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AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

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Abstract. An inequality of Ostrowski type for twice differentiable mappings whose derivatives are bounded and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

1 Introduction

In 1938, Ostrowski (see for example [2, p. 468]) proved the following integral inequality:

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^{\circ}$ with $a < b$. If $f' : (a, b) \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f'\|_{\infty} := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty} \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For some applications of Ostrowski's inequality to some special means and numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for n-time differentiable mappings (see for example [2, p.468]) from which we would like to mention only the case of twice differentiable mappings [2, p. 470].

Theorem 1.2. *Let $f : [a, b] \to \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f''\|_{\infty} = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\left| \frac{1}{2} \left[f(x) + \frac{(x-a) f(a) + (b-x) f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{1.2}$$

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$$\leq \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

for all $x\in[a,b]$.

In this paper we point out an inequality of Ostrowski's type which is similar, in a sense, to Milovanović-Pečarić result and apply it for Special Means and in Numerical Integration.

2 Some Integral Inequalities

The following result holds.

Theorem 2.1. *Let* $f:[a,b]\to\mathbf{R}$ *be a twice differentiable mapping on* (a,b) *and* $f'':(a,b)\to\mathbf{R}$ *is bounded, i.e.,* $\|f''\|_\infty=\sup_{t\in(a,b)}|f''(t)|<\infty$. *Then we have the inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.1}$$

$$\leq\left[\frac{1}{24}(b-a)^2+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_\infty\leq\frac{(b-a)^2}{6}\|f''\|_\infty$$

for all $x\in[a,b]$.

Proof. Let us define the mapping $K(\cdot,\cdot):[a,b]^2\to\mathbf{R}$ given by

$$K(x,t):=\begin{cases}\dfrac{(t-a)^2}{2} & \text{if } t\in[a,x]\\[2ex] \dfrac{(t-b)^2}{2} & \text{if } t\in(x,b]\end{cases}.$$

Integrating by parts, we have successively

$$\int_a^b K(x,t)f''(t)\,dt=\int_a^x\frac{(t-a)^2}{2}f''(t)\,dt+\int_x^b\frac{(t-b)^2}{2}f''(t)\,dt$$

$$=\left.\frac{(t-a)^2}{2}f'(t)\right|_a^x-\int_a^x(t-a)f'(t)\,dt+\left.\frac{(t-b)^2}{2}f'(t)\right|_x^b-\int_x^b(t-b)f'(t)\,dt$$

$$=\frac{(x-a)^2}{2}f'(x)-\left[(t-a)f(t)|_a^x-\int_a^x f(t)\,dt\right]$$

$$-\frac{(b-x)^2}{2}f'(x)-\left[(t-b)f(t)|_x^b-\int_x^b f(t)\,dt\right]$$

$$=\frac{1}{2}\left[(x-a)^2-(b-x)^2\right]f'(x)$$

$$-(x-a)f(x)+\int_a^x f(t)\,dt+(x-b)f(x)+\int_x^b f(t)\,dt$$

$$=(b-a)\left(x-\frac{a+b}{2}\right)f'(x)-(b-a)f(x)+\int_a^b f(t)\,dt$$

from which we get the integral identity:

(2.2)
$$\int_a^b f(t)\,dt=(b-a)f(x)-(b-a)\left(x-\frac{a+b}{2}\right)f'(x)+\int_a^b K(x,t)f''(t)\,dt$$

for all $x\in[a,b]$.

