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GRÜSS INEQUALITY IN INNER PRODUCT SPACES

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Dedicated to the memory of my grandfather Teodor Radu.

Abstract. A generalization of Grüss integral inequality in inner product spaces is given.

1 Introduction

In 1935, G. Grüss proved the following integral inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right|$$

$$\leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$\phi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma \quad \text{for all } x \in [a,b].$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x) = g(x) = sgn\left(x - \frac{a+b}{2}\right).$$

For other similar results, generalizations for positive linear functionals, discrete versions, determinantal versions etc. see the Chapter X of the book [1] by Mitrinović, Pečarić and Fink where further references are given.

In this paper we point out a version of Grüss' inequality in inner product spaces.

2 The Results

The following theorem holds

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Theorem 2.1. *Let $(X;(.,.))$ be a real inner product space and $e \in X, \|e\| = 1$. If $\phi, \gamma, \Phi, \Gamma$ are real numbers and x, y are vectors in X so that the condition*

$$(\Phi e - x, x - \phi e) \geq 0 \quad \text{and} \quad (\Gamma e - y, y - \gamma e) \geq 0 \tag{2.1}$$

holds, then we have the inequality

$$|(x,y) - (x,e)(e,y)| \leq \frac{1}{4} |\Phi - \phi| \cdot |\Gamma - \gamma| . \tag{2.2}$$

The constant $\frac{1}{4}$ is the best possible.

Proof. Firstly, let observe that

$$(x,y) - (x,e)(e,y) = (x - (x,e)e, y - (y,e)e).$$

Using Schwarz's inequality in inner product spaces, we have

$$\begin{aligned} & |(x - (x,e)e, y - (y,e)e)|^2 \\ & \leq \|x - (x,e)e\|^2 \cdot \|y - (y,e)e\|^2 \\ & = \left(\|x\|^2 - |(x,e)|^2\right)\left(\|y\|^2 - |(y,e)|^2\right). \end{aligned} \tag{2.3}$$

On the other hand, a simple computation shows that

$$\begin{aligned} & (\Phi - (x,e)) \cdot ((x,e) - \phi) - (\Phi e - x, x - \phi e) \\ & = \|x\|^2 - |(x,e)|^2 \end{aligned}$$

and

$$\begin{aligned} & (\Gamma - (y,e)) \cdot ((y,e) - \gamma) - (\Gamma e - y, y - \gamma e) \\ & = \|y\|^2 - |(y,e)|^2 . \end{aligned}$$

From the condition (2.1) we deduce now

$$\|x\|^2 - |(x,e)|^2 \leq (\Phi - (x,e)) \cdot ((x,e) - \phi) \tag{2.4}$$

and

$$\|y\|^2 - |(y,e)|^2 \leq (\Gamma - (y,e)) \cdot ((y,e) - \gamma) . \tag{2.5}$$

Using the elementary inequality $4ab \leq (a+b)^2$ holding for each real numbers a, b; for $a := \Phi - (x,e)$ and $b := (x,e) - \phi$, we get

$$(\Phi - (x,e)) \cdot ((x,e) - \phi) \leq \frac{1}{4} (\Phi - \phi)^2 \tag{2.6}$$

and, similarly,

$$(\Gamma - (y,e)) \cdot ((y,e) - \gamma) \leq \frac{1}{4} (\Gamma - \gamma)^2 . \tag{2.7}$$

Consequently, using the inequalities (2.3) – (2.7), we have successively

$$|(x,y) - (x,e)(e,y)|^2 \leq \left(\|x\|^2 - |(x,e)|^2 \right) \left(\|y\|^2 - |(y,e)|^2 \right)$$

$$\leq (\Phi - (x,e)) \cdot ((x,e) - \phi) \cdot (\Gamma - (y,e)) \cdot ((y,e) - \gamma)$$

$$\leq \frac{1}{16} (\Phi - \phi)^2 (\Gamma - \gamma)^2$$

from where we get the desired inequality (2.2).

To prove that the constant $\frac{1}{4}$ is sharp, let $e, m \in X$ with $\|e\| = \|m\| = 1, e \perp m$ and assume that $\phi, \gamma, \Phi, \Gamma$ are real numbers. Define the vectors

$$x := \frac{\phi + \Phi}{2} e + \frac{\Phi - \phi}{2} m$$

and

$$y := \frac{\Gamma + \gamma}{2} e + \frac{\Gamma - \gamma}{2} m.$$

Then

$$(\Phi e - x, x - \phi e) = \left(\frac{\Phi - \phi}{2} \right)^2 (e - m, e + m) = 0$$

and, similarly,

$$(\Gamma e - y, y - \gamma e) = 0,$$

i.e., the condition (2.1) holds.

