operator on a Hilbert space H, then

(i) $\langle A^{r}x,x\rangle\ge\langle Ax,x\rangle^{r}$ for all $r>1$ and $x\in H$ with $\|x\|=1$;

(ii) $\langle A^{r}x,x\rangle\le\langle Ax,x\rangle^{r}$ for all $0<r<1$ and $x\in H$ with $\|x\|=1$;

(iii) If A is invertible, then $\langle A^{r}x,x\rangle\ge\langle Ax,x\rangle^{r}$ for all $r<0$ and $x\in H$ with $\|x\|=1$.

In [2] (see also [3, p. 16]) we obtained the following additive reverse of (1.6):

Theorem 2. *Let I be an interval and $f:I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A)\subset\mathring{I}$, then*

$$(1.7)\qquad (0\le)\,\langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\le\langle f'(A)Ax,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle$$

for any $x\in H$ with $\|x\|=1$.

This is a generalization of the scalar discrete inequality obtained in [7]. For other reverse inequalities of this type see [3, p. 16].

Motivated by the above results, in this paper we obtain some multiplicative refinements and reverses of Jensen's inequality for positive convex or concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

2. Refinements

We have:

Theorem 3. *Let $f:I\to[0,\infty)$ be continuous on the interval I and assume that for some $\nu\in(0,1)$ the function f^{ν} is convex on I, then f is convex on I and we have the inequality*

$$(2.1)\qquad f(\langle Ax,x\rangle)\le f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle\le\langle f(A)x,x\rangle,$$

where A is a selfadjoint operator with $\operatorname{Sp}(A)\subset I$ and $x\in H$ with $\|x\|=1$.

Proof. Let $t,\ s\in I$ and $\lambda\in[0,1]$. Since f^{ν} is convex on I, then

$$f^{\nu}((1-\lambda)t+\lambda s)\le(1-\lambda)f^{\nu}(t)+\lambda f^{\nu}(s)$$

and by taking the power $\frac{1}{\nu}>1$ and using the convexity of power function $g(t)=t^r$ with exponent $r=\frac{1}{\nu}>1$ we have

$$\begin{aligned} f((1-\lambda)t+\lambda s) &\le [(1-\lambda) f^{\nu}(t)+\lambda f^{\nu}(s)]^{\frac{1}{\nu}} \\ &\le (1-\lambda)(f^{\nu}(t))^{\frac{1}{\nu}}+\lambda (f^{\nu}(s))^{\frac{1}{\nu}}=(1-\lambda) f(t)+\lambda f(s), \end{aligned}$$

which proves the convexity of f on I.

Now by Hölder-McCarthy inequality we have for any $x\in H$ with $\|x\|=1$

$$\langle f^{\nu}(A)x,x\rangle \le \langle f(A)x,x\rangle^{\nu}$$

and by Young's inequality that

$$\begin{aligned} f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle &\le f^{1-\nu}(\langle Ax,x\rangle)\langle f(A)x,x\rangle^{\nu} \\ &\le (1-\nu) f(\langle Ax,x\rangle)+\nu\langle f(A)x,x\rangle \end{aligned} \tag{2.2}$$

for any $x\in H$ with $\|x\|=1$.

The last inequality follows now by Jensen's inequality for the convex function f.

Applying the Jensen's inequality for the convex function f^{ν} we also have

$$f^{\nu}(\langle Ax,x\rangle)\le \langle f^{\nu}(A)x,x\rangle,$$

which implies that

$$f(\langle Ax,x\rangle)=f^{1-\nu}(\langle Ax,x\rangle) f^{\nu}(\langle Ax,x\rangle)\le f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle$$

that proves the first inequality in (2.1). □

The case of concave functions is as follows:

Theorem 4. *Let $f:I\to[0,\infty)$ be continuous and concave on the interval I. Then for any $\nu\in(0,1)$ the function f^{ν} is concave on I and we have the inequality*

$$\begin{aligned} \left\langle f^{1/2}(A)x,x\right\rangle^2 &\le \left\langle f^{1-\nu}(A)x,x\right\rangle\langle f^{\nu}(A)x,x\rangle \\ &\le f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle \le f(\langle Ax,x\rangle), \end{aligned} \tag{2.3}$$

where A is a selfadjoint operator with $\operatorname{Sp}(A)\subset I$ *and $x\in H$ with* $\|x\|=1$.