Using the identity (2.2), we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{2.3}$$

$$=\frac{1}{b-a}\left|\int_a^b K(x,t)f''(t)\,dt\right|\le\frac{1}{b-a}\|f''\|_\infty\int_a^b|K(x,t)|\,dt$$

$$=\frac{1}{b-a}\|f''\|_\infty\left[\int_a^x\frac{(t-a)^2}{2}dt+\int_x^b\frac{(t-b)^2}{2}dt\right]$$

$$=\frac{1}{b-a}\|f''\|_\infty\left[\left.\frac{(t-a)^3}{6}\right|_a^x+\left.\frac{(t-b)^3}{6}\right|_x^b\right]$$

$$=\frac{1}{b-a}\|f''\|_\infty\left[\frac{(x-a)^3+(b-x)^3}{6}\right].$$

Now, observe that

$$(x-a)^3+(b-x)^3=(b-a)\left[(x-a)^2+(b-x)^2-(x-a)(b-x)\right]$$

$$= (b-a)[(x-a+b-x)^2 - 3(x-a)(b-x)]$$

$$= (b-a)[(b-a)^2 + 3[x^2 - (a+b)x + ab]]$$

$$= (b-a)\left[(b-a)^2 + 3\left[\left(x - \frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2\right]\right]$$

$$= (b-a)\left[\left(\frac{b-a}{2}\right)^2 + 3\left(x - \frac{a+b}{2}\right)^2\right].$$

Using (2.3), we get the desired inequality (2.1). ∎

Corollary 2.2. *Under the above assumptions, we have the mid-point inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{(b-a)^2}{24}\left\|f''\right\|_\infty .$$

This follows by Theorem 2.1, choosing $x = \frac{a+b}{2}$.

Corollary 2.3. *Under the above assumptions we have the following trapezoid like inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{b-a}{4}(f'(b) - f'(a)) \right|$$

$$\le \frac{(b-a)^2}{6}\left\|f''\right\|_\infty .$$

This follows using Theorem 2.1 with $x = a, \quad x = b$, adding the results and using the triangle inequality for the modulus.

3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}] \quad (i = 0, ..., n-1)$. We have the following quadrature formula:

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping on (a,b) whose second derivative $f'' : (a,b) \to \mathbf{R}$ is bounded, i.e., $\left\|f''\right\|_\infty < \infty$. Then we have the following :*

$$\int_a^b f(x)\,dx = A(f, f', \boldsymbol{\xi}, I_n) + R(f, f', \boldsymbol{\xi}, I_n) \tag{3.1}$$

where

$$A(f,f',\boldsymbol{\xi},I_n)=\sum_{i=0}^{n-1}h_i f(\xi_i)-\sum_{i=0}^{n-1}f'(\xi_i)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)h_i$$

and the remainder satisfies the estimation:

$$|R(f,f',\boldsymbol{\xi},I_n)|\le\left[\frac{1}{24}\sum_{i=0}^{n-1}h_i^3+\frac{1}{2}\sum_{i=0}^{n-1}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\|f''\|_\infty \tag{3.2}$$

$$\le\frac{\|f''\|}{6}\sum_{i=0}^{n-1}h_i^3.$$

for all ξ_i *as above, where* $h_i:=x_{i+1}-x_i \quad (i=0,...,n-1)$.

Proof. Apply Theorem 2.1 on the interval $[x_i,x_{i+1}] \quad (i=0,...,n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}}f(t)\,dt-h_i f(\xi_i)+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)h_i f'(\xi_i)\right|$$

$$\le\left[\frac{1}{24}h_i^3+\frac{1}{2}h_i\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\|f''\|_\infty\le\frac{\|f''\|_\infty}{6}h_i^3.$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce the desired estimation. ∎

Remark 3.1. *Choosing* $\xi_i=\frac{x_i+x_{i+1}}{2}$, *we recapture the midpoint quadrature formula*

$$\int_a^b f(x)\,dx=A_M(f,I_n)+R_M(f,I_n)$$

where the remainder $R_M(f,I_n)$ *satisfies the estimation*

$$|R_M(f,I_n)|\le\frac{\|f''\|_\infty}{24}\sum_{i=0}^{n-1}h_i^3.$$

4 Applications for Special Means

Let us recall the following means :

