SOME OSTROWSKI TYPE INEQUALITIES FOR N-TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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Abstract. Some generalizations of the Ostrowski inequality for n-time differentiable mappings are given. Applications in Numerical Integration and for power series expansions are also presented.

1 Introduction

In 1938, Ostrowski (see for example [3, p.468]) proved the following integral inequality:

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°(I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić (see for example [3, p. 468]), proved the following generalization of Ostrowski's result:

Let $f : [a,b] \to R$ be an n-times differentiable function, $n \ge 1$, and such that $\left\|f^{(n)}\right\|_\infty := \sup_{t\in(a,b)} \left|f^{(n)}(t)\right| < \infty$. Then

$$\left| \frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

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$$\leq \frac{\left\|f^{(n)}\right\|_{\infty}}{n(n+1)!} \cdot \frac{(x-a)^{n+1}+(b-x)^{n+1}}{b-a}$$

for all $x \in [a,b]$.

In [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following Ostrowski type inequality for twice differentiable mappings:

Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping on (a,b) and $f'' : (a,b) \to \mathbf{R}$ is bounded, i.e., $\|f''\|_{\infty} = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right) f'(x) \right|$$

$$\leq \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_{\infty} \leq \frac{(b-a)^2}{6}\|f''\|_{\infty}$$

for all $x \in [a,b]$.

In this paper we establish another generalization of Ostrowski inequality for n-time differentiable mappings which naturally generalizes the result from [1] and apply it in Numerical Integration and for power series expansions of functions on an interval.

2 Integral identities

The following lemma holds:

Lemma 2.1. *Let $f : [a,b] \to \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$ we have the identity:*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) \tag{2.1}$$

$$+ (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt$$

where the kernel $K_n : [a,b]^2 \to \mathbf{R}$ is given by

$$K_n(x,t) := \begin{cases} \dfrac{(t-a)^n}{n!} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^n}{n!} & \text{if } t \in (x,b] \end{cases}, x \in [a,b] \tag{2.2}$$

and n is a natural number, $n \geq 1$.

Proof. The proof is by mathematical induction.

For $n = 1$, we have to prove the equality

$$\int_a^b f(t)\,dt = (b-a) f(x) - \int_a^b K_1(x,t) f^{(1)}(t)\,dt \tag{2.3}$$

where

$$K_1(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases}.$$

Integrating by parts, we have:

$$\int_a^b K_1(x,t) f^{(1)}(t)\,dt = \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt$$

$$= (t-a) f(t)\big|_a^x - \int_a^x f(t)\,dt + (t-b) f(t)\big|_x^b - \int_x^b f(t)\,dt$$

$$= (x-a) f(x) + (b-x) f(x) - \int_a^b f(t)\,dt = (b-a) f(x) - \int_a^b f(t)\,dt$$

and the identity (2.3) is proved.

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \tag{2.4}$$

$$+ (-1)^{n+1} \int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt$$

where, obviously,

$$K_{n+1}(t) := \begin{cases} \dfrac{(t-a)^{n+1}}{(n+1)!} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^{n+1}}{(n+1)!} & \text{if } t \in (x,b] \end{cases}.$$

We have

$$\int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt = \int_a^x \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t)\,dt + \int_x^b \frac{(t-b)^{n+1}}{(n+1)!} f^{(n+1)}(t)\,dt$$

and integrating by parts gives

$$\int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt = \left.\frac{(t-a)^{n+1}}{(n+1)!} f^{(n)}(t)\right|_a^x - \frac{1}{n!}\int_a^x (t-a)^n f^{(n)}(t)\,dt$$

$$+\left.\frac{(t-b)^{n+1}}{(n+1)!} f^{(n)}(t)\right|_x^b - \frac{1}{n!}\int_x^b (t-b)^n f^{(n)}(t)\,dt$$

$$= \frac{(x-a)^{n+1} + (-1)^{n+2}(b-x)^{n+1}}{(n+1)!} f^{(n)}(x) - \int_a^b K_n(x,t) f^{(n)}(t)\,dt.$$

