

AN OSTROWSKI TYPE INEQUALITY FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $L_p(A, B)$ AND APPLICATIONS

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ABSTRACT. An inequality of the Ostrowski type for twice differentiable mappings whose derivatives belong to $L_p(a, b)$ ($p > 1$) and applications in Numerical Integration are investigated.

1 INTRODUCTION

The following inequality is well known in the literature as Ostrowski's integral inequality (see for example [1, p. 468])

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \rightarrow \mathbf{R}$ is bounded, i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$, then we have the inequality:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in (a, b)$.

The constant $\frac{1}{4}$ is the best possible.

For a simple proof and some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper [2] by S.S. Dragomir and A. Wang.

In [3], the same authors considered another inequality of Ostrowski type for $\|\cdot\|_p$ -norm ($p > 1$) as follows:

Theorem 1.2. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_p(a, b)$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1$) then we have the inequality:*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p$$

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for all $x \in [a, b]$, where

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}},$$

is the $L_p(a, b)$ -norm.

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski inequality for n -times differentiable mappings (see for example [1, p. 468]). The case of twice differentiable mappings [1, p. 470] is as follows:

Theorem 1.3. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) , i.e., $\|f''\|_\infty := \sup_{t \in (a, b)} |f''(t)| < \infty$. Then we have the inequality:*

$$(1.3) \quad \left| \frac{1}{2} \left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{\|f''\|_\infty}{4} (b-a)^2 \left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right]$$

for all $x \in [a, b]$.

In this paper, we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|\cdot\|_p$ -norm of the second derivative f'' and apply it in Numerical Integration.

2 SOME INTEGRAL INEQUALITIES

The following inequality of Ostrowski type for mappings which are twice differentiable, holds:

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a twice differentiable mapping on (a, b) and $f'' \in L_p(a, b)$ ($p > 1$). Then we have the inequality:*

$$(2.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ \leq \frac{1}{2(b-a)(2q+1)^{\frac{1}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{\frac{1}{q}} \|f''\|_p \\ \leq \frac{(b-a)^{1+\frac{1}{q}} \|f''\|_p}{2(2q+1)^{\frac{1}{q}}}$$

for all $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let us define the mapping $K(\cdot, \cdot) : [a, b]^2 \rightarrow \mathbf{R}$ given by

$$K(x, t) := \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x, b] \end{cases}.$$

Integrating by parts, we have successively,

$$\begin{aligned} \int_a^b K(x, t) f''(t) dt &= \int_a^x \frac{(t-a)^2}{2} f''(t) dt + \int_x^b \frac{(t-b)^2}{2} f''(t) dt \\ &= \frac{(t-a)^2}{2} f'(t) \Big|_a^x - \int_a^x (t-a) f'(t) dt + \frac{(t-b)^2}{2} f'(t) \Big|_x^b - \int_x^b (t-b) f'(t) dt \\ &= \frac{(x-a)^2}{2} f'(x) - \left[(t-a) f(t) \Big|_a^x - \int_a^x f(t) dt \right] \\ &\quad - \frac{(b-x)^2}{2} f'(x) - \left[(t-b) f(t) \Big|_x^b - \int_x^b f(t) dt \right] \\ &= \frac{1}{2} \left[(x-a)^2 - (b-x)^2 \right] f'(x) \\ &\quad - (x-a) f(x) + \int_a^x f(t) dt + (x-b) f(x) + \int_x^b f(t) dt \\ &= (b-a) \left(x - \frac{a+b}{2} \right) f'(x) - (b-a) f(x) + \int_a^b f(t) dt \end{aligned}$$

from which we get the integral identity

$$(2.2) \quad \int_a^b f(t) dt = (b-a) f(x) - (b-a) \left(x - \frac{a+b}{2} \right) f'(x) + \int_a^b K(x, t) f''(t) dt$$

for all $x \in [a, b]$.

Using (2.2), we have, by Hölder's integral inequality, that

$$\begin{aligned}
 (2.3) \quad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\
 &= \frac{1}{b-a} \left| \int_a^b K(x,t) f''(t) dt \right| \leq \frac{1}{b-a} \left(\int_a^b K^q(x,t) dt \right)^{\frac{1}{q}} \|f''\|_p \\
 &= \frac{1}{b-a} \left[\int_a^x \frac{(t-a)^{2q}}{2^q} dt + \int_x^b \frac{(t-b)^{2q}}{2^q} dt \right]^{\frac{1}{q}} \|f''\|_p \\
 &= \frac{1}{b-a} \left[\frac{(x-a)^{2q+1}}{2^q(2q+1)} + \frac{(b-x)^{2q+1}}{2^q(2q+1)} \right]^{\frac{1}{q}} \|f''\|_p \\
 &= \frac{1}{2(b-a)} \frac{1}{(2q+1)^{\frac{1}{q}}} \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{\frac{1}{q}} \|f''\|_p
 \end{aligned}$$

