

## AN INEQUALITY OF OSTROWSKI'S TYPE FOR CUMULATIVE DISTRIBUTION FUNCTIONS

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ABSTRACT. The main aim of this paper is to establish an Ostrowski type inequality for the cumulative distribution function of a random variable taking values in a finite interval  $[a, b]$ . An application for a Beta random variable is given.

### 1 INTRODUCTION

In [1], S.S. Dragomir and S. Wang proved the following version of Ostrowski's inequality for differentiable mappings whose derivatives belong to  $L_1(a, b)$ :

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbf{R}$  belongs to  $L_1(a, b)$ . Then we have the inequality:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \|f'\|_1$$

for all  $x \in [a, b]$ .

Note that the classical Ostrowski's integral inequality states that (see e.g. [3, p.468]):

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all  $x \in [a, b]$  provided  $f' \in L_\infty(a, b)$ .

In the above paper [1], the authors have applied inequality (1.1) to Numerical Integration obtaining estimations for the error bounds of general Riemann's quadrature formulae in terms of  $\|f'\|_1$ .

Applications of Ostrowski's inequality for the same problems in Numerical Integration have been pointed out by the same authors in [2].

The main aim of the present work is to establish an Ostrowski like inequality for the cumulative distribution function and expectation of a random variable.

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## 2 THE RESULTS

Let  $X$  be a random variable taking values in the finite interval  $[a, b]$ , with cumulative distribution function  $F(x) = \Pr(X \leq x)$ .

The following theorem holds

**Theorem 2.1.** *Let  $X$  and  $F$  be as above. Then we have the inequality*

$$\begin{aligned}
 (2.1) \quad & \left| \Pr(X \leq x) - \frac{b - E(X)}{b - a} \right| \\
 & \leq \frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \\
 & \leq \frac{1}{b - a} [(b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x)] \\
 & \leq \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{(b - a)}
 \end{aligned}$$

for all  $x \in [a, b]$ . All the inequalities in (2.1) are sharp and the constant  $\frac{1}{2}$  is the best possible.

*Proof.* Consider the kernel  $p : [a, b]^2 \rightarrow \mathbf{R}$  given by

$$(2.2) \quad p(x, t) := \begin{cases} t - a & \text{if } t \in [a, x] \\ t - b & \text{if } t \in (x, b] \end{cases}.$$

Then the Riemann-Stieltjes integral  $\int_a^b p(x, t) dF(t)$  exists for any  $x \in [a, b]$  and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$\begin{aligned}
 (2.3) \quad & \int_a^b p(x, t) dF(t) = \int_a^x (t - a) dF(t) + \int_x^b (t - b) dF(t) \\
 & = (t - a) F(t) \Big|_a^x - \int_a^x F(t) dt + (t - b) F(t) \Big|_x^b - \int_x^b F(t) dt \\
 & = (b - a) F(x) - \int_a^b F(t) dt.
 \end{aligned}$$

On the other hand, the integration by parts formula for Riemann-Stieltjes integral also gives:

$$(2.4) \quad \begin{aligned} E(X) &:= \int_a^b t dF(t) = tF(t)|_a^b - \int_a^b F(t) dt \\ &= bF(b) - aF(a) - \int_a^b F(t) dt = b - \int_a^b F(t) dt. \end{aligned}$$

Now, using (2.3) and (2.4), we get the equality

$$(2.5) \quad (b-a)F(x) + E(X) - b = \int_a^b p(x,t) dF(t)$$

for all  $x \in [a, b]$ .