That is

$$\int_a^b K_n(x,t) f^{(n)}(t)\,dt = \frac{(x-a)^{n+1} + (-1)^{n+2}(b-x)^{n+1}}{(n+1)!} f^{(n)}(x)$$

$$-\int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt.$$

Now, using the mathematical induction hypothesis, we get

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)$$

$$+\frac{(b-x)^{n+1} + (-1)^n (x-a)^{n+1}}{(n+1)!} f^{(n)}(x) - (-1)^n \int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt$$

$$= \sum_{k=0}^{n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)$$

$$+(-1)^{n+1}\int_a^b K_{n+1}(x,t) f^{(n+1)}(t)\,dt.$$

That is, the identity (2.4); and the theorem is thus proved. ∎

Corollary 2.2. *With the above assumptions, we have the representation*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\frac{(b-a)^{k+1}}{2^{k+1}}f^{(k)}\left(\frac{a+b}{2}\right) \tag{2.5}$$

$$+(-1)^n\int_a^b M_n(t)\,f^{(n)}(t)\,dt$$

where

$$M_n(t) := \begin{cases} \dfrac{(t-a)^n}{n!} & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ \dfrac{(t-b)^n}{n!} & \text{if } t\in\left(\frac{a+b}{2},b\right] \end{cases}.$$

The proof follows by Lemma 2.1 by choosing $x=\frac{a+b}{2}$.

Corollary 2.3. *With the above assumptions, we have the representation:*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right] \tag{2.6}$$

$$+\int_a^b T_n(t)\,f^{(n)}(t)\,dt$$

where

$$T_n(t) := \frac{1}{n!}\left[\frac{(b-t)^n+(-1)^n(t-a)^n}{2}\right], \qquad t\in[a,b].$$

Proof. Firstly, choose in (2.1) $x=a$ to get

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)+(-1)^n\int_a^b\frac{(t-b)^n}{n!}f^{(n)}(t)\,dt$$

$$=\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(a)+\int_a^b\frac{(b-t)^n}{n!}f^{(n)}(t)\,dt.$$

Also, if we put $x=b$ in (2.1), we get

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}(-1)^k\frac{(b-a)^{k+1}}{(k+1)!}f^{(k)}(b)+(-1)^n\int_a^b\frac{(t-a)^n}{n!}f^{(n)}(t)\,dt.$$

Summing the above two identities and dividing by 2, we get

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n} \frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]$$

$$+\int_a^b T_n(t)\, f^{(n)}(t)\,dt$$

and the corollary is proved. ∎

The following Taylor-like formula with integral remainder also holds:

Corollary 2.4. *Let $g:[a,y]\to\mathbf{R}$ be a mapping such that $g^{(n)}$ is absolutely continuous on $[a,y]$. Then for all $x\in[a,y]$, we have the identity*

$$g(y) = g(a) + \sum_{k=0}^{n-1} \frac{\left[(y-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]}{(k+1)!} g^{(k+1)}(x) \tag{2.7}$$

$$+(-1)^n \int_a^y K_n(y,t)\, g^{(n+1)}(t)\,dt.$$

The proof is obvious by Lemma 2.1 choosing $f=g'$, and $b=y$.

Remark 2.1. *If we choose $n=1$ in (2.1), we get the identity*

$$\int_a^b f(t)\,dt = (b-a) f(x) - \int_a^b K_1(x,t)\, f'(t)\,dt \tag{2.8}$$

for all $x\in[a,b]$, where

$$K_1(x,t) := \begin{cases} t-a & \textit{if } t\in[a,x] \\ t-b & \textit{if } t\in(x,b] \end{cases}$$

which is the identity employed by S.S. Dragomir and S. Wang to prove an Ostrowski type inequality in paper [2].

If in (2.5) we choose $n=1$, then we get

$$\int_a^b f(t)\,dt = (b-a) f\left(\frac{a+b}{2}\right) - \int_a^b M_1(t)\, f'(t)\,dt \tag{2.9}$$

where

$$M_1(t) = \begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ t-b & \text{if } t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

which gives the mid-point type inequality useful in Numerical Analysis.

Also, if we put $n = 1$ in (2.6), we get the trapezoid identity

$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) + \int_a^b T_1(t) f'(t)\,dt \tag{2.10}$$

where

$$T_1(t) = \frac{a+b}{2} - t, \qquad t \in [a,b].$$

Finally, if in the Taylor-like formula (2.7) we put $n = 1$, we get

$$g(y) = g(a) + (y-a) g'(x) - \int_a^y K_1(y,t) g^{(2)}(t)\,dt$$

where $x \in [a,y]$.

