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A LOWER BOUND FOR CONTINUOUS CONVEX MAPPINGS ON NORMED LINEAR SPACES

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Abstract. A lower bound for continuous convex mappings defined on normed linear spaces in terms of norm derivatives and best approximants is given.

1 Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives

$$(x,y)_{i(s)} = \lim_{t \to -(+)0} \left(\|y+tx\|^2 - \|y\|^2 \right)/2t.$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1], [3]):

(i) $(x,y)_i = -(-x,y)_s$ if x, y are in X;

(ii) $(x,x)_p = \|x\|^2$ for all x in X;

(iii) $(\alpha x, \beta y)_p = \alpha\beta(x,y)_p$ for all x, y in X and $\alpha\beta \geq 0$;

(iv) $(\alpha x + y, x)_p = \alpha\|x\|^2 + (y,x)_p$ for all x, y in X and α a real number ;

(v) $(x+y,z)_p \leq \|x\| \cdot \|z\| + (y,z)_p$ for all x, y, z in X;

(vi) the element x in X is Birkhoff orthogonal over y in X (we denote $x \perp y(B)$), i.e., $\|x+ty\| \geq \|x\|$ for all t a real number iff $(y,x)_i \leq 0 \leq (y,x)_s$;

(vii) the space X is smooth iff $(y,x)_i = (y,x)_s$ for all x, y in X iff $(\cdot,\cdot)_p$ is linear in the first variable;

(viii) we have the representation:

$$(y,x)_i = \inf\{f(y) : f \in J(x)\} \quad \text{and} \quad (y,x)_s = \sup\{f(y) : f \in J(x)\}$$

where J is the *normalized duality mapping*, i.e.,

$$J(x) = \{f \in X^* : f(x) = \|f\| \cdot \|x\|, \|f\| = \|x\|\},$$

where $p = s$ or $p = i$.

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Now, let $(X, \|\cdot\|)$ be a normed linear space and G a nondense subset in X. Suppose $x_0 \in X \setminus Cl(G)$ and $g_0 \in G$.

Definition 1. *The element g_0 will be called the best approximation element of x_0 in G if*

$$\|x_0 - g_0\| = \inf_{g \in G} \|x_0 - g\| \tag{1.1}$$

and we shall denote by $\mathcal{P}_G(x_0)$ the set of all elements which satisfy (1.1).

The main aim of this paper is to prove some characterization of best approximants from convex subsets in normed linear spaces. A lower bound for convex mappings in terms of norm derivatives is also given.

For the classical results in domain, see the monograph [4] due to Ivan Singer.

2 The Results

We shall consider the concept of sub-orthogonality in the sense of Birkhoff introduced by the author in the paper [1]:

Definition 2. *Let $(X, \|\cdot\|)$ be a normed linear space and $x, y \in X$. The element x will be called* sub-orthogonal in the sense of Birkhoff *over y if $(y, x)_i \leq 0$. We shall denote this by $x \perp_S y(B)$.*

The following elementary properties of sub-orthogonality hold:

(i) $0 \perp_S y(B)$ and $x \perp_S 0(B)$ for all $x, y \in X$;

(ii) $x \perp_S y(B)$ implies $(\alpha x) \perp_S (\beta y)(B)$ for $\alpha\beta \geq 0$;

(iii) $x \perp_S x(B)$ implies $x = 0$.

The following characterization of best approximants from convex sets in normed linear spaces which completes the classical results from the book [4] holds.

Theorem 2.1. *Let C be a nondense convex set in the normed linear spaces X. If $x_0 \in X \setminus Cl(C)$ and $g_0 \in C$, then the following statements are equivalent:*

(i) $g_0 \in P_G(x_0)$;

(ii) We have the relation:

$$x_0 - g_0 \perp_S (C - g_0)(B); \tag{2.1}$$

(iii) The following inclusion holds

$$C - g_0 \subset \cup_{f \in J(x_0 - g_0)} K_-(f); \tag{2.2}$$

where J is the normalized duality mapping and $K_-(f)$ is the half space $\{x \in X : f(x) \leq 0\}$;

(iv) We have the bound

(2.3) $$\inf_{g\in C}(g-x_0,g_0-x_0)_s=\|g_0-x_0\|^2.$$

Proof. "$(i)\Rightarrow(ii)$". If $g_0\in\mathcal{P}_G(x_0)$, then $\|x_0-g_0\|=\inf_{g\in G}\|x_0-g\|$, which implies that

$$\|x_0-g_0\|^2\le\|x_0-((1-t)g_0+tg)\|^2$$

for each $g\in C$ and $t\in[0,1]$.