Proof. From (2.2) and the concavity of f we have

$$\begin{aligned} f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle &\le f^{1-\nu}(\langle Ax,x\rangle)\langle f(A)x,x\rangle^{\nu} \\ &\le (1-\nu) f(\langle Ax,x\rangle)+\nu\langle f(A)x,x\rangle \\ &\le f(\langle Ax,x\rangle) \end{aligned}$$

for any $\nu\in(0,1)$ and $x\in H$ with $\|x\|=1$, that proves the last part of (2.3).

Now, let t, $s\in I$ and $\lambda\in[0,1]$. Since f is concave on I, then

$$f((1-\lambda)t+\lambda s)\ge (1-\lambda) f(t)+\lambda f(s).$$

By taking the power $\nu\in(0,1)$ and using the concavity of power function $g(t)=t^r$ for the exponent $r=\nu\in(0,1)$ we have

$$f^{\nu}((1-\lambda)t+\lambda s)\ge((1-\lambda) f(t)+\lambda f(s))^{\nu}\ge(1-\lambda) f^{\nu}(t)+\lambda f^{\nu}(s)$$

that shows that f^{ν} is concave on I.

Applying the Jensen's inequality for the concave function $f^{1-\nu}$ we have

$$\left\langle f^{1-\nu}(A)x,x\right\rangle\le f^{1-\nu}(\langle Ax,x\rangle)$$

for $x\in H$ with $\|x\|=1$, that proves the second inequality in (2.3).

Now, by using the Schwarz type inequality for continuous functions of selfadjoint operators

$$\langle g(A)h(A)x,x\rangle^2 \le \langle g^2(A)x,x\rangle \langle h^2(A)x,x\rangle$$

for $x \in H$ with $\|x\| = 1$, then by choosing $g = f^{\frac{1-\nu}{2}}$ and $h = f^{\frac{\nu}{2}}$ we get the first inequality in (2.3). □

3. Upper Bounds

The following reverse inequalities also hold:

Theorem 5. *Let $f : [m, M] \to [0, \infty)$ be a continuous function and assume that*

$$0 < \gamma = \min_{t\in[m,M]} f(t) < \max_{t\in[m,M]} f(t) = \Gamma < \infty. \tag{3.1}$$

Then for any A a selfadjoint operator with

$$m1_H \le A \le M1_H \tag{3.2}$$

we have the inequality

$$(1-\nu) f(\langle Ax,x\rangle) + \nu \langle f(A)x,x\rangle \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^\nu(A)x,x\rangle f^{1-\nu}(\langle Ax,x\rangle) \tag{3.3}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in [0,1]$.

Moreover, if f is convex on $[m, M]$, then

$$f^\nu(\langle Ax,x\rangle) \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^\nu(A)x,x\rangle \tag{3.4}$$

while, if f is concave on $[m, M]$, then

$$\langle f(A)x,x\rangle \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^\nu(A)x,x\rangle f^{1-\nu}(\langle Ax,x\rangle) \tag{3.5}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By Tominaga's inequality we have

$$(1-\nu) + \nu s \le S(s) s^\nu$$

for any $s > 0$ and $\nu \in [0,1]$.

If $s \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right]$, then

$$(1-\nu) + \nu s \le S(s) s^\nu \le s^\nu \max_{s\in\left[\frac{\gamma}{\Gamma},\frac{\Gamma}{\gamma}\right]} S(s) = S\left(\frac{\Gamma}{\gamma}\right) s^\nu \tag{3.6}$$

and since for any $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$ we have

$$\frac{f(t)}{f(\langle Ax,x\rangle)} \in \left[\frac{\gamma}{\Gamma}, \frac{\Gamma}{\gamma}\right],$$

hence by (3.6) we get

$$(1-\nu) f(\langle Ax,x\rangle) + \nu f(t) \le S\left(\frac{\Gamma}{\gamma}\right) f^\nu(t) f^{1-\nu}(\langle Ax,x\rangle) \tag{3.7}$$

for any $t \in [m, M]$ and $x \in H$ with $\|x\| = 1$.

If we use the functional calculus for the operator A we have by (3.7) that

$$(1-\nu) f(\langle Ax,x\rangle) + \nu f(A) \le S\left(\frac{\Gamma}{\gamma}\right) f^\nu(A) f^{1-\nu}(\langle Ax,x\rangle) \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

If we take the inner product over $y \in H$ with $\|y\| = 1$ in (3.8), then we get

$$(1-\nu) f(\langle Ax, x\rangle) + \nu \langle f(A) y, y\rangle \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^{\nu}(A) y, y\rangle f^{1-\nu}(\langle Ax, x\rangle)$$

that for $y = x$ we get the desired inequality.