Remark 2.2. *If we choose $n = 2$ in (2.1), we get the identity:*

$$\int_a^b f(t)\,dt = (b-a) f(x) - \left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b K_2(x,t) f''(t)\,dt \tag{2.11}$$

where

$$K_2(x,t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in (x,b] \end{cases}$$

and $x \in [a,b]$, which is the identity employed by P. Cerone, S.S. Dragomir and J. Roumeliotis to prove some Ostrowski type inequalities for twice differentiable mappings in the paper [1].

If in (2.5) we choose $n = 2$, then we get

$$\int_a^b f(t)\,dt = (b-a) f\left(\frac{a+b}{2}\right) + \int_a^b M_2(t) f''(t)\,dt \tag{2.12}$$

where

$$M_2(t) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in \left[a, \frac{a+b}{2}\right] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

which is the classical mid-point identity.

Also, if we put $n = 2$ in (2.6), we get the identity

$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{2} \cdot \frac{f'(a)-f'(b)}{2} + \int_a^b T_2(t) f''(t)\,dt \tag{2.13}$$

where

$$T_2(t) = \frac{1}{2} \cdot \frac{(b-t)^2 + (t-a)^2}{2}, \qquad t \in [a,b].$$

Finally, if we put $n = 2$ in (2.7), we get

$$g(y) = g(a) + (y-a) g'(x) - (y-a)\left(x - \frac{a+y}{2}\right) g''(x) + \int_a^y K_2(y,t) g^{(3)}(t)\,dt, \tag{2.14}$$

where K_2 is as above and $a \le x \le y$.

3 Some integral inequalities for $\|\cdot\|_\infty$ −norm

The following theorem holds:

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_\infty[a,b]$. Then for all $x \in [a,b]$, we have the inequality:*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \tag{3.1}$$

$$\le \frac{\|f^{(n)}\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right] \le \frac{\|f^{(n)}\|_\infty (b-a)^{n+1}}{(n+1)!}$$

where

$$\left\| f^{(n)} \right\|_\infty := \sup_{t \in [a,b]} \left| f^{(n)}(t) \right| < \infty.$$

Proof. Using the identity (2.1), we have:

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right|$$

$$= \left| \int_a^b K_n(x,t) f^{(n)}(t)\, dt \right| \leq \left\| f^{(n)} \right\|_\infty \int_a^b |K_n(x,t)|\, dt$$

$$= \left\| f^{(n)} \right\|_\infty \left[\int_a^x \frac{(t-a)^n}{n!} dt + \int_x^b \frac{(b-t)^n}{n!} dt \right]$$

$$= \frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right],$$

and the first part of inequality (3.1) is proved.

To prove the second inequality in (3.1), we observe that

$$(x-a)^{n+1} + (b-x)^{n+1} \leq (b-a)^{n+1}, \tag{3.2}$$

for all $x \in [a,b]$. ∎

Taking into account the fact that the mapping $h_n : [a,b] \to \mathbf{R}$, $h_n(x) := (x-a)^{n+1} + (b-x)^{n+1}$, has the property

$$\inf_{x\in[a,b]} h_n(x) = h_n\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n},$$

then the best inequality we can get from (3.1) is the one for which $x = \frac{a+b}{2}$.

In this way, we can state the following corollary.

Corollary 3.2. *Assume that f is as in Theorem 3.1. Then we have the inequality:*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \left[\frac{1+(-1)^k}{(k+1)!} \right] \frac{(b-a)^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{a+b}{2}\right) \right| \tag{3.3}$$

$$\leq \frac{\left\| f^{(n)} \right\|_\infty (b-a)^{n+1}}{2^n (n+1)!}.$$

Another result which generalizes the trapezoid inequality, is the following one:

Corollary 3.3. *With the above assumptions, we have the inequality:*

$$\left| \int_a^b f(t)\, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \tag{3.4}$$

$$\leq \frac{(b-a)^{n+1}}{(n+1)!} \left\| f^{(n)} \right\|_\infty \times \begin{cases} 1 & \text{if } n = 2r \\ \dfrac{2^{2r+1}-1}{2^{2r}} & \text{if } n = 2r+1 \end{cases}.$$