Now, assume that  $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b$  is a sequence of divisions with  $\nu(\Delta_n) \rightarrow 0$  as  $n \rightarrow \infty$ , where

$$\nu(\Delta_n) := \max \{x_{i+1}^{(n)} - x_i^{(n)} : i = 0, \dots, n-1\}.$$

If  $p : [a, b] \rightarrow \mathbf{R}$  is continuous on  $[a, b]$  and  $\nu : [a, b] \rightarrow \mathbf{R}$  is monotonous nondecreasing, then the Riemann-Stieltjes integral  $\int_a^b p(x) d\nu(x)$  exists and

$$(2.6) \quad \begin{aligned} \left| \int_a^b p(x) d\nu(x) \right| &= \left| \lim_{\nu(\Delta_n) \rightarrow 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)}) [\nu(x_{i+1}^{(n)}) - \nu(x_i^{(n)})] \right| \\ &\leq \lim_{\nu(\Delta_n) \rightarrow 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})| (\nu(x_{i+1}^{(n)}) - \nu(x_i^{(n)})) \\ &= \int_a^b |p(x)| d\nu(x). \end{aligned}$$

Using (2.6) we have:

$$(2.7) \quad \begin{aligned} \left| \int_a^b p(x,t) dF(t) \right| &= \left| \int_a^x (t-a) dF(t) + \int_x^b (t-b) dF(t) \right| \\ &\leq \left| \int_a^x (t-a) dF(t) \right| + \left| \int_x^b (t-b) dF(t) \right| \leq \int_a^x |t-a| dF(t) + \int_x^b |t-b| dF(t) \end{aligned}$$

$$\begin{aligned}
&= \int_a^x (t-a) dF(t) + \int_x^b (b-t) dF(t) \\
&= (t-a) F(t)|_a^x - \int_a^x F(t) dt - (b-t) F(t)|_x^b + \int_x^b F(t) dt \\
&= \left[ [2x - (a+b)] F(x) - \int_a^x F(t) dt + \int_x^b F(t) dt \right] \\
&= [2x - (a+b)] F(x) + \int_a^b \operatorname{sgn}(t-x) F(t) dt.
\end{aligned}$$

Using the identity (2.5) and the inequality (2.7), we deduce the first part of (2.1).

We know that

$$\int_a^b \operatorname{sgn}(t-x) F(t) dt = - \int_a^x F(t) dt + \int_x^b F(t) dt.$$

As  $F(\cdot)$  is monotonous nondecreasing on  $[a, b]$ , we can state that

$$\int_a^x F(t) dt \geq (x-a) F(a) = 0$$

and

$$\int_x^b F(t) dt \leq (b-x) F(b) = b-x$$

and then

$$\int_a^b \operatorname{sgn}(t-x) F(t) dt \leq b-x \quad \text{for all } x \in [a, b].$$

Consequently, we have the inequality

$$[2x - (a+b)] F(x) + \int_a^b \operatorname{sgn}(t-x) F(t) dt$$

$$\begin{aligned} &\leq [2x - (a + b)] F(x) + (b - x) = (b - x)(1 - F(x)) + (x - a) F(x) \\ &= (b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x) \end{aligned}$$

and the second part of (2.1) is proved.

Finally,

$$\begin{aligned} &(b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x) \\ &\leq \max\{b - x, x - a\} [\Pr(X \geq x) + \Pr(X \leq x)] \\ &= \frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \end{aligned}$$

and the last part of (2.1) is also proved.

Now, assume that the inequality (2.1) holds with a constant  $c > 0$  instead of  $\frac{1}{2}$ , i.e.,

$$\begin{aligned} (2.8) \quad &\left| \Pr(X \leq x) - \frac{b - E(X)}{b - a} \right| \\ &\leq \frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \\ &\leq \frac{1}{b - a} [(b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x)] \\ &\leq c + \frac{\left| x - \frac{a + b}{2} \right|}{b - a} \end{aligned}$$

for all  $x \in [a, b]$ .