If f is convex, then

$$f(\langle Ax, x\rangle) \le \langle f(A) x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$, then by (3.3) we get

$$f(\langle Ax, x\rangle) \le S\left(\frac{\Gamma}{\gamma}\right) \langle f^{\nu}(A) x, x\rangle f^{1-\nu}(\langle Ax, x\rangle)$$

for any $x \in H$ with $\|x\| = 1$, which is equivalent to (3.4).

If f is concave, then

$$\langle f(A) x, x\rangle \le f(\langle Ax, x\rangle)$$

and by (3.3) we get (3.5). □

Remark 1. *If for some $\nu \in (0,1)$ the function f^{ν} is convex on $[m, M]$, then according with Theorem 3 f is convex on I and the inequality (3.4) is trivially satisfied since $S\left(\frac{\Gamma}{\gamma}\right) \ge 1$.*

If f is convex on I and for some $\nu \in (0,1)$ the function f^{ν} is concave on $[m, M]$, then from (3.4) we have the meaningful inequality

$$1 \le \frac{f^{\nu}(\langle Ax, x\rangle)}{\langle f^{\nu}(A) x, x\rangle} \le S\left(\frac{\Gamma}{\gamma}\right) \tag{3.9}$$

for any $x \in H$ with $\|x\| = 1$.

The inequality (3.5) can be written in equivalent form as

$$\frac{\langle f(A) x, x\rangle}{f(\langle Ax, x\rangle)} \le S\left(\frac{\Gamma}{\gamma}\right) \frac{\langle f^{\nu}(A) x, x\rangle}{f^{\nu}(\langle Ax, x\rangle)} \tag{3.10}$$

for any $x \in H$ with $\|x\| = 1$.

We also have:

Theorem 6. *With the assumptions of Theorem 5 we have the inequality*

$$(1-\nu) f(\langle Ax, x\rangle) + \nu \langle f(A) x, x\rangle \le K^{R}\left(\frac{\Gamma}{\gamma}\right) \langle f^{\nu}(A) x, x\rangle f^{1-\nu}(\langle Ax, x\rangle) \tag{3.11}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in [0,1]$ and $R = \max\{\nu, 1-\nu\}$.

Moreover, if f is convex on $[m, M]$, then

$$\frac{f^{\nu}(\langle Ax, x\rangle)}{\langle f^{\nu}(A) x, x\rangle} \le K^{R}\left(\frac{\Gamma}{\gamma}\right) \tag{3.12}$$

while, if f is concave on $[m, M]$, then

$$\frac{\langle f(A) x, x\rangle}{f(\langle Ax, x\rangle)} \le K^{R}\left(\frac{\Gamma}{\gamma}\right) \frac{\langle f^{\nu}(A) x, x\rangle}{f^{\nu}(\langle Ax, x\rangle)} \tag{3.13}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. By inequality (1.5) we have

$$(1-\nu)+\nu s\leq K^{R}(s)\, s^{\nu}$$

for any $s>0$ and $\nu\in[0,1]$.

If $s\in\left[\frac{\gamma}{\Gamma},\frac{\Gamma}{\gamma}\right]$, then

$$(1-\nu)+\nu s\leq S(s)\, s^{\nu}\leq s^{\nu}\max_{s\in\left[\frac{\gamma}{\Gamma},\frac{\Gamma}{\gamma}\right]}K^{R}(s)=K^{R}\left(\frac{\Gamma}{\gamma}\right)s^{\nu} \tag{3.14}$$

and since, for any $t\in[m,M]$ and $x\in H$ with $\|x\|=1$ we have

$$\frac{f(t)}{f(\langle Ax,x\rangle)}\in\left[\frac{\gamma}{\Gamma},\frac{\Gamma}{\gamma}\right],$$

hence by (3.14) we get

$$(1-\nu)f(\langle Ax,x\rangle)+\nu f(t)\leq K^{R}\left(\frac{\Gamma}{\gamma}\right)f^{\nu}(t)\, f^{1-\nu}(\langle Ax,x\rangle)$$

for any $t\in[m,M]$ and $x\in H$ with $\|x\|=1$.