Denoting $w_0:=x_0-g_0$ and $u_0:=g_0-g$ we get $\|w_0\|^2\le\|w_0+tu_0\|^2$ for all $t\in[0,1]$, which implies

$$(\|w_0+tu_0\|^2-\|w_0\|^2)/2t\ge 0 \text{ for all } t\in(0,1].$$

Letting $t\to 0+$ we deduce $(u_0,w_0)_s\ge 0$ which is equivalent to $(g-g_0,x_0-x_0)_i\le 0$ for all $g\in C$ and then the relation (2.1) holds.

"$(ii)\Leftrightarrow(iii)$". If $w_0\perp_S(C-g_0)$, then $(g-g_0,w_0)_i\le 0$ for all $g\in C$ and then there exists (see the property (viii) from introduction) a continuous linear functional f so that $f\in J(w_0)$ and $f(g-g_0)=(g-g_0,w_0)_i$ and then $f(g-g_0)\le 0$, *i.e.*, $g-g_0\in K_-(f)$. Consequently the inclusion (2.2) holds.

Conversly, if the inclusion (2.2) holds, then for each $g\in C$ there exists a functional $f_0\in J(x_0-g_0)$ so that $g-g_0\in K_-(f_0)$. But, by property (viii) stated above, we have

$$(g-g_0,x_0-g_0)_i=\inf\{f_0(g-g_0):f\in J(x_0-g_0)\}$$

and as $f_0\in J(x_0-g_0)$ and $f_0(g-g_0)\le 0$ it follows that $(g-g_0,x_0-g_0)_i\le 0$. Consequently the relation (2.1) holds and the implication is proved.

"$(ii)\Rightarrow(iv)$". Relation (2.1) is equivalent to

$$(g_0-g,x_0-g_0)_s\ge 0 \text{ for all } g\in C.$$

A simple calculation shows that

$$(g_0-g,x_0-g_0)_s=(x_0-g-(x_0-g_0),x_0-g_0)_s$$

$$=(x_0-g,x_0-g_0)_s-\|x_0-g_0\|^2$$

$$=(g-x_0,g_0-x_0)_s-\|x_0-g_0\|^2$$

and then, by the above inequality, we deduce

$$(g-x_0,g_0-x_0)_s\ge\|g_0-x_0\|^2$$

for all $g\in C$, which is equivalent to (2.3).

"$(iv)\Rightarrow(i)$". Using the properties of semi-inner product $(,)_s$, we have

$$(g-x_0,g_0-x_0)_s\le\|g-x_0\|\cdot\|g_0-x_0\|$$

for each $g\in C$. From (2.3) we get

$$\|g_0-x_0\|^2\le(g-x_0,g_0-x_0)_s$$

for each $g\in C$, consequently, by the previous two inequalities we deduce that $\|g_0-x_0\|\le\|g-x_0\|$ for all $g\in C$, *i.e.*, $g_0\in\mathcal{P}_G(x_0)$. ∎

Remark 2.1. *The relation (2.3) is equivalent to the fact that the element $g_0 \in C$ minimizes the (nonlinear) functional*

$$F_{x_0,g_0} : C \to \mathbf{R}, \quad F_{x_0,g_0}(u) := (u - x_0, g_0 - x_0)_s.$$

The following corollary holds.

Corollary 2.2. *Let G be a nondense linear subspace in X. If $x_0 \in X \backslash Cl(G)$ and $g_0 \in G$, then the following statement are equivalent:*

(i) $g_0 \in P_G(x_0)$,

(ii) $x_0 - g_0 \perp G(B)$,

(iii) $G \subset \cup_{f \in J(x_0 - g_0)} K_-(f)$.

The equivalence "$(i) \Leftrightarrow (ii)$" is a well known result due to Singer and follows from the fact that a vector is sub-orthogonal on a linear subspace iff it is orthogonal on that subspace.

Now, let denote by

$$F^{\leq}(r) := \{x \in X : F(x) \leq r\}, \quad r \in \mathbf{R}$$

the $r-$ *level set* of F and assume that r is so that $F^{\leq}(r)$ is nonempty.

The following theorem characterizes best approximants by elements of the level set $F^{\leq}(r)$. This result can also be viewed as an estimation theorem for the continuous convex mappings defined on a normed space in terms of semi-inner product $(.,.)_i$.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a normed linear space, $F : X \to \mathbf{R}$ a continuous convex mapping on X, $r \in \mathbf{R}$ so that $F^{\leq}(r) \neq \emptyset, x_0 \in X \backslash F^{\leq}(r)$ and $g_0 \in F^{\leq}(r)$. The following statements are equivalent:*

(i) $g_0 \in P_{F^{\leq}(r)}(x_0)$;

(ii) We have the estimation:

$$F(x) \geq r + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i \tag{2.4}$$

for all $x \in F^{\leq}(r)$, or, equivalently, the estimation

$$F(x) \geq F(x_0) + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - x_0, x_0 - g_0)_i \tag{2.5}$$

for all $x \in F^{\leq}(r)$.