Choose the random variable  $X$  such that  $F : [0, 1] \rightarrow \mathbf{R}$ ,

$$F(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, 1] \end{cases}.$$

Then we have:

$$E(X) = 0, \quad \int_0^1 \operatorname{sgn}(t) F(t) dt = 1$$

and by (2.8), for  $x = 0$ , we get

$$1 \leq c + \frac{1}{2}$$

which shows that  $c = \frac{1}{2}$  is the best possible value. ■

**Remark 2.1.** Taking into account the fact that

$$\Pr(X \geq x) = 1 - \Pr(X \leq x),$$

then from (2.1) we get the equivalent inequality

$$\begin{aligned} (2.9) \quad & \left| \Pr(X \geq x) - \frac{E(X) - a}{b - a} \right| \\ & \leq \frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \\ & \leq \frac{1}{b - a} [(b - x) \Pr(X \geq x) + (x - a) \Pr(X \leq x)] \\ & \leq \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b - a} \end{aligned}$$

for all  $x \in [a, b]$ .

**Remark 2.2.** The following particular cases are also interesting:

$$(2.10) \quad \left| \Pr\left(X \leq \frac{a+b}{2}\right) - \frac{b - E(X)}{b - a} \right| \leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) F(t) dt \leq \frac{1}{2}$$

and

$$(2.11) \quad \left| \Pr\left(X \geq \frac{a+b}{2}\right) - \frac{E(X) - a}{b - a} \right| \leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) F(t) dt \leq \frac{1}{2}.$$

The following corollary could be useful in practice

**Corollary 2.2.** Under the above assumptions, we have

$$\begin{aligned} (2.12) \quad & \frac{1}{b - a} \left[ \frac{a + b}{2} - E(X) \right] \\ & \leq \Pr\left(X \leq \frac{a + b}{2}\right) \leq \frac{1}{b - a} \left[ \frac{a + b}{2} - E(X) \right] + 1. \end{aligned}$$

*Proof.* From the inequality (2.10), we get

$$-\frac{1}{2} + \frac{b - E(X)}{b - a} \leq \Pr\left(X \leq \frac{a + b}{2}\right) \leq \frac{1}{2} + \frac{b - E(X)}{b - a}.$$

But

$$-\frac{1}{2} + \frac{b - E(X)}{b - a} = \frac{-b + a + 2b - 2E(X)}{2(b - a)} = \frac{1}{b - a} \left[ \frac{a + b}{2} - E(X) \right]$$

and

$$\begin{aligned} \frac{1}{2} + \frac{b - E(X)}{b - a} &= 1 + \frac{b - E(X)}{b - a} - \frac{1}{2} = 1 + \frac{2b - 2E(X) - b + a}{2(b - a)} \\ &= 1 + \frac{1}{b - a} \left[ \frac{a + b}{2} - E(X) \right] \end{aligned}$$

and the inequality is proved. ■

**Remark 2.3.** Let  $1 \geq \varepsilon \geq 0$ , and assume that

$$(2.13) \quad E(X) \geq \frac{a + b}{2} + (1 - \varepsilon)(b - a)$$

then

$$(2.14) \quad \Pr\left(X \leq \frac{a + b}{2}\right) \leq \varepsilon.$$

Indeed, if (2.13) holds, then by the right-hand side of (2.12) we get

$$\Pr\left(X \leq \frac{a + b}{2}\right) \leq \frac{1}{b - a} \left[ \frac{a + b}{2} - E(X) \right] + 1 \leq \frac{(\varepsilon - 1)(b - a)}{b - a} + 1 = \varepsilon.$$

**Remark 2.4.** Also, if

$$(2.15) \quad E(X) \leq \frac{a + b}{2} - \varepsilon(b - a)$$

then, by the right-hand side of (2.12),

$$\Pr\left(X \leq \frac{a + b}{2}\right) \geq \left[ \frac{a + b}{2} - E(X) \right] \cdot \frac{1}{b - a} \geq \frac{\varepsilon(b - a)}{(b - a)} = \varepsilon$$

i.e.,

$$(2.16) \quad \Pr\left(X \leq \frac{a + b}{2}\right) \geq \varepsilon \quad (\varepsilon \in [0, 1]).$$

The following corollary is also interesting:

**Corollary 2.3.** Under the above assumptions of Theorem 2.1, we have the inequality:

$$(2.17) \quad \begin{aligned} \frac{1}{b - x} \int_a^b \left[ \frac{1 + \operatorname{sgn}(t - x)}{2} \right] F(t) dt &\geq \Pr(X \geq x) \\ &\geq \frac{1}{x - a} \int_a^b \left[ \frac{1 - \operatorname{sgn}(t - x)}{2} \right] F(t) dt \end{aligned}$$

for all  $x \in (a, b)$ .