Proof. Using the identity (2.6), we get

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(b-a)^{k+1}}{(k+1)!}\left[\frac{f^{(k)}(a)+(-1)^k f^{(k)}(b)}{2}\right]\right|$$

$$=\left|\int_a^b T_n(t)\,f^{(n)}(t)\,dt\right|\le\left\|f^{(n)}\right\|_\infty\int_a^b|T_n(t)|\,dt.$$

If $n=2r$, then

$$\int_a^b|T_n(t)|\,dt=\frac{1}{(2r)!}\int_a^b\left[\frac{(b-t)^{2r}+(t-a)^{2r}}{2}\right]dt \tag{3.5}$$

$$=\frac{1}{(2r)!}\cdot\frac{1}{2}\left[\frac{(b-a)^{2r+1}}{(2r+1)}+\frac{(b-a)^{2r+1}}{(2r+1)}\right]=\frac{(b-a)^{2r+1}}{(2r+1)!}.$$

For $n=2r+1$, put $h_{2r+1}(t):=(b-t)^{2r+1}-(t-a)^{2r+1}$, $t\in[a,b]$. Observe that $h_{2r+1}(t)=0$ iff $t=\frac{a+b}{2}$ and $h_{2r+1}(t)>0$ if $t\in[a,\frac{a+b}{2})$ and $h_{2r+1}(t)<0$ if $t\in(\frac{a+b}{2},b]$.

Then

$$\int_a^b|T_{2r+1}(t)|\,dt$$

$$=\int_a^{\frac{a+b}{2}}\left[(b-t)^{2r+1}-(t-a)^{2r+1}\right]dt+\int_{\frac{a+b}{2}}^b\left[(t-a)^{2r+1}-(b-t)^{2r+1}\right]dt$$

$$=2\frac{(b-a)^{2r+2}}{2r+2}-\frac{4\left(\frac{b-a}{2}\right)^{2r+2}}{2r+2}$$

$$=\frac{1}{2r+2}\left[2(b-a)^{2r+2}-\frac{(b-a)^{2r+2}}{2^{2r}}\right]=\frac{(b-a)^{2r+2}}{2r+2}\left(2-\frac{1}{2^{2r}}\right)$$

$$=\frac{(b-a)^{2r+2}}{2r+2}\cdot\frac{2^{2r+1}-1}{2^{2r}}.$$

Using (3.4) we get the desired inequality (3.3). ∎

The following inequality in terms of $\|\cdot\|_\infty$ − norm for the Taylor like expansion (2.7) also holds:

Corollary 3.4. *Let g be as in Corollary 2.4. Then we have the inequality:*

$$\left|g(y)-g(a)-\sum_{k=0}^{n-1}\frac{\left[(y-x)^{k+1}+(-1)^k(x-a)^{k+1}\right]}{(k+1)!}g^{(k+1)}(x)\right| \tag{3.6}$$

$$\leq \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!}\left[(y-x)^{n+1}+(x-a)^{n+1}\right] \leq \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!}(y-a)^{n+1}$$

for all $a \leq x \leq y$.

Remark 3.1. *It is well known that for the classical Taylor expansion around a we have the inequality*

$$\left|g(y)-\sum_{k=0}^{n}\frac{(y-a)^k}{k!}g^{(k)}(a)\right| \leq \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!}(y-a)^{n+1} \tag{3.7}$$

for all $y \geq a$. It is clear now that the above approximation (3.6) *around the arbitrary point $x \in [a,y]$ provides a better approximation for the mapping g at the point y than the classical Taylor expansion around the point a.*

If in (3.6) we choose $x = \frac{a+y}{2}$, then we get

$$\left|g(y)-g(a)-\sum_{k=1}^{n}\frac{\left[1+(-1)^{k-1}\right]}{k!}\frac{(y-a)^k}{2^k}g^{(k)}\left(\frac{a+y}{2}\right)\right| \tag{3.8}$$

$$\leq \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!2^n}(y-a)^{n+1}.$$

The above inequality (3.8) shows that for $g \in C^\infty[a,b]$ the series

$$g(a)+\sum_{k=0}^{\infty}\frac{\left[1+(-1)^k\right]}{(k+1)!}\frac{(y-a)^{k+1}}{2^{k+1}}g^{(k+1)}\left(\frac{a+y}{2}\right)$$

converges more rapidly to $g(y)$ than the usual one

$$\sum_{k=0}^{\infty}\frac{(y-a)^k g^{(k)}(a)}{k!}$$

which comes from Taylor's expansion. Further, it should be noted that (3.8) only involves odd derivatives of g.