Proof. "$(i) \Rightarrow (ii)$". Firstly, let observe as $x_0 \in X \backslash F^{\leq}(r)$ we have that $F(x_0) > r$.

Now, let $x \in F^{\leq}(r)$. Then $F(x) \leq r$ and if we choose $\alpha := F(x_0) - r, \beta := r - F(x)$, then obviously $\alpha > 0, \beta \geq 0$ and $0 < \alpha + \beta = F(x_0) - F(x)$.

Let consider the element

$$u := \frac{\alpha x \ + \ \beta x_0}{\alpha + \beta}.$$

Then, by the convexity of F we have:

$$F(u) \leq \frac{\alpha F(x) + \beta F(x_0)}{\alpha + \beta} = \frac{(F(x_0) - r)F(x) + (r - F(x))F(x_0)}{F(x_0) - F(x)}$$

which shows that $u \in F^{\leq}(r)$.

As $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$ and as $F^{\leq}(r)$ is a convex set, we get (see Theorem 2.1, "$(i) \Rightarrow (ii)$") that

$$(g - g_0, x_0 - x_0)_i \leq 0$$

for all $g \in F^{\leq}(r)$.

Choose $g = u$, where u is defined as above. Then

$$\left(\frac{(F(x_0) - r)x + (r - F(x))x_0}{F(x_0) - F(x)} - g_0, x_0 - g_0 \right)_i \leq 0 \tag{2.6}$$

for all $x \in F^{\leq}(r)$. But

$$\left(\frac{(F(x_0) - r)x + (r - F(x))x_0}{F(x_0) - F(x)} - g_0, x_0 - g_0 \right)_i$$

$$= \frac{1}{F(x_0) - F(x)} ((r - F(x))(x_0 - g_0) + (F(x_0) - r)(x - g_0), x_0 - g_0)_i$$

$$= \frac{1}{F(x_0) - F(x)} ((r - F(x))\|x_0 - g_0\|^2 + (F(x_0) - r)(x - g_0, x_0 - g_0)_i)$$

and then, by (2.6), we get

$$(r - F(x))\|x_0 - g_0\|^2 + (F(x_0) - r)(x - g_0, x_0 - g_0)_i \geq 0$$

which is equivalent with the desired estimation (2.4).

Now, let observe that

$$r + \frac{F(x_0) \ - \ r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i$$

$$= r + \frac{F(x_0) \ - \ r}{\|x_0 - g_0\|^2}(x - x_0 + x_0 - g_0, x_0 - g_0)_i$$

$$= r + \frac{F(x_0) \ - \ r}{\|x_0 - g_0\|^2}[(x - x_0, x_0 - g_0)_i + \|x_0 - g_0\|^2]$$

$$= r + F(x_0) - r + \frac{F(x_0) \ - \ r}{\|x_0 - g_0\|^2}(x - x_0, x_0 - g_0)_i$$

$$= F(x_0) + \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - x_0, x_0 - g_0)_i$$

which shows that (2.4) and (2.5) are equivalent.

"$(ii) \Rightarrow (i)$". As $x \in F^{\leq}(r)$, then $0 \geq F(x) - r$. On the other hand, by (2.4), we have

$$F(x) - r \geq \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i$$

for all $x \in F^{\leq}(r)$, consequently

$$0 \geq \frac{F(x_0) - r}{\|x_0 - g_0\|^2}(x - g_0, x_0 - g_0)_i$$

for all $x \in F^{\leq}(r)$. As $F(x_0) - r > 0$, we get

$$0 \geq (x - g_0, x_0 - g_0)_i$$

for all $x \in F^{\leq}(r)$. Now, using the implication "$(ii) \Rightarrow (i)$" of Theorem 2.1, we deduce that $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$, and the theorem is proved. ∎

Remark 2.2. *If $g_0 \in \mathcal{P}_{F^{\leq}(r)}(x_0)$, then $F(g_0) = r$.*

Indeed, as $g_0 \in F^{\leq}(r)$, then $F(g_0) \leq r$. On the other hand, choosing $x = g_0$ in (2.4) we get $F(g_0) \geq r$, and then the required equality holds.

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