Now, let observe that

$$(x,y) = \left(\frac{\phi + \Phi}{2} \right) \cdot \left(\frac{\Gamma + \gamma}{2} \right) + \left(\frac{\Phi - \phi}{2} \right) \cdot \left(\frac{\Gamma - \gamma}{2} \right)$$

and

$$(x,e)(e,y) = \left(\frac{\phi + \Phi}{2} \right) \cdot \left(\frac{\Gamma + \gamma}{2} \right).$$

Consequently,

$$|(x,y) - (x,e)(e,y)| = \frac{1}{4} |\Phi - \phi| \cdot |\Gamma - \gamma|$$

which shows that the constant $\frac{1}{4}$ is sharp. ∎

3 Some Applications

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbf{R}\cup\{\infty\}$. Denote $L^2(\Omega)$ the Hilbert space of all real valued functions x defined on Ω and 2-integrable on Ω , i.e., $\int_\Omega |x(s)|^2 d\mu(s) < \infty$.

Proposition 3.1. *Let $f, g \in L^2(\Omega)$, $m, M, n, N \in \mathbf{R}$ and $e \in L^2(\Omega)$ is so that $\int_\Omega |e(s)|^2 d\mu(s) = 1$. If the following condition holds*

$$me \leq f \leq Me, \qquad ne \leq g \leq Ne \qquad a.e. \text{ on } \Omega,$$

then we have the Grüss type inequality

$$\left| \int_\Omega f(s)g(s)d\mu(s) - \int_\Omega f(s)e(s)d\mu(s) \cdot \int_\Omega e(s)g(s)d\mu(s) \right| \leq \frac{1}{4}(M-m)(N-n). \tag{3.1}$$

The constant $\frac{1}{4}$ is the best possible.

Proof. Consider the inner product

$$(f, g) = \int_\Omega f(s)g(s)d\mu(s).$$

Then we have

$$(Me - f, f - me) = \int_\Omega (Me(s) - f(s))(f(s) - me(s))\, d\mu(s) \geq 0$$

and, similarly,

$$(Ne - g, g - ne) \geq 0.$$

Applying Theorem 2.1 for the Hilbert space $L^2(\Omega)$, we get the desired inequality (3.1). ∎

Now, if we assume that $\mu(\Omega) < \infty$, then we can obtain the following Grüss inequality for integral means:

Proposition 3.2. *Let $L^2(\Omega)$ be as above and $\mu(\Omega) < \infty$. If $f, g \in L^2(\Omega)$ and p, P, q, Q are real numbers so that*

$$p \leq f \leq P, \qquad q \leq g \leq Q \qquad a.e. \text{ on } \Omega,$$

then we have the inequality

$$\left| \frac{1}{\mu(\Omega)} \int_\Omega f(s)\, g(s)\, d\mu(s) - \frac{1}{\mu(\Omega)} \int_\Omega f(s)\, d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_\Omega g(s)\, d\mu(s) \right| \leq \frac{1}{4}(P-p)(Q-q).$$

The constant $\frac{1}{4}$ is sharp.

Proof. The proof follows by the above proposition choosing

$$e = \frac{1}{[\mu(\Omega)]^{1/2}},$$

and

$$M = [\mu(\Omega)]^{1/2} P, \quad m = [\mu(\Omega)]^{1/2} p, \quad N = [\mu(\Omega)]^{1/2} Q \quad \text{and} \quad n = [\mu(\Omega)]^{1/2} q.$$

We omit the details. ∎

Remark 3.1. *It is important to observe that our Grüss type inequality also holds for integrals considered on infinite intervals.*

If $\rho : (-\infty, +\infty) \to (0, \infty)$ is a probability density function, i.e., $\int_{-\infty}^{+\infty} \rho(t)dt = 1$, then $\rho^{\frac{1}{2}} \in L^2(-\infty, +\infty)$ and obviously $\|\rho^{\frac{1}{2}}\|_2 = 1$. Consequently, if we assume that $f, g \in L^2(-\infty, +\infty)$ and

$$\alpha\rho^{\frac{1}{2}} \le f \le \Psi\rho^{\frac{1}{2}}, \quad \beta\rho^{\frac{1}{2}} \le g \le \Theta\rho^{\frac{1}{2}} \quad \text{a.e. on } (-\infty, +\infty),$$

then we have the inequality

$$\left| \int_{-\infty}^{+\infty} f(t)g(t)dt - \int_{-\infty}^{+\infty} f(t)\rho^{1/2}(t)dt \cdot \int_{-\infty}^{+\infty} f(t)\rho^{1/2}(t)dt \right| \le \frac{1}{4}(\Psi - \alpha)(\Theta - \beta). \tag{3.2}$$

Finally, we would like to note that, in this way, we can state many Grüss type inequalities by choosing the following well known probability distributions

$$\rho(t) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}}, \quad t > 0, \lambda > 0 \quad \text{(Exponential distribution)}$$

$$\rho(t) = \frac{1}{2\lambda} e^{-\frac{|t-\theta|}{\lambda}}, \quad \lambda > 0, \quad -\infty < t, \quad \theta < \infty \quad \text{(Laplace distribution)}$$

or, Cauchy, Gamma, Erlang, Logistic, Maxwell-Boltzman, Pareto, Rayleigh distributions etc...

We omit the details.

References

[1] MITRINOVIĆ, D.S. ; PEČARIĆ, J.E. ; FINK , A.M.; *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

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