Remark 3.2. *If in the inequality* (3.1) *we choose $n = 1$ we get*

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)\right| \leq \frac{(x-a)^2+(b-x)^2}{2}\left\|f'\right\|_\infty.$$

As a simple calculation shows that

$$\frac{1}{2}\left[(x-a)^2+(b-x)^2\right]=\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2$$

consequently we obtain the Ostrowski inequality:

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^2\|f'\|_\infty \tag{3.9}$$

for all $x\in[a,b]$.

If in (3.2) we put $n=1$, we get the mid-point inequality

$$\left|\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a)\right|\leq\frac{1}{4}(b-a)^2\|f'\|_\infty . \tag{3.10}$$

From the inequality (3.4), for $n=1$, we get the trapezoid inequality

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\leq\frac{1}{2}(b-a)^2\|f'\|_\infty . \tag{3.11}$$

Also, from (3.6) we deduce

$$|g(y)-g(a)-(y-a)g'(x)|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+y}{2}\right)^2}{(y-a)^2}\right]\|g''\|_\infty \tag{3.12}$$

for all $a\leq x\leq y$.

Remark 3.3. *If in the inequality* (3.1) *we choose* $n=2$, *then we get*

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)+\left(x-\frac{a+b}{2}\right)(b-a)f'(x)\right|$$

$$\leq\frac{1}{6}\left[(x-a)^3+(b-x)^3\right]\|f''\|_\infty$$

for all $x\in[a,b]$.

Now, observe that

$$\begin{aligned}(x-a)^3+(b-x)^3&=(b-a)\left[(x-a)^2+(b-x)^2-(x-a)(b-x)\right]\\&=(b-a)\left[(x-a+b-x)^2-3(x-a)(b-x)\right]\\&=(b-a)\left[(b-a)^2+3\left[x^2-(a+b)x+ab\right]\right]\\&=(b-a)\left[(b-a)^2+3\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right]\right]\\&=(b-a)\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]\end{aligned}$$

and then we recapture the result obtained in [1], namely

$$\left|\int_a^b f(t)\,dt-(b-a)f(x)+\left(x-\frac{a+b}{2}\right)(b-a)f'(x)\right| \tag{3.13}$$

$$\leq\left[\frac{1}{24}+\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)^3\|f''\|_\infty .$$

If we put $n=2$ in (3.3), we get the classical mid-point inequality

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{24}(b-a)^3\|f''\|_\infty . \tag{3.14}$$

Now, if in (3.4) we put $n=2$, then we get the inequality:

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)-\frac{(b-a)^2}{2}\cdot\frac{f'(a)-f'(b)}{2}\right| \tag{3.15}$$

$$\leq\frac{(b-a)^3}{6}\|f''\|_\infty .$$

Finally, if we put $n=2$ in (3.6), then we get the inequality:

$$\left|g(y)-g(a)-(y-a)g'(x)+(y-a)\left(x-\frac{a+y}{2}\right)g''(x)\right|$$

$$\leq\left[\frac{1}{24}+\frac{1}{2}\cdot\frac{\left(x-\frac{a+y}{2}\right)^2}{(y-a)^2}\right](y-a)^3\|g'''\|_\infty .$$

4 Applications for numerical integration

Consider the partition $I_m : a=x_0<x_1<...<x_{m-1}<x_m=b$ of the interval $[a,b]$ and the intermediate points $\xi=\left(\xi_0,...,\xi_{m-1}\right)$ where $\xi_j\in[x_j,x_{j+1}]$ $j=0,...,m-1$. Define the formula

$$F_{m,k}(f,I_m,\boldsymbol{\xi}):=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{\left[\left(x_{j+1}-\xi_j\right)^{k+1}+(-1)^k\left(\xi_j-x_j\right)^{k+1}\right]}{(k+1)!}f^{(k)}\left(\xi_j\right)$$

which can be regarded as a perturbation of Riemann's sum

$$\Gamma(f,I_m,\boldsymbol{\xi})=\sum_{j=0}^{m-1}f\left(\xi_j\right)h_j$$

where $h_j:=x_{j+1}-x_j,\quad j=0,...,m-1.$

The following theorem holds.