Now the proof goes along the lines of the proof in Theorem 5 and we omit the details. □

In the recent paper [4], the author obtained the following reverse of Young's inequality

$$\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{3.15}$$

for any $a,\, b>0$ and $\nu\in[0,1]$, where K is Kantorovich's constant defined in (1.4).

If $a,\, b\in[m,M]$, then by the properties of K we have the upper bound:

$$\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{M}{m}\right)-1\right)\right], \tag{3.16}$$

for any and $\nu\in[0,1]$.

Using a similar argument as in Theorem 3 and the inequality (3.16) we can also state:

Theorem 7. *With the assumptions of Theorem 5 we have the inequality*

$$\begin{aligned}&(1-\nu)f(\langle Ax,x\rangle)+\nu\langle f(A)x,x\rangle\\&\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right]\langle f^{\nu}(A)x,x\rangle f^{1-\nu}(\langle Ax,x\rangle)\end{aligned} \tag{3.17}$$

for any $x\in H$ *with* $\|x\|=1$, *where* $\nu\in[0,1]$.

Moreover, if f *is convex on* $[m,M]$, *then*

$$\frac{f^{\nu}(\langle Ax,x\rangle)}{\langle f^{\nu}(A)x,x\rangle}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right] \tag{3.18}$$

while, if f *is concave on* $[m,M]$, *then*

$$\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{\Gamma}{\gamma}\right)-1\right)\right]\frac{\langle f^{\nu}(A)x,x\rangle}{f^{\nu}(\langle Ax,x\rangle)} \tag{3.19}$$

for any $x\in H$ *with* $\|x\|=1$.

The proof follows as above, however the details are not presented here.

The interested reader may establish similar results by employing the following multiplicative reverses of Young inequalities:

Lemma 1 (see [5])**.** *If* $a,\ b\in[\gamma,\Gamma]\subset(0,\infty)$ *and* $\nu\in[0,1]$, *then we have*

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}}\leq\max\left\{\kappa_{\gamma,\Gamma}(\nu),\kappa_{\gamma,\Gamma}(1-\nu)\right\}\tag{3.20}$$

where

$$\kappa_{\gamma,\Gamma}(\nu):=\frac{(1-\nu)\,\gamma+\nu\Gamma}{\gamma^{1-\nu}\Gamma^{\nu}}\tag{3.21}$$

and

Lemma 2 (see [6])**.** *If* $a,\ b\in[\gamma,\Gamma]\subset(0,\infty)$ *and* $\nu\in[0,1]$, *then we have*

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\Gamma}{\gamma}-1\right)^{2}\right].\tag{3.22}$$

The inequality (3.22) can also be obtained from inequality (2.9) from the paper [1].

4. Some Examples

Now, let $\nu\in(0,1)$ and $r\geq\frac{1}{\nu}>1$. Consider the function $f:[0,\infty)\to[0,\infty)$ defined by $f(t)=t^{r}$. Then the function f^{ν} is convex on $[0,\infty)$ and if A is a positive operator on the Hilbert space H, then by the inequality (2.1) we have the following refinement of Hölder-McCarthy inequality

$$\langle Ax,x\rangle^{r}\leq\langle Ax,x\rangle^{(1-\nu)r}\langle A^{\nu r}x,x\rangle\leq\langle A^{r}x,x\rangle,\tag{4.1}$$

for any $x\in H$ with $\|x\|=1$.

If $r\geq2$, then by (4.1) we have

$$\langle Ax,x\rangle^{r}\leq\langle Ax,x\rangle^{r/2}\left\langle A^{r/2}x,x\right\rangle\leq\langle A^{r}x,x\rangle,\tag{4.2}$$

for any $x\in H$ with $\|x\|=1$.

Let $\nu\in(0,1)$ and $\alpha\in\mathbb{R}$. Consider the function $f(t)=\exp(\alpha t)$, $t\in\mathbb{R}$. Then the function $f^{\nu}(t)=\exp(\nu\alpha t)$ is convex on $\mathbb{R}$ and for any selfadjoint operator A on H we have

$$\exp(\alpha\langle Ax,x\rangle)\leq\exp((1-\nu)\,\alpha\langle Ax,x\rangle)\langle\exp(\nu\alpha A)\,x,x\rangle\leq\langle\exp(\alpha A)\,x,x\rangle,\tag{4.3}$$

for any $x\in H$ with $\|x\|=1$.

