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AN OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS AND APPLICATIONS FOR CUBATURE FORMULAE

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Abstract. An inequality of the Ostrowski type for double integrals and applications in Numerical Analysis in connection with cubature formulae are given.

1 Introduction

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468]

Theorem 1.1. *Let $f:[a,b]\to\mathbf{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$. Then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$.
The constant $\frac{1}{4}$ is the best possible.

For some generalizations of this classic fact see the book [5, p. 468-484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in the recent paper [3] by S.S. Dragomir and S. Wang. For other results of Ostrowski's type see the papers [1], [2] and [4].

In 1975, G.N. Milovanović generalized Theorem 1.1 where f is a function of several variables [5, p. 468]

Theorem 1.2. *Let $f:\mathbf{R}^m\to\mathbf{R}$ be a differentiable function defined on $D=\{(x_1,...,x_m)\,|a_i\le x_i\le b_i\,(i=1,...,m)\}$ and let $\left|\frac{\partial f}{\partial x_i}\right|\le M_i$ $(M_i>0, i=1,...,m)$ in D. Furthermore, let function $x\longmapsto p(x)$ be integrable and $p(x)>0$ for every $x\in D$. Then for every $x\in D$, we have the inequality:*

$$\left|f(x)-\frac{\int_D p(y)f(y)\,dy}{\int_D p(y)\,dy}\right| \le \frac{\sum_{i=1}^m M_i\int_D p(y)\,|x_i-y_i|\,dy}{\int_D p(y)\,dy}. \tag{1.2}$$

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In the present paper we point out an Ostrowski type inequality for double integrals and apply it in Numerical Integration obtaining a general cubature formula.

2 The Results

The following inequality of Ostrowski's type for mappings of two variables holds:

Theorem 2.1. *Let* $f:[a,b]\times[c,d]\to\mathbf{R}$ *continuous on* $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ *exists on* $(a,b)\times(c,d)$ *and is bounded, i.e.,*

$$\left\|f''_{s,t}\right\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty$$

then we have the inequality:

$$(2.1)\qquad \left|\int_a^b\int_c^d f(s,t)\,dsdt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds\right.\right.$$

$$\left.\left.-(d-c)(b-a)f(x,y)\right]\right|$$

$$\le\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty$$

for all $(x,y)\in[a,b]\times[c,d]$.

Proof. We have the equality:

$$(2.2)\qquad \int_a^x\int_c^y (s-a)(t-c)\,f''_{s,t}(s,t)\,dtds$$

$$=\int_a^x (s-a)\left[f'_s(s,y)(y-c)-\int_c^y f'_s(s,t)\,dt\right]ds$$

$$=(y-c)\int_a^x (s-a)\,f'_s(s,y)\,ds-\int_c^y\left(\int_a^x (s-a)\,f'_s(s,t)\,ds\right)dt$$

$$=(y-c)\left[(x-a)f(x,y)-\int_a^x f(s,y)\,ds\right]-\int_c^y\left[(x-a)f(x,t)-\int_a^x f(s,t)\,ds\right]dt$$

$$=(y-c)(x-a)f(x,y)-(y-c)\int_a^x f(s,y)\,ds$$

$$-(x-a)\int_{c}^{y} f(x,t)\,dt+\int_{a}^{x}\int_{c}^{y} f(s,t)\,dsdt.$$

Also, by similar computations we have

$$\int_{a}^{x}\int_{y}^{d}(s-a)(t-d)\,f''_{s,t}(s,t)\,dsdt \tag{2.3}$$

$$=\int_{a}^{x}(s-a)\left[(d-y)\,f'_{s}(s,y)-\int_{y}^{d} f'_{s}(s,t)\,dt\right]ds$$

$$=(d-y)\int_{a}^{x}(s-a)\,f'_{s}(s,y)\,ds-\int_{y}^{d}\left(\int_{a}^{x}(s-a)\,f'_{s}(s,t)\,ds\right)dt$$

$$=(d-y)\left[(x-a)\,f(x,y)-\int_{a}^{x} f(s,y)\,ds\right]-\int_{y}^{d}\left[(x-a)\,f(x,t)-\int_{a}^{x} f(s,t)\,ds\right]dt$$