*Proof.* From the inequality (2.1) we have:

$$\begin{aligned} & \Pr(X \leq x) - \frac{b - E(X)}{b - a} \\ & \leq \frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \end{aligned}$$

which is equivalent to:

$$\begin{aligned} & (b - a) \Pr(X \leq x) - [2x - (a + b)] \Pr(X \leq x) \\ & \leq b - E(X) + \int_a^b \operatorname{sgn}(t - x) F(t) dt, \end{aligned}$$

i.e.,

$$2(b - x) \Pr(X \leq x) \leq b - E(X) + \int_a^b \operatorname{sgn}(t - x) F(t) dt.$$

As (see the Proof of Theorem 2.1):

$$b - E(X) = \int_a^b F(t) dt$$

then from the above inequality we deduce the first part of (2.17).

The second part of (2.17) follows by a similar argument from

$$\begin{aligned} & \Pr(X \leq x) - \frac{b - E(X)}{b - a} \\ & \geq -\frac{1}{b - a} \left[ [2x - (a + b)] \Pr(X \leq x) + \int_a^b \operatorname{sgn}(t - x) F(t) dt \right] \end{aligned}$$

and we shall omit the details. ■

**Remark 2.5.** If we put  $x = \frac{a+b}{2}$  in (2.17), then we get

$$\begin{aligned} (2.18) \quad & \frac{1}{b - a} \int_a^b \left[ 1 + \operatorname{sgn} \left( t - \frac{a+b}{2} \right) \right] F(t) dt \geq \Pr \left( X \geq \frac{a+b}{2} \right) \\ & \geq \frac{1}{b - a} \int_a^b \left[ 1 - \operatorname{sgn} \left( t - \frac{a+b}{2} \right) \right] F(t) dt. \end{aligned}$$



## 3 APPLICATIONS FOR A BETA RANDOM VARIABLE

A Beta random variable  $X$  with parameters  $(p, q)$  has the probability density function

$$f(x; p, q) := \frac{x^{p-1} (1-x)^{q-1}}{B(p, q)}; \quad 0 < x < 1$$

where  $\Omega = \{(p, q) : p, q > 0\}$  and  $B(p, q) := \int_0^1 t^{p-1} (1-t)^{q-1} dt$ .

Let us compute the expectation of  $X$ .

We have

$$E(X) = \frac{1}{B(p, q)} \int_0^1 x \cdot x^{p-1} (1-x)^{q-1} dx = \frac{B(p+1, q)}{B(p, q)},$$

i.e.,

$$E(X) = \frac{p}{p+q}.$$

The following proposition holds:

**Proposition 3.1.** *Let  $X$  be a Beta random variable with parameters  $(p, q)$ . Then we have the inequalities:*

$$\left| \Pr(X \leq x) - \frac{q}{p+q} \right| \leq \frac{1}{2} + \left| x - \frac{1}{2} \right|$$

and

$$\left| \Pr(X \geq x) - \frac{p}{p+q} \right| \leq \frac{1}{2} + \left| x - \frac{1}{2} \right|$$

for all  $x \in [0, 1]$  and particularly:

$$\left| \Pr\left(X \leq \frac{1}{2}\right) - \frac{q}{p+q} \right| \leq \frac{1}{2}$$

and

$$\left| \Pr\left(X \geq \frac{1}{2}\right) - \frac{p}{p+q} \right| \leq \frac{1}{2}$$

respectively.

The proof follows by Theorem 2.1 applied for a Beta random variable,  $X$ .

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