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AN OSTROWSKI'S TYPE INEQUALITY FOR A RANDOM VARIABLE WHOSE PROBABILITY DENSITY FUNCTION BELONGS TO $L_\infty[A, B]$

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. An inequality of Ostrowski's type for a random variable whose probability density function is in $L_\infty[a,b]$ in terms of the cumulative distribution function and expectation is given. An application for a Beta random variable is also given.

1 Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [1, p. 469]

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping in I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In [2], S.S. Dragomir and S. Wang applied Ostrowski's inequality in Numerical Analysis obtaining an estimation of the error bound for the quadrature rules of Riemann type in terms of the infinity norm $\|\cdot\|_\infty$. Application for special means : logarithmic mean, identric mean, p-logarithmic mean etc... were also given.

The main aim of this paper is to give an Ostrowski's type inequality for random variables whose probability density functions are in $L_\infty[a,b]$. An application for a Beta Random Variable is also given.

2 The Results

Let X be a random variable with the probability density function $f : [a,b] \subset \mathbf{R} \to \mathbf{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \le x)$.

The following theorem holds

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Theorem 2.1. *Let* $f \in L_\infty[a,b]$ *and put* $\|f\|_\infty = \sup_{t\in[a,b]} f(t) < \infty$*. Then we have the inequality:*

$$\left|\Pr(X \le x) - \frac{b - E(X)}{b-a}\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{2.1}$$

or equivalently,

$$\left|\Pr(X \ge x) - \frac{E(X) - a}{b-a}\right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{2.2}$$

for all $x \in [a,b]$*. The constant* $\frac{1}{4}$ *in* (2.1) *and* (2.2) *is sharp.*

Proof. Let $x, y \in [a,b]$. Then

$$|F(x) - F(y)| = \left|\int_x^y f(t)\,dt\right| \le |x-y|\,\|f\|_\infty$$

which shows that F is $\|f\|_\infty$ −Lipschitzian on $[a,b]$.

Consider the kernel $p : [a,b]^2 \to \mathbf{R}$ given by

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases}.$$

Then the Riemann-Stieltjes integral $\int_a^b p(x,t)\,dF(t)$ exists for any $x \in [a,b]$ and the formula of integration by parts for Riemann-Stieltjes integral gives:

$$\int_a^b p(x,t)\,dF(t) = \int_a^x (t-a)\,dF(t) + \int_x^b (t-b)\,dF(t) \tag{2.3}$$

$$= (t-a)F(t)\big|_a^b - \int_a^x F(t)\,dt + (t-b)F(t)\big|_x^b - \int_x^b F(t)\,dt$$

$$= (b-a)F(x) - \int_a^b F(t)\,dt.$$

The integration by parts formula for Riemann-Stieltjes integral also gives

$$E(X) = \int_a^b t\,dF(t) = tF(t)\big|_a^b - \int_a^b F(t)\,dt \tag{2.4}$$

$$= bF(b) - aF(a) - \int_a^b F(t)\,dt = b - \int_a^b F(t)\,dt.$$

Now, using (2.3) and (2.4), we get the equality

$$(b-a)F(x) + E(X) - b = \int_a^b p(x,t)\,dF(t), \tag{2.5}$$

for all $x \in [a,b]$.

Now, assume that

$$\Delta_n := a = x_0^{(n)} < x_1^{(n)} < \dots < x_{n-1}^{(n)} < x_n^{(n)} = b$$

is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where

$$\nu(\Delta_n) := \max\left\{x_{i+1}^{(n)} - x_i^{(n)} : i = 0, \dots, n-1\right\}.$$

If $p : [a,b] \to \mathbf{R}$ is Riemann integrable on $[a,b]$ and $\nu : [a,b] \to \mathbf{R}$ is L-lipschitzian (lipschitzian with the constant L), then :

$$\left|\int_a^b p(x)\,d\nu(x)\right| = \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)\right]\right| \tag{2.6}$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left(x_{i+1}^{(n)} - x_i^{(n)}\right) \left|\frac{\nu\left(x_{i+1}^{(n)}\right) - \nu\left(x_i^{(n)}\right)}{x_{i+1}^{(n)} - x_i^{(n)}}\right|$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p\left(\xi_i^{(n)}\right)\right| \left(x_{i+1}^{(n)} - x_i^{(n)}\right) = L \int_a^b |p(x)|\,dx.$$

Applying the inequality (2.6) for the mappings $p(x,\cdot)$ and $F(\cdot)$, we get :

$$\left|\int_a^b p(x,t)\,dF(t)\right| \leq \|f\|_\infty \int_a^b |p(x,t)|\,dt$$

$$= \|f\|_\infty \left[\int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt\right] = \|f\|_\infty \left[\frac{(x-a)^2 + (b-x)^2}{2}\right]$$

$$= \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f\|_\infty$$

for all $x \in [a,b]$.

Finally, by the identity (2.5) we deduce:

$$\left|F(x)-\frac{b-E(X)}{b-a}\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty$$

for all $x\in[a,b]$ which is (2.1).