$$=(x-a)(d-y)\,f(x,y)-(d-y)\int_{a}^{x} f(s,y)\,ds$$

$$-(x-a)\int_{y}^{d} f(x,t)\,dt+\int_{a}^{x}\int_{c}^{y} f(s,t)\,dsdt.$$

Now,

$$\int_{x}^{b}\int_{y}^{d}(s-b)(t-d)\,f''_{s,t}(s,t)\,dsdt \tag{2.4}$$

$$=\int_{x}^{b}(s-b)\left[(d-y)\,f'_{s}(s,y)-\int_{y}^{d} f'_{s}(s,t)\,dt\right]ds$$

$$=(d-y)\int_{x}^{b}(s-b)\,f'_{s}(s,y)\,ds-\int_{y}^{d}\left(\int_{x}^{b}(s-b)\,f'_{s}(s,t)\,ds\right)dt$$

$$= (d-y)\left[(b-x)f(x,y)-\int_x^b f(s,y)\,ds\right]-\int_y^d\left[(b-x)f(x,t)-\int_x^b f(s,t)\,ds\right]dt$$

$$= (d-y)(b-x)f(x,y)-(d-y)\int_x^b f(s,y)\,ds$$

$$-(b-x)\int_y^d f(x,t)\,dt+\int_x^b\int_y^d f(s,t)\,dsdt$$

and finally

$$\int_x^b\int_c^y (s-b)(t-c)\,f''_{s,t}(s,t)\,dsdt \tag{2.5}$$

$$= \int_x^b (s-b)\left[(y-c)f'_s(s,y)-\int_c^y f'_s(s,t)\,dt\right]ds$$

$$= (y-c)\int_x^b (s-b)\,f'_s(s,y)\,ds-\int_c^y\left(\int_x^b (s-b)\,f'_s(s,t)\,ds\right)dt$$

$$= (y-c)\left[(b-x)f(x,y)-\int_x^b f(s,y)\,ds\right]-\int_c^y\left[(b-x)f(x,t)-\int_x^b f(s,t)\,ds\right]dt$$

$$= (y-c)(b-x)f(x,y)-(y-c)\int_x^b f(s,y)\,ds$$

$$-(b-x)\int_c^y f(x,t)\,dt+\int_x^b\int_c^y f(s,t)\,dsdt.$$

If we add the equalities (2.2) – (2.5) we get in the right membership:

$$[(y-c)(x-a)+(x-a)(d-y)+(d-y)(b-x)+(y-c)(b-x)]\,f(x,y)$$

$$-(d-c)\int_a^x f(s,y)\,ds-(d-c)\int_x^b f(s,y)\,ds-(b-a)\int_c^y f(x,t)\,dt$$

$$-(b-a)\int_y^d f(x,t)\,dt+\int_a^x\int_c^y f(s,t)\,dsdt+\int_a^x\int_y^d f(s,t)\,dsdt$$

$$+\int_x^b\int_y^d f(s,t)\,dsdt+\int_x^b\int_c^y f(s,t)\,dsdt$$

$$=(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(s,y)\,ds$$

$$-(b-a)\int_c^b f(x,t)\,dt+\int_a^b\int_c^d f(s,t)\,dsdt.$$

For the first membership, let us define the kernels: $p:[a,b]^2\to\mathbf{R}$, $q:[c,d]^2\to\mathbf{R}$ given by:

$$p(x,s):=\begin{cases} s-a & \text{if } s\in[a,x] \\ s-b & \text{if } s\in(x,b] \end{cases}$$

and

$$q(y,t):=\begin{cases} t-c & \text{if } t\in[c,y] \\ t-d & \text{if } t\in(y,d] \end{cases}.$$

Now, using these notations, we deduce that the left membership can be represented as :

$$\int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,dsdt.$$

Consequently, we get the identity

$$\int_a^b\int_c^d p(x,s)\,q(y,t)\,f''_{s,t}(s,t)\,dsdt \tag{2.6}$$

$$=(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(x,y)\,ds-(b-a)\int_c^d f(x,t)\,dt+\int_a^b\int_c^d f(s,t)\,dsdt$$

for all $(x,y) \in [a,b] \times [c,d]$.