For $\alpha=1$, we get

$$\exp(\langle Ax,x\rangle)\leq\exp((1-\nu)\langle Ax,x\rangle)\langle\exp(\nu A)\,x,x\rangle\leq\langle\exp(A)\,x,x\rangle,\tag{4.4}$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

Consider $q\in(0,1)$, then the function $f(t)=t^{q}$ is concave on $[0,\infty)$ and if A is a positive operator on the Hilbert space H, then by the inequality (2.3) we have

$$\begin{aligned}\left\langle A^{q/2}x,x\right\rangle^{2}&\leq\left\langle A^{(1-\nu)q}x,x\right\rangle\langle A^{\nu q}x,x\rangle\leq\langle Ax,x\rangle^{(1-\nu)q}\langle A^{\nu q}x,x\rangle\\&\leq\langle Ax,x\rangle^{q},\end{aligned}\tag{4.5}$$

for any $x\in H$ with $\|x\|=1$.

If we take in (4.5) $q\to 1$, $q<1$ we get

$$\left\langle A^{1/2}x,x\right\rangle^{2}\leq\left\langle A^{1-\nu}x,x\right\rangle\left\langle A^{\nu}x,x\right\rangle\leq\left\langle Ax,x\right\rangle^{1-\nu}\left\langle A^{\nu}x,x\right\rangle \leq\left\langle Ax,x\right\rangle, \tag{4.6}$$

for any $x\in H$ with $\|x\|=1$.

The function $f(t)=\ln(t+1)$ is positive concave on $[0,\infty)$ and if A is a positive operator on the Hilbert space H, then by the inequality (2.3) we have

$$\begin{aligned}\left\langle\sqrt{\ln(A+1_H)}x,x\right\rangle^{2}&\leq\left\langle\ln^{1-\nu}(A+1_H)x,x\right\rangle\left\langle\ln^{\nu}(A+1_H)x,x\right\rangle\\&\leq\ln^{1-\nu}(\left\langle Ax,x\right\rangle+1)\left\langle\ln^{\nu}(A+1_H)x,x\right\rangle\\&\leq\ln(\left\langle Ax,x\right\rangle+1),\end{aligned} \tag{4.7}$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

Consider the function $f:\mathbb{R}\to(0,\infty)$, $f(t)=\exp\left(-\frac{1}{2}t^2\right)$. Then

$$f'(t)=-t\exp\left(-\frac{1}{2}t^2\right)\text{ and }f''(t)=\exp\left(-\frac{1}{2}t^2\right)\left(t^2-1\right),$$

which shows that the function is concave on $[-1,1]$.

Now, if A is selfadjoint and $-1_H\leq A\leq 1_H$, then by (2.3) we have that

$$\begin{aligned}&\left\langle\exp\left(-\frac{1}{4}A^2\right)x,x\right\rangle^{2}\\&\leq\left\langle\exp\left(-\frac{1}{2}(1-\nu)A^2\right)x,x\right\rangle\left\langle\exp\left(-\frac{1}{2}\nu A^2\right)x,x\right\rangle\\&\leq\exp\left(-\frac{1}{2}(1-\nu)\left\langle Ax,x\right\rangle^{2}\right)\left\langle\exp\left(-\frac{1}{2}\nu A^2\right)x,x\right\rangle\\&\leq\exp\left(-\frac{1}{2}\left\langle Ax,x\right\rangle^{2}\right),\end{aligned} \tag{4.8}$$

for any $x\in H$ with $\|x\|=1$.

Now, let $\nu\in(0,1)$ and $\frac{1}{\nu}>r\geq 1$. Consider the function $f:[0,\infty)\to[0,\infty)$ defined by $f(t)=t^r$. Then f is convex while function f^{ν} is concave on $[0,\infty)$ and if A is a positive operator on the Hilbert space H satisfying the condition (3.2) then by taking $\gamma=m^r$ and $\Gamma=M^r$ in (3.9) we get

$$1\leq\frac{\left\langle Ax,x\right\rangle^{r\nu}}{\left\langle A^{r\nu}x,x\right\rangle}\leq S\left(\left(\frac{M}{m}\right)^{r}\right) \tag{4.9}$$

for any $x\in H$ with $\|x\|=1$, where $\nu\in(0,1)$ and $\frac{1}{\nu}>r\geq 1$.