Now, taking into account the fact that

$$\Pr(X\ge x)=1-\Pr(X\le x)$$

the inequality (2.2) is also obtained.

To prove the sharpness of the constant $k=\frac{1}{4}$, assume that the inequality (2.1) holds with a constant $c>0$, i.e.,

$$\left|\Pr(X\le x)-\frac{b-E(X)}{b-a}\right|\le\left[c+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{2.7}$$

for all $x\in[a,b]$.

Assume that X_0 is a random variable having the probability density function $f_0:[0,1]\to\mathbf{R}$, $f_0(t)=1$. Then

$$\Pr(X_0\ge x)=x,\qquad x\in[0,1],$$

$$E(X_0)=\frac{1}{2},$$

and

$$\|f_0\|_\infty=1.$$

Consequently, (2.7) becomes

$$\left|x-\frac{1}{2}\right|\le c+\left(x-\frac{1}{2}\right)^2\qquad\text{for all } x\in[0,1].$$

Choosing $x=0$, we get $c\ge\frac{1}{4}$ and the sharpness of the constant is thus proved. ∎

The above theorem has some interesting corollaries for the expectation of X as follows:

Corollary 2.2. *Under the above assumptions, we have the double inequality:*

$$b-\frac{1}{2}(b-a)^2\|f\|_\infty\le E(X)\le a+\frac{1}{2}(b-a)^2\|f\|_\infty. \tag{2.8}$$

Proof. We know that

$$a\le E(X)\le b.$$

Now, choose $x=a$ in (2.1) to obtain

$$\left|\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}(b-a)\|f\|_\infty$$

i.e.,

$$b - E(X) \leq \frac{1}{2}(b-a)^2 \|f\|_\infty$$

which is equivalent to the first inequality in (2.8).

Also, choose $x = b$ in (2.1) to get

$$\left|1 - \frac{b - E(X)}{b-a}\right| \leq \frac{1}{2}(b-a)\|f\|_\infty$$

i.e.,

$$E(X) - a \leq \frac{1}{2}(b-a)^2 \|f\|_\infty$$

and the inequality (2.8) is proved. ∎

Remark 2.1. *We know that*

$$1 = \int_a^b f(x)\,dx \leq (b-a)\|f\|_\infty$$

which gives us

$$\|f\|_\infty \geq \frac{1}{b-a}.$$

Now, if we assume that $\|f\|_\infty$ is not too large, i.e.,

$$\|f\|_\infty \leq \frac{2}{b-a} \tag{2.9}$$

then

$$a + \frac{1}{2}(b-a)^2 \|f\|_\infty \leq b$$

and

$$b - \frac{1}{2}(b-a)^2 \|f\|_\infty \geq a$$

which shows that the inequality (2.8) is a tighter inequality than $a \leq E(X) \leq b$ when (2.9) holds.

Another equivalent inequality to (2.8) which can be more useful in practice is the following one:

Corollary 2.3. *With the above assumptions, we have the inequality*

$$\left|E(X) - \frac{a+b}{2}\right| \leq \frac{(b-a)^2}{2}\left(\|f\|_\infty - \frac{1}{b-a}\right). \tag{2.10}$$

Proof. From the inequality (2.8) we have :

$$b-\frac{a-b}{2}-\frac{1}{2}(b-a)^2\|f\|_\infty \le E(X)-\frac{a+b}{2}$$

$$\le a-\frac{a+b}{2}+\frac{1}{2}(b-a)^2\|f\|_\infty$$

i.e.,

$$-\frac{(b-a)^2}{2}\left(\|f\|_\infty-\frac{1}{b-a}\right)\le E(X)-\frac{a+b}{2}$$

$$\le \frac{(b-a)^2}{2}\left(\|f\|_\infty-\frac{1}{b-a}\right)$$

which is exactly (2.10). ∎

This corollary provides the possibility of finding a sufficient condition in terms of $\|f\|_\infty$ for the expectation $E(X)$ to be close to the mean value $\frac{a+b}{2}$.

Corollary 2.4. *Let X and f be as above and $\varepsilon>0$. If*

$$\|f\|_\infty \le \frac{1}{b-a}+\frac{2\varepsilon}{(b-a)^2} \tag{2.11}$$

then

$$\left|E(X)-\frac{a+b}{2}\right|\le\varepsilon.$$

The proof is obvious and we shall omit the details.