Now, using the identity (2.6) we get

$$\left| \int_a^b \int_c^d f(s,t)\,dsdt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(x,y)\,ds \right.\right. \tag{2.7}$$

$$\left.\left. -(d-c)(b-a) f(x,y)\right] \right|$$

$$\leq \int_a^b \int_c^d |p(x,s)|\,|q(y,t)|\,\left|f''_{s,t}(s,t)\right| dsdt \leq \left\|f''_{s,t}\right\|_\infty \int_a^b \int_c^d |p(x,s)|\,|q(y,t)|\,dsdt.$$

Now, observe that

$$\int_a^b |p(x,s)|\,ds = \int_a^x (s-a)\,ds + \int_x^b (b-s)\,ds$$

$$= \frac{(x-a)^2 + (b-x)^2}{2} = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2$$

and, similarly,

$$\int_c^d |q(y,t)|\,dt = \frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2.$$

Finally, using (2.7), we get the desired inequality (2.1). ∎

Corollary 2.2. *Under the above assumptions, we have the inequality:*

$$\left| \int_a^b \int_c^d f(s,t)\,dsdt - \left[(b-a)\int_c^d f\left(\frac{a+b}{2}, t\right) dt \right.\right. \tag{2.8}$$

$$\left.\left. + (d-c)\int_a^b f\left(s, \frac{c+d}{2}\right) ds - (d-c)(b-a) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]\right|$$

$$\leq \frac{1}{16}(b-a)^2 (d-c)^2 \left\|f''_{s,t}\right\|_\infty.$$

Remark 2.1. *The constants $\frac{1}{4}$ from the first and the second bracket are optimal in the sense that not both of them can be less than $\frac{1}{4}$.*

Indeed, if we had assumed that there exists $c_1, c_2 \in \left(0, \frac{1}{4}\right)$ so that

$$\text{(2.9)} \quad \left| \int_a^b \int_c^d f(s,t)\, dsdt - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds - (d-c)(b-a) f(x,y) \right] \right|$$

$$\leq \left[c_1 (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[c_2 (d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right] \left\| f''_{s,t} \right\|_\infty$$

for all f as in Theorem 2.1 and $(x,y) \in [a,b] \times [c,d]$, then we would have had for $f(s,t) = st$ and $x = a, y = c$ that:

$$\int_a^b \int_c^d f(s,t)\, dsdt = \frac{\left(b^2 - a^2\right)\left(d^2 - c^2\right)}{4},$$

$$\int_c^d f(x,t)\, dt = a \cdot \frac{d^2 - c^2}{2}, \int_a^b f(s,y)\, ds = c \cdot \frac{b^2 - a^2}{2},$$

$$\left\| f''_{s,t} \right\|_\infty = 1$$

and by (2.9), the inequality:

$$\left| \frac{\left(b^2 - a^2\right)\left(d^2 - c^2\right)}{4} - (b-a)\, a \cdot \frac{d^2 - c^2}{2} - (d-c)\, c \cdot \frac{b^2 - a^2}{2} + (d-c)(b-a)\, ac \right|$$

$$\leq (b-a)^2 \left(c_1 + \frac{1}{4} \right) (d-c)^2 \left(c_2 + \frac{1}{4} \right)$$

i.e.

$$\frac{(b-a)^2 (d-c)^2}{4} \leq (b-a)^2 (d-c)^2 \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right)$$

i.e.

$$\text{(2.10)} \quad \frac{1}{4} \leq \left(c_1 + \frac{1}{4} \right) \left(c_2 + \frac{1}{4} \right).$$

Now, as we have assumed that $c_1, c_2 \in \left(0, \frac{1}{4}\right)$, we get

$$c_1 + \frac{1}{4} < \frac{1}{2}, c_2 + \frac{1}{4} < \frac{1}{2}$$

and then $\left(c_1 + \frac{1}{4}\right)\left(c_2 + \frac{1}{4}\right) < \frac{1}{4}$ which contradicts the inequality (2.10), and the statement is proved. ∎