Theorem 4.1. *Let $f:[a,b]\to\mathbf{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and I_m a partitioning of $[a,b]$ as above. Then we have the quadrature formula*

$$\int_a^b f(x)\,dx = F_{m,k}(f,I_m,\boldsymbol{\xi}) + R_{m,k}(f,I_m,\boldsymbol{\xi}) \tag{4.1}$$

where $F_{m,k}$ is defined above and the remainder $R_{m,k}$ satisfies the estimation:

$$|R_{m,k}(f,I_m,\boldsymbol{\xi})| \le \frac{\|f^{(n)}\|_\infty}{(n+1)!}\sum_{j=0}^{m-1}\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right] \tag{4.2}$$

$$\le \frac{\|f^{(n)}\|_\infty}{(n+1)!}\sum_{j=0}^{m-1} h_j^{n+1}$$

for all $\boldsymbol{\xi}$ as above.

Proof. Apply Theorem 3.1 on the interval $[x_j, x_{j+1}]$ to get

$$\left|\int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1}\left[\frac{\left[\left(x_{j+1}-\xi_j\right)^{k+1} + (-1)^k\left(\xi_j - x_j\right)^{k+1}\right]}{(k+1)!}\right] f^{(k)}(\xi_j)\right|$$

$$\le \frac{\|f^{(n)}\|_\infty}{(n+1)!}\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right]$$

$$\le \frac{\|f^{(n)}\|_\infty}{(n+1)!} h_j^{n+1}$$

for all $j\in\{0,...,m-1\}$.

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we deduce the desired estimation (4.2). ∎

As an interesting particular case, we can consider the following perturbed mid-point formula

$$M_{m,k}(f,I_m) := \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\left[\frac{1+(-1)^k}{(k+1)!}\right]\frac{h_j^{k+1}}{2^{k+1}} f^{(k)}\left(\frac{x_j + x_{j+1}}{2}\right),$$

which in effect involves only even k.

We state the following result concerning the estimation of the remainder term.

Corollary 4.2. *Let f and I_m be as in Theorem 4.1. Then we have*

$$\int_a^b f(t)\,dt = M_{m,k}(f,I_m) + R_{m,k}(f,I_m) \tag{4.3}$$

and the remainder term $R_{m,k}$ satisfies the estimation

$$|R_{m,k}(f,I_m)| \leq \frac{\left\|f^{(n)}\right\|_\infty}{2^n (n+1)!} \sum_{j=0}^{m-1} h_j^{n+1}. \tag{4.4}$$

We can consider the following perturbed version of the trapezoid formula:

$$T_{m,k}(f,I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2}\right].$$

By the use of Corollary 3.3, we have the following approximation of the integral $\int_a^b f(t)\,dt$ in terms of $T_{m,k}(f,I_m)$:

Corollary 4.3. *Let f and I_m be as in Theorem 4.1. Then we have*

$$\int_a^b f(t)\,dt = T_{m,k}(f,I_m) + \tilde{R}_{m,k}(f,I_m) \tag{4.5}$$

and the remainder $\tilde{R}_{m,k}(f,I_m)$ satisfies the inequality

$$\left|\tilde{R}_{m,k}(f,I_m)\right| \leq \frac{C_n}{(n+1)!} \left\|f^{(n)}\right\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}$$

where

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \dfrac{2^{2r+1}-1}{2^{2r}} & \text{if } n = 2r+1 \end{cases}.$$

Remark 4.1. *a). If we choose $n = 1$ in the above quadrature formulae* (4.1) *and* (4.3)*, we recapture some results from the paper [2].*

b). If we put $n = 2$, then by the above Theorem 4.1 and Corollary 4.2, we recover some results from the paper [1].