and the first inequality in (2.1) is proved. The second inequality is obvious taking into account that

$$(x-a)^{2q+1} + (b-x)^{2q+1} \leq (b-a)^{2q+1}$$

for all $x \in [a, b]$. ■

The following particular case for euclidean norms is interesting

Corollary 2.2. *Let $f : [a, b] \rightarrow \mathbf{R}$ be as above and $f'' \in L_2(a, b)$. Then we have the inequality:*

$$\begin{aligned}
 (2.4) \quad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\
 &\leq \frac{(b-a)^{\frac{3}{2}}}{2} \left[\frac{1}{80} + \frac{1}{2} \cdot \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{\left(x - \frac{a+b}{2}\right)^4}{(b-a)^4} \right]^{\frac{1}{2}} \|f''\|_2.
 \end{aligned}$$

Proof. Apply inequality (2.1) for $p = q = 2$, to get

$$(2.5) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right|$$

$$\leq \frac{1}{2(b-a)\sqrt{5}} \left[(x-a)^5 + (b-x)^5 \right]^{\frac{1}{2}} \|f''\|_2.$$

Denote $t := x - \frac{a+b}{2}$. Then

$$x-a = t + \frac{b-a}{2}, b-x = \frac{b-a}{2} - t.$$

Let us compute

$$I := (x-a)^5 + (b-x)^5 = \left(t + \frac{b-a}{2} \right)^5 + \left(\frac{b-a}{2} - t \right)^5.$$

We know that, for numbers $A, B \in \mathbf{R}$, we have

$$\begin{aligned} A^5 + B^5 &= (A+B)(A^4 - A^3B + A^2B^2 - AB^3 + B^4) \\ &= (A+B)[A^4 + B^4 - AB(A^2 + B^2) + A^2B^2] \\ &= (A+B)[(A^2 + B^2)^2 - A^2B^2 - AB(A^2 + B^2)]. \end{aligned}$$

Now, if we put $A := t + \frac{b-a}{2}$, $B := \frac{b-a}{2} - t$, then we get

$$A^2 + B^2 = 2t^2 + \frac{(b-a)^2}{2}, AB = \frac{(b-a)^2}{4} - t^2$$

and then

$$\begin{aligned} J &:= (A^2 + B^2)^2 - A^2B^2 - AB(A^2 + B^2) \\ &= \left[2t^2 + \frac{(b-a)^2}{2} \right]^2 - \left[t^2 - \frac{(b-a)^2}{4} \right]^2 - \left[\frac{(b-a)^2}{4} - t^2 \right] \left[2t^2 + \frac{(b-a)^2}{2} \right] \\ &= 5t^4 + \frac{5}{2}(b-a)^2 t^2 + \frac{(b-a)^4}{16} = 5 \left[t^4 + 2 \left(\frac{b-a}{2} \right)^2 t^2 + \frac{1}{5} \left(\frac{b-a}{2} \right)^4 \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} I &= (b-a) \left[5 \left(x - \frac{a+b}{2} \right)^4 + \frac{5}{2} (b-a)^2 \left(x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^4}{16} \right] \\ &= 5(b-a)^5 \left[\frac{1}{80} + \frac{1}{2} \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} + \frac{\left(x - \frac{a+b}{2} \right)^4}{(b-a)^4} \right]. \end{aligned}$$

Finally, using the inequality (2.5), we get the desired result (2.4) ■

Remark 2.1. Let $f : [a, b] \rightarrow \mathbf{R}$ be as above. Then we have the midpoint inequality:

$$(2.6) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8(2q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_p.$$

Taking into account the fact that the mapping

$$h : [a, b] \rightarrow \mathbf{R}, \quad h(x) = (x - a)^{2q+1} + (b - x)^{2q+1}$$

has the property that

$$\inf_{x \in [a, b]} h(x) = h\left(\frac{a+b}{2}\right) = \frac{(b-a)^{2q+1}}{2^{2q}}$$

and

$$\sup_{x \in [a, b]} h(x) = h(a) = h(b) = (b-a)^{2q+1};$$

then, the best estimation we can get from (2.1) is that one for which $x = \frac{a+b}{2}$, obtaining the inequality (2.6).