The following corollary of Theorem 2.1 also holds:

Corollary 2.5. *Let X and f be as above. Then we have the inequality:*

$$\left|\Pr\left(X\le\frac{a+b}{2}\right)-\frac{1}{2}\right| \tag{2.12}$$

$$\le\frac{1}{4}(b-a)\|f\|_\infty+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|$$

$$\le\frac{3}{4}(b-a)\|f\|_\infty-\frac{1}{2}.$$

Proof. If we choose in (2.1) $x=\frac{a+b}{2}$, we get

$$\left|\Pr\left(X\le\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|\le\frac{1}{4}(b-a)\|f\|_\infty$$

which is clearly equivalent to

$$\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right| \leq\frac{1}{4}\left(b-a\right)\left\|f\right\|_{\infty}.$$

Now, using the triangle inequality, we get

$$\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right|$$

$$=\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|$$

$$\leq\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|+\left|\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|$$

$$\leq\frac{1}{4}\left(b-a\right)\left\|f\right\|_{\infty}+\frac{1}{b-a}\left|E\left(X\right)-\frac{a+b}{2}\right|\leq\frac{3}{4}\left(b-a\right)\left\|f\right\|_{\infty}-\frac{1}{2}$$

and the desired inequality is proved. ∎

Remark 2.2. *A similar result applies for*

$$\Pr\left(X\geq\frac{a+b}{2}\right).$$

We shall omit the details.

Finally, the following result holds:

Corollary 2.6. *Let X and f be as above. Then we have the inequality:*

$$\left|E\left(X\right)-\frac{a+b}{2}\right|\leq\frac{1}{4}\left(b-a\right)^{2}\left\|f\right\|_{\infty}+\left(b-a\right)\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right|. \tag{2.13}$$

Proof. As in the above corollary, we have:

$$\frac{1}{b-a}\left|E\left(X\right)-\frac{a+b}{2}\right|$$

$$\leq\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E\left(X\right)-\frac{a+b}{2}\right)\right|+\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right|$$

$$\leq\frac{1}{4}\left(b-a\right)\left\|f\right\|_{\infty}+\left|\Pr\left(X\leq\frac{a+b}{2}\right)-\frac{1}{2}\right|$$

from whence we get (2.13). ∎

Remark 2.3. *If we assume that $f \in C[a,b]$, then F is differentiable on (a,b) and by Ostrowski's inequality* (1.1) *applied for F we get*

$$\left| F(x) - \frac{1}{b-a}\int_a^b F(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f\|_\infty$$

for all $x \in [a,b]$.

Now, using the identity (2.4) we recapture the inequality (2.1) and (2.2) for random variables whose probability density functions are continuous on $[a,b]$.

3 Application for a Beta Random Variable

A Beta random variable X, with parameters (p,q) has the probability density function

$$f(x:p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}; \qquad 0 < x < 1$$

where

$$\Omega = \{(p,q) : p,q > 0\}$$

and

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt.$$

We observe that for $0 < p < 1$,

$$\|f(\cdot,p,q)\|_\infty = \sup_{x\in(0,1)} \left[\frac{x^{p-1}(1-x)^{q-1}}{B(p,q)} \right] = \infty.$$

Assume that $p,q \ge 1$. Then

$$\frac{df(x,p,q)}{dx} = \frac{1}{B(p,q)}\left[(p-1)x^{p-2}(1-x)^{q-1} - (q-1)x^{p-1}(1-x)^{q-2} \right]$$

$$= \frac{x^{p-2}(1-x)^{q-2}}{B(p,q)}\left[(p-1)(1-x) - (q-1)x \right]$$

$$= \frac{x^{p-2}(1-x)^{q-2}}{B(p,q)}\left[-(p+q-2)x + (p-1) \right].$$

We observe that

$$\frac{df(x,p,q)}{dx} = 0$$

iff $x_0 = \frac{p-1}{p+q-2} \in (0,1)$ (for $p,q>1$) and then $\frac{df(x,p,q)}{dx} > 0$ on $(0,x_0)$ and $\frac{df(x,p,q)}{dx} < 0$ on $(x_0,1)$. Consequently,

$$\|f(\cdot,p,q)\|_\infty = f(x_0;p,q) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

On the other hand we have

$$E(X) = \frac{1}{B(p,q)} \int_0^1 x \cdot x^{p-1}(1-x)^{q-1}\,dx = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}.$$

Now, using Theorem 2.1 we can state the following proposition:

Proposition 3.1. *Let X be a Beta random variable with the parameters (p,q), $p,q \geq 1$. Then we have the inequality :*

$$\begin{aligned}&\left|\Pr(X \leq x) - \frac{q}{p+q}\right| \\ &\leq \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right] \times \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}\end{aligned}$$

and

$$\begin{aligned}&\left|\Pr(X \geq x) - \frac{p}{p+q}\right| \\ &\leq \left[\frac{1}{4} + \left(x - \frac{1}{2}\right)^2\right] \times \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}\end{aligned}$$

where $x \in [0,1]$. Particularly, we have

$$\left|\Pr\left(X \leq \frac{1}{2}\right) - \frac{q}{p+q}\right| \leq \frac{1}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}$$

and

$$\left|\Pr\left(X \geq \frac{1}{2}\right) - \frac{p}{p+q}\right| \leq \frac{1}{4} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}}.$$

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MCMC Melbourne, Victoria, 8001, AUSTRALIA.

E-mail address: {neil,sever}@matilda.vut.edu.au