We omit the details.

5 Applications for some particular mappings

a) Consider $g : \mathbf{R} \to \mathbf{R}$, $g(x) = e^x$. Then $g^{(n)}(x) = e^x$, $n \in \mathbf{N}$ and

$$\left\|g^{(n+1)}\right\|_\infty = \sup_{t\in[a,y]} \left|g^{(n+1)}(t)\right| = e^y.$$

Using inequality (3.6), we have

$$\left|e^y - e^a - e^x \sum_{k=0}^{n-1} \frac{\left[(y-x)^{k+1} + (-1)^k (x-a)^{k+1}\right]}{(k+1)!}\right| \tag{5.1}$$

$$\leq \frac{e^y}{(n+1)!}\left[(y-x)^{n+1}+(x-a)^{n+1}\right] \leq \frac{e^y}{(n+1)!}(y-a)^{n+1}$$

for all $a \leq x \leq y$.

Particularly, if we choose $a=0$, then we get

$$\left|e^y - 1 - e^x \sum_{k=0}^{n-1} \frac{\left[(y-x)^{k+1}+(-1)^k x^{k+1}\right]}{(k+1)!}\right| \tag{5.2}$$

$$\leq \frac{e^y}{(n+1)!}\left[(y-x)^{n+1}+x^{n+1}\right] \leq \frac{e^y}{(n+1)!} y^{n+1}.$$

Moreover, if we choose $x=\frac{y}{2}$, then we get

$$\left|e^y - 1 - e^{\frac{y}{2}} \sum_{k=0}^{n-1} \frac{1+(-1)^k}{(k+1)!} \cdot \frac{y^{k+1}}{2^{k+1}}\right| \leq \frac{e^y y^{n+1}}{2^n (n+1)!} \tag{5.3}$$

for all $y \geq 0$.

b) Consider $g:(0,\infty) \to \mathbf{R}$, $g(x)=\ln x$. Then

$$g^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n}, \qquad n \geq 1, x>0$$

and

$$\left\|g^{(n+1)}\right\|_\infty = \sup_{t\in[a,y]}\left|\frac{(-1)^n n!}{t^{n+1}}\right| = \frac{n!}{a^{n+1}}, \qquad a>0.$$

Using the inequality (3.6) we can state:

$$\left|\ln y - \ln a - \sum_{k=0}^{n-1} \frac{(y-x)^{k+1}+(-1)^k (x-a)^{k+1}}{(k+1)!} \cdot \frac{(-1)^k k!}{x^{k+1}}\right|$$

$$\leq \frac{n!}{(n+1)!a^{n+1}}\left[(y-x)^{n+1}+(x-a)^{n+1}\right] \leq \frac{n!}{(n+1)!a^{n+1}}(y-a)^{n+1}$$

which is equivalent to

$$\left|\ln\left(\frac{y}{a}\right) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{(x-a)^{k+1}+(-1)^k (y-x)^{k+1}}{x^{k+1}}\right| \tag{5.4}$$

$$\leq \frac{(y-x)^{n+1}+(x-a)^{n+1}}{(n+1)a^{n+1}} \leq \frac{1}{(n+1)a^{n+1}}(y-a)^{n+1}.$$

Now, if we choose in (5.4) $y=z+1, x=w+1, a=1, z \geq w \geq 0$, then we get

$$\left|\ln(z+1) - \sum_{k=0}^{n-1} \frac{1}{k+1} \cdot \frac{w^{k+1}+(-1)^k (z-w)^{k+1}}{(w+1)^{k+1}}\right| \tag{5.5}$$

$$\le \frac{(z-w)^{n+1}+w^{n+1}}{n+1} \le \frac{1}{(n+1)} z^{n+1}.$$

Finally, if we choose in (5.4), $y=ua, x=wa$ with $u \ge w > 1,$ then we have

$$\left| \ln u - \sum_{k=0}^{n-1} \frac{1}{k+1} \frac{(w-1)^{k+1}+(-1)^k (u-w)^{k+1}}{w^{k+1}} \right|$$

$$\le \frac{(u-w)^{n+1}+(w-1)^{m+1}}{n+1} \le \frac{(u-1)^{n+1}}{(n+1)}.$$

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