Remark 2.2. *Now, if we assume that* $f(s,t) = h(s)h(t)$, $h : [a,b] \to \mathbf{R}$, h *is continuous and suppose that* $\|h'\|_\infty < \infty$, *then from* (2.1) *we get (for* $x = y$*)*

$$\Bigg| \int_a^b h(s)\,ds \int_a^b h(s)\,ds - h(x)(b-a)\int_a^b h(s)\,ds$$

$$-h(x)(b-a)\int_a^b h(s)\,ds + (b-a)^2 h^2(x) \Bigg|$$

$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]^2 \|h\|_\infty^2$$

i.e.

$$\left[\int_a^b h(s)\,ds - h(x)(b-a)\right]^2 \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)\right]^2 \|h\|_\infty^2$$

which is clearly equivalent to Ostrowski's inequality.

Consequently (2.1) can be also regarded as a generalization for double integrals of the classical result due to Ostrowski.

3 Applications for Cubature Formulae

Let us consider the arbitrary division $I_n : a = x_1 < x_1 < ... < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < ... < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ be intermediate points. Consider the sum

$$C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\,dt \tag{3.1}$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\,ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j)$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s,t)\,ds dt,$$

and

$$h_i := x_{i+1} - x_i \,(i = 0, ..., n-1), \qquad l_j := y_{j+1} - y_j \quad (j = 0, ..., m-1).$$

With this assumption, we can state the following cubature formula:

Theorem 3.1. *Let* $f : [a,b] \times [c,d] \to \mathbf{R}$ *be as in Theorem 2.1 and* $I_n, J_m, \boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *be as above. Then we have the cubature formula:*

$$\int_a^b \int_c^d f(s,t)\, dsdt = C(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) + R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta}) \tag{3.2}$$

where the remainder term $R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})$ *satisfies the estimation:*

$$|R(f, I_n, J_m, \boldsymbol{\xi}, \boldsymbol{\eta})| \tag{3.3}$$

$$\leq \left\|f''_{s,t}\right\|_\infty \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)^2\right]$$

$$\leq \frac{1}{4}\left\|f''_{s,t}\right\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ $(i = 0, ..., n-1; j = 0, ..., m-1)$ to get:

$$\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(s,t)\, dsdt - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t)\, dt + l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j)\, ds - h_i l_j f(\xi_i, \eta_j)\right]\right|$$

$$\leq \left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\left[\frac{1}{4}l_j^2 + \left(\eta_j - \frac{y_j + y_{j+1}}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty$$

for all $i = 0, ..., n-1; \quad j = 0, ..., m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality we deduce the first inequality in (3.3).

For the second part we observe that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{1}{2}h_i \quad \text{and} \quad \left|\eta_j - \frac{y_j + y_{j+1}}{2}\right| \leq \frac{1}{2}l_j$$

for all i, j as above. ∎

Remark 3.1. *As*

$$\sum_{i=0}^{n-1} h_i^2 \leq \nu(h) \sum_{i=0}^{n-1} h_i = (b-a)\,\nu(h)$$

and

$$\sum_{j=0}^{m-1} l_j^2 \leq \mu(l) \sum_{j=0}^{m-1} l_j = (d-c)\,\mu(l)$$

where

$$\nu(h) = \max\{h_i : i = 0, ..., n-1\},$$

$$\mu(l) = \max\{l_j : j = 0, ..., m-1\},$$

the right membership of (3.3) *can be bounded by*

$$\frac{1}{4}\left\|f''_{s,t}\right\|_{\infty}(b-a)(d-c)\,\nu(h)\,\mu(l)$$

which is of order 2 *precision.*

Now, define the sum

$$C_M(f, I_n, J_m) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i + x_{i+1}}{2}, t\right) dt$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j + y_{j+1}}{2}\right) ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right).$$

Then we have the best cubature formula we can get from (3.2).

Corollary 3.2. *Under the above assumptions we have*

$$\int_a^b \int_c^d f(s,t)\,dsdt = C_M(f, I_n, J_m) + R(f, I_n, J_m) \tag{3.4}$$

when the remainder $R(f, I_n, J_m)$ *satisfies the estimation:*

$$|R(f, I_n, J_m)| \leq \frac{\left\|f''_{s,t}\right\|_{\infty}}{16} \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_j^2.$$

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