a) The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2},\qquad a,b\ge 0;$$

b) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \qquad a,b \geq 0;$$

c) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \qquad a,b \geq 0;$$

d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a \quad \text{if } a = b, \qquad a,b > 0; \\ \dfrac{b-a}{\ln b - \ln a} \quad \text{if } a \neq b \end{cases}$$

e) The identric mean:

$$I = I(a,b) := \begin{cases} a \quad \text{if } a = b, \qquad a,b > 0; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} \quad \text{if } a \neq b \end{cases}$$

f) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} \quad \text{if } a \neq b; \\ a \quad \text{if } a = b \end{cases}$$

where $p \in \mathbf{R} \backslash \{-1, 0\}, \quad a,b > 0.$

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

(1) Consider the mapping $f : (0, \infty) \to \mathbf{R}, \quad f(x) = x^r, \quad r \in \mathbf{R} \backslash \{-1, 0\}.$ Then, we have, for $0 < a < b$:

$$\frac{1}{b-a} \int_a^b f(x)\,dx = L_r^r(a,b)$$

and

$$\|f''\|_\infty = |r(r-1)|\,\delta_r(a,b), \qquad r \in \mathbf{R} \backslash \{-1, 0\};$$

where

$$\delta_r(a,b) := \begin{cases} b^{r-1} & \text{if } r \in (1,\infty) \\ a^{r-1} & \text{if } r \in (-\infty,1) \setminus \{-1,0\} \end{cases}.$$

Using the inequality (2.1) we have the result:

$$\left|x^r - L_r^r(a,b) - r(x-A)x^{r-1}\right| \tag{4.1}$$

$$\leq \frac{|r(r-1)|}{6}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right]\delta_r(a,b)$$

$$\leq \frac{|r(r-1)|(b-a)^2}{6}\delta_r(a,b)$$

for all $x \in [a,b]$. If in (4.1) we choose $x = A$, we get

$$|A^r - L_r^r| \leq \frac{|r(r-1)|(b-a)^2}{24}\delta_r(a,b). \tag{4.2}$$

(2) Consider the mapping $f(x) = \frac{1}{x}$, $x \in [a,b] \subset (0,\infty)$. Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_{-1}^{-1}(a,b)$$

and

$$\|f''\|_\infty = \frac{2}{a^3}.$$

Applying the inequality (2.1) for the above mapping, we get

$$\left|\frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2}\right| \leq \frac{1}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right]$$

$$\leq \frac{(b-a)^2}{3a^3}$$

which is equivalent to

$$|x(L-x) - L(A-x)| \leq \frac{x^2L}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right] \tag{4.3}$$

$$\leq \frac{x^2L(b-a)^2}{3a^3}$$

for all $x \in [a,b]$. Now, if we choose in (4.3), $x = A$, we get

$$0 \leq A - L \leq \frac{(b-a)^2 AL}{12a^3}. \tag{4.4}$$

If in (4.3) we choose $x = L$, we get

$$0 \leq A - L \leq \frac{L^2}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(L-A)^2\right]. \tag{4.5}$$

(3) Let us consider the mapping

$$f(x) = \ln x, \qquad x \in [a,b] \subset (0,\infty).$$

Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \ln I(a,b),$$

and

$$\|f''\|_\infty = \frac{1}{a^2}.$$

Inequality (2.1) gives us

$$\left|\ln x - \ln I - \frac{x-A}{x}\right| \tag{4.6}$$

$$\le \frac{1}{6a^2}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right] \le \frac{(b-a)^2}{6a^2}.$$

Now, if in (4.6) we choose $x = A$, we get

$$1 \le \frac{A}{I} \le \exp\left[\frac{1}{24a^2}(b-a)^2\right]. \tag{4.7}$$

If in (4.6) we choose $x = I$, we get

$$0 \le A - I \le \frac{I}{6a^2}\left[\frac{1}{4}(b-a)^2 + 3(A-I)^2\right]. \tag{4.8}$$

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