Remark 2.2. *If in (2.1) we choose $x = a$ we get*

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{b-a}{2} f'(a) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p$$

and putting $x = b$, we also get

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{2} f'(b) \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p.$$

Summing the above two inequalities, using the triangle inequality and dividing by 2, we get the perturbed trapezoid formula

$$(2.7) \quad \left| \frac{f(a) + f(b)}{2} - \frac{b-a}{4} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^{1+\frac{1}{q}}}{2(2q+1)^{\frac{1}{q}}} \|f''\|_p.$$

Remark 2.3. *If $p = q = 2$, then we get for the euclidean norm, from (2.6),*

$$(2.8) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{\sqrt{5}(b-a)^{\frac{3}{2}}}{40} \|f''\|_2$$

and, from (2.7),

$$(2.9) \quad \left| \frac{f(a) + f(b)}{2} - \frac{b-a}{4} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{\sqrt{5}(b-a)^{\frac{3}{2}}}{10} \|f''\|_2.$$

3 APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$). We have the following quadrature formula:

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a twice differentiable mapping on (a, b) whose second derivative $f'' : (a, b) \rightarrow \mathbf{R}$ belongs to $L_p(a, b)$ ($p > 1$), i.e.,*

$$\|f''\|_p := \left(\int_a^b |f''(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

Then the following perturbed Riemann type quadrature formula holds:

$$(3.1) \quad \int_a^b f(x) dx = A(f, f', \xi, I_n) + R(f, f', \xi, I_n)$$

where $A(f, f', \xi, I_n)$ is given by

$$A(f, f', \xi, I_n) := \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i$$

and the remainder satisfies the estimation:

$$(3.2) \quad |R(f, f', \xi, I_n)| \leq \frac{1}{2(2q+1)^{\frac{1}{q}}} \left(\sum_{i=0}^{n-1} (\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p \leq \frac{1}{2(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p,$$

for all $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$).

Proof. Apply inequality (2.1) on the interval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) to get

$$\left| f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t) dt - \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) h_i \right| \leq \frac{1}{2(2q+1)^{1/q}} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right]^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p dt \right)^{\frac{1}{p}}$$

for all $i \in \{0, \dots, n-1\}$.

Summing over i from 0 to $n - 1$, using the generalized triangle inequality and Hölder's discrete inequality, we get:

$$\begin{aligned}
 & |R(f, f', \xi, I_n)| \\
 & \leq \sum_{i=0}^{n-1} \left| f(\xi_i) h_i - \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t) dt \right| \\
 & \leq \frac{1}{2(2q+1)^{1/q}} \sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right]^{\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f''(t)|^p dt \right)^{\frac{1}{p}} \\
 & \leq \frac{1}{2(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right]^{\frac{1}{q}} \right)^q \\
 & \quad \times \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} |f''(t)|^p dt \right)^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\
 & = \frac{1}{2(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \right] \right)^{\frac{1}{q}} \|f''\|_p
 \end{aligned}$$

and the first inequality in (3.2) is proved.

The last part is obvious from the fact that

$$(\xi_i - x_i)^{2q+1} + (x_{i+1} - \xi_i)^{2q+1} \leq h_i^{2q+1}$$

for all $i \in \{0, \dots, n-1\}$. ■

Now, if we consider the midpoint formula

$$M(f, I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

then we have

$$(3.3) \quad \int_a^b f(t) dt = M(f, I_n) + R(f, I_n)$$

and the remainder $R(f, I_n)$ can be estimated in terms of the p -norm of f'' as follows:

$$(3.4) \quad |R(f, I_n)| \leq \frac{1}{8(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p$$

which is, in a certain sense, the best estimation we can obtain from (3.2).

Also, we can construct the following perturbed trapezoid formula

$$T_p(f, f', I_n) := \frac{1}{2} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h_i + \frac{1}{4} \sum_{i=0}^{n-1} h_i^2 (f'(x_i) - f'(x_{i+1})).$$

Then we have

$$(3.5) \quad \int_a^b f(t) dt = T_p(f, f', I_n) + R_p(f, f', I_n)$$

and the remainder can be estimated (see the inequality (2.7)) as follows:

$$(3.6) \quad |R_p(f, f', I_n)| \leq \frac{1}{2(2q+1)^{1/q}} \left(\sum_{i=0}^{n-1} h_i^{2q+1} \right)^{\frac{1}{q}} \|f''\|_p.$$

Remark 3.1. To derive the corresponding results for the euclidean norm $\|f''\|_2$, we put in the above $p = q = 2$.

We omit the details.

Remark 3.2. The reader can obtain the corresponding quadrature formulae for equidistant partitioning by choosing $x_i = a + i \cdot \frac{b-a}{n}$ ($i = 0, \dots, n-1$).

Remark 3.3. If we consider equidistant partitioning of $[a, b]$ then the perturbed trapezoid formula we considered above will involve the calculation for f' only at the endpoints a and b , which is a good advantage for practical applications.

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