If we take $r=1$ in (4.9), then we get

$$1\leq\frac{\left\langle Ax,x\right\rangle^{\nu}}{\left\langle A^{\nu}x,x\right\rangle}\leq S\left(\frac{M}{m}\right) \tag{4.10}$$

for any $x\in H$ with $\|x\|=1$ and $\nu\in(0,1)$.

Consider $q\in(0,1)$, then the function $f(t)=t^q$ is concave on $[0,\infty)$ and if A is a positive operator on the Hilbert space H satisfying the condition (3.2) then by taking $\gamma=m^q$ and $\Gamma=M^q$ in (3.5) we get

$$\left\langle A^{q}x,x\right\rangle\leq S\left(\left(\frac{M}{m}\right)^{q}\right)\left\langle A^{\nu q}x,x\right\rangle\left\langle Ax,x\right\rangle^{(1-\nu)q} \tag{4.11}$$

for any $x \in H$ with $\|x\| = 1$ and $\nu \in (0,1)$.

This inequality can be written in equivalent form as

$$\frac{\langle A^q x, x\rangle}{\langle Ax, x\rangle^q} \le S\left(\left(\frac{M}{m}\right)^q\right) \frac{\langle A^{\nu q} x, x\rangle}{\langle Ax, x\rangle^{\nu q}} \tag{4.12}$$

for any $x \in H$ with $\|x\| = 1$ and $\nu \in (0,1)$.

Consider the function $f(t) = \exp\left(-\frac{1}{2}t^2\right)$ on the concavity interval $[-1,1]$. We have that

$$\max_{t\in[-1,1]} f(t) = f(0) = 1 \text{ and } \min_{t\in[-1,1]} f(t) = f(\pm 1) = \exp\left(-\frac{1}{2}\right).$$

If A is selfadjoint and $-1_H \le A \le 1_H$, then by the inequality (3.10) we have

$$\frac{\left\langle \exp\left(-\frac{1}{2}A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\langle Ax, x\rangle^2\right)} \le S\left(\exp\left(\frac{1}{2}\right)\right) \frac{\left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\nu\langle Ax, x\rangle^2\right)} \tag{4.13}$$

for any $x \in H$ with $\|x\| = 1$.

If A is a positive operator on the Hilbert space H satisfying the condition (3.2) then by (3.12) we get

$$1 \le \frac{\langle Ax, x\rangle^{r\nu}}{\langle A^{r\nu} x, x\rangle} \le K^R\left(\left(\frac{M}{m}\right)^r\right) \tag{4.14}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in (0,1)$ with $\frac{1}{\nu} > r \ge 1$ and $R = \max\{\nu, 1-\nu\}$.

From (3.13) we have

$$\frac{\langle A^q x, x\rangle}{\langle Ax, x\rangle^q} \le K^R\left(\left(\frac{M}{m}\right)^q\right) \frac{\langle A^{\nu q} x, x\rangle}{\langle Ax, x\rangle^{\nu q}} \tag{4.15}$$

where $q \in (0,1)$, $\nu \in (0,1)$ and $R = \max\{\nu, 1-\nu\}$.

If A is selfadjoint and $-1_H \le A \le 1_H$, then by the inequality (3.13) we have

$$\frac{\left\langle \exp\left(-\frac{1}{2}A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\langle Ax, x\rangle^2\right)} \le K^R\left(\exp\left(\frac{1}{2}\right)\right) \frac{\left\langle \exp\left(-\frac{1}{2}\nu A^2\right)x, x\right\rangle}{\exp\left(-\frac{1}{2}\nu\langle Ax, x\rangle^2\right)} \tag{4.16}$$

for any $x \in H$ with $\|x\| = 1$, where $\nu \in (0,1)$ and $R = \max\{\nu, 1-\nu\}$.

References

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63** (2015), Issue 3, 622-635.

[2] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces. *J. Inequal. Appl.* **2010**, Art. ID 496821, 15 pp.

[3] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[4] S. S. Dragomir, A note on Young's inequality, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas,* to appear, see http://link.springer.com/article/10.1007/s13398-016-0300-8.

[5] S. S. Dragomir, Two new reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 158. [http://rgmia.org/papers/v18/v18a158.pdf].

[6] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 131. [http://rgmia.org/papers/v18/v18a131.pdf].

[7] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78. MR:1325895 (96c:26012).

[8] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[9] C. A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.
[10] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.
[11] J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[12] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.
[13] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa