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AN OSTROWSKI TYPE INEQUALITY FOR WEIGHTED MAPPINGS WITH BOUNDED SECOND DERIVATIVES

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Abstract. A weighted integral inequality of Ostrowski type for mappings whose second derivatives are bounded is proved. The inequality is extended to account for applications in numerical integration.

1 Introduction

In 1938, Ostrowski (see for example [7, p. 468]) proved the following inequality

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in I^o (I^o is the interior of I), and let $a, b \in I^o$ with $a < b$. If $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in (a, b)$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

A similar result for twice differentiable mappings [5] is given below.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x) + \left(x - \frac{a+b}{2}\right)f'(x)\right| \le \left[\frac{1}{24} + \frac{\left(x-\frac{a+b}{2}\right)^2}{2(b-a)^2}\right](b-a)^2\|f''\|_\infty \tag{1.2}$$

for all $x \in [a, b]$.

In this paper, we extend the above result and develop an Ostrowski-type inequality for weighted integrals. Applications to special weight functions and numerical integration are investigated.

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2 Preliminaries

In the next section weighted (or product) integral inequalities are constructed. The weight function (or density) is assumed to be non-negative and integrable over its entire domain. The following generic quantitative measures of the weight are defined.

Definition 3. *Let $w : (a, b) \to [0, \infty)$ be an integrable function, i.e. $\int_a^b w(t)\,dt < \infty$, then define*

$$m_i(a, b) = \int_a^b t^i w(t)\,dt, \qquad i = 0, 1, \ldots \tag{2.1}$$

as the i^{th} moment *of w.*

Definition 4. *Define the* mean *of the interval $[a, b]$ with respect to the density w as*

$$\mu(a, b) = \frac{m_1(a, b)}{m_0(a, b)} \tag{2.2}$$

and the variance *by*

$$\sigma^2(a, b) = \frac{m_2(a, b)}{m_0(a, b)} - \mu^2(a, b). \tag{2.3}$$

3 The Results

3.1 1-point inequality

Theorem 3.1. *Let $f, w : (a, b) \to \mathbb{R}$ be two mappings on (a, b) with the following properties:*

(1) $\sup_{t\in(a,b)} |f''(t)| < \infty$,

(2) $w(t) \geq 0 \quad \forall t \in (a, b)$,

(3) $\int_a^b w(t)\,dt < \infty$,

then the following inequalities hold

$$\left|\frac{1}{m_0(a, b)}\int_a^b w(t)f(t)\,dt - f(x) + \big(x - \mu(a, b)\big)f'(x)\right|$$

$$\leq \frac{\|f''\|_\infty}{2}\left[\big(x - \mu(a, b)\big)^2 + \sigma^2(a, b)\right] \tag{3.1}$$

$$\leq \frac{\|f''\|_\infty}{2}\left(\left|x - \frac{a + b}{2}\right| + \frac{b - a}{2}\right)^2 \tag{3.2}$$

for all $x \in [a, b]$.

Proof. Define the mapping $K(\cdot, \cdot) : [a, b]^2 \to \mathbb{R}$ by

$$K(x, t) := \begin{cases} \int_a^t (t - u)w(u)\,du, & a \leq t \leq x, \\ \int_b^t (t - u)w(u)\,du, & x < t \leq b. \end{cases}$$

Integrating by parts gives

$$\begin{aligned}\int_a^b K(x,t)f''(t)\,dt &= \int_a^x\int_a^t (t-u)w(u)f''(t)\,dudt + \int_x^b\int_b^t (t-u)w(u)f''(t)\,dudt\\ &= f'(x)\int_a^b (x-u)w(u)\,du\\ &\quad -\int_a^x\int_a^t (t-u)w(u)f'(t)\,dudt - \int_x^b\int_b^t (t-u)w(u)f'(t)\,dudt\\ &= \int_a^b w(t)f(t)\,dt + f'(x)\int_a^b (x-u)w(u)\,du - f(x)\int_a^b w(u)\,du\end{aligned}$$

providing the identity

$$\begin{aligned}(3.3)\qquad &\int_a^b K(x,t)f''(t)\,dt\\ &= \int_a^b w(t)f(t)\,dt - m_0(a,b)f(x) + m_0(a,b)\big(x-\mu(a,b)\big)f'(x)\end{aligned}$$

that is valid for all $x\in[a,b]$.

Now taking the modulus of (3.3) we have,

$$\begin{aligned}(3.4)\qquad &\left|\int_a^b w(t)f(t)\,dt - m_0(a,b)f(x) + m_0(a,b)\big(x-\mu(a,b)\big)f'(x)\right|\\ &= \left|\int_a^b K(x,t)f''(t)\,dt\right|\\ &\le \|f''\|_\infty\int_a^b |K(x,t)|\,dt\\ &= \|f''\|_\infty\left[\int_a^x\int_a^t (t-u)w(u)\,dudt + \int_x^b\int_b^t (t-u)w(u)\,dudt\right]\\ &= \frac{\|f''\|_\infty}{2}\int_a^b (x-t)^2 w(t)\,dt.\end{aligned}$$

The last line being computed by reversing the order of integration and evaluating the inner integrals. To obtain the desired result (3.1) observe that

$$\int_a^b (x-t)^2 w(t)\,dt = m_0(a,b)\Big[\big(x-\mu(a,b)\big)^2 + \sigma^2(a,b)\Big].$$

To obtain (3.2) note that

$$\begin{aligned}\int_a^b (x-t)^2\,dt &\le \sup_{t\in[a,b]}(x-t)^2 m_o(a,b)\\ &= \max\{(x-a)^2,(x-b)^2\}m_0(a,b)\\ &= \frac{1}{2}\left((x-a)^2+(x-b)^2+\left|(x-a)^2-(x-b)^2\right|\right)m_0(a,b)\\ &= \left(\left|x-\frac{a+b}{2}\right| + \frac{b-a}{2}\right)^2 m_0(a,b)\end{aligned}$$

which upon susbitution into (3.4) furnishes the result. ■

Note also that the inequality (3.1) is valid even for unbounded w or interval $[a, b]$. This is not the case with (1.2).

Corollary 3.2. *The inequality* (3.1) *is minimized at* $x = \mu(a, b)$ *producing the generalized "mid-point" inequality*

$$\left|\frac{1}{m_0(a,b)}\int_a^b w(t)f(t)\,dt - f(\mu(a,b))\right| \le \|f''\|_\infty \frac{\sigma^2(a,b)}{2}. \tag{3.5}$$

Proof. Substituting $\mu(a, b)$ for x in (3.1) produces the desired result. Note that $x = \mu(a, b)$ not only minimizes the bound of the inequality (3.1), but also causes the derivative term to vanish. ■

The optimal point (2.2) can be interpreted in many ways. In a physical context, $\mu(a, b)$ represents the centre of mass of a one dimensional rod with mass density w. Equivalently, this point can be viewed as that which minimizes the error variance for the probability density w (see [4] for an application). Finally (2.2) is also the Gauss node point for a one-point rule [12]. The bound in (3.5) is directly proportional to the variance of the density w. So that the tightest bound is achieved by sampling at the mean point of the interval (a, b), while its value is given by the variance.

3.2 2-point inequality

Here a two point analogy of (3.1) is developed where the result is extended to create an inequality with two independent parameters x_1 and x_2. This is mainly used (Section 5) to find an optimal grid for composite weighted-quadrature rules.

Theorem 3.3. *Let the conditions of Theorem 3.1 hold, then the following 2-point inequality is obtained*

$$\begin{aligned}\Big|\int_a^b w(t)f(t)\,dt &- m_0(a,\xi)f(x_1) + m_0(a,\xi)\big(x_1-\mu(a,\xi)\big)f'(x_1)\\ &- m_0(\xi,b)f(x_2) + m_0(\xi,b)\big(x_2-\mu(\xi,b)\big)f'(x_2)\Big|\\ &\le \frac{\|f''\|_\infty}{2}\Big\{m_0(a,\xi)\Big[\big(x_1-\mu(a,\xi)\big)^2+\sigma^2(a,\xi)\Big]\\ &\qquad + m_0(\xi,b)\Big[\big(x_2-\mu(\xi,b)\big)^2+\sigma^2(\xi,b)\Big]\Big\}\end{aligned} \tag{3.6}$$

for all $a \le x_1 < \xi < x_2 \le b$.

Proof. Define the mapping $K(\cdot,\cdot,\cdot,\cdot) : [a, b]^4 \to \mathbb{R}$ by

$$K(x_1,x_2,\xi,t) := \begin{cases}\int_a^t (t-u)w(u)\,du, & a \le t \le x_1,\\ \int_\xi^t (t-u)w(u)\,du, & x_1 < t, \xi < x_2,\\ \int_b^t (t-u)w(u)\,du, & x_2 \le t \le b.\end{cases}$$

With this kernel, the proof is almost identical to that of Theorem 3.1.

Integrating by parts produces the integral identity

$$\begin{aligned}\int_a^b K(x_1,x_2,\xi,t) f''(t)\,dt &= \int_a^b w(t)f(t)\,dt - m_0(a,\xi)f(x_1) + m_0(a,b)\big(x-\mu(a,\xi)\big)f'(x_1)\\ &\qquad - m_0(\xi,b)f(x_2) + m_0(\xi,b)\big(x-\mu(\xi,b)\big)f'(x_2).\end{aligned} \tag{3.7}$$

Re-arranging and taking bounds produces the result (3.6). ∎

Corollary 3.4. *The optimal location of the points x_1, x_2 and ξ satisfy*

$$x_1 = \mu(a,\xi), \qquad x_2 = \mu(\xi,b), \qquad \xi = \frac{\mu(a,\xi)+\mu(\xi,b)}{2} \tag{3.8}$$

Proof. By inspection of the right hand side of (3.6) it is obvious that choosing

$$x_1 = \mu(a,\xi) \qquad \text{and} \qquad x_2 = \mu(\xi,b) \tag{3.9}$$

minimizes this quantity. To find the optimal value for ξ write the expression in braces in (3.6) as

$$\begin{aligned}2\int_a^b |K(x_1,x_2,\xi,t)|\,dt &= m_0(a,\xi)\Big[\big(x_1-\mu(a,\xi)\big)^2 + \sigma^2(a,\xi)\Big]\\ &\quad + m_0(\xi,b)\Big[\big(x_2-\mu(\xi,b)\big)^2 + \sigma^2(\xi,b)\Big]\\ &= \int_a^\xi (x_1-t)^2 w(t)\,dt + \int_\xi^b (x_2-t)^2 w(t)\,dt.\end{aligned} \tag{3.10}$$

Substituting (3.9) into the right hand side of (3.10) and differentiating with respect to ξ gives

$$\frac{d}{d\xi}\int_a^b |K(\mu(a,\xi),\mu(\xi,b),\xi,t)|\,dt = \big(\mu(\xi,b) - \mu(\xi,a)\big)\left(\xi - \frac{\mu(a,\xi)+\mu(\xi,b)}{2}\right) w(\xi).$$

Assuming $w(\xi) \neq 0$, then this equation possesses only one root. A minimum exists at this root since (3.10) is convex, and so the corollary is proved. ∎

Equation (3.8) shows not only where sampling should occur within each subinterval (i.e. x_1 and x_2), but how the domain should be divided to make up these subintervals (ξ).

4 Some Weighted Integral Inequalities

Integration with weight functions are used in countless mathematical problems. Two main areas are: (i) approximation theory and spectral analysis and (ii) statistical analysis and the theory of distributions.

In this section (3.1) is evaluated for the more popular weight functions. In each case (1.2) cannot be used since the weight $w(t)$ or the interval $(b-a)$ is unbounded. The optimal point (2.2) is easily identified.

4.1 Uniform (Legendre)

Substituting $w(t) = 1$ into (2.2) and (2.3) gives

$$\mu(a,b) = \frac{\int_a^b t\,dt}{\int_a^b dt} = \frac{a+b}{2} \tag{4.1}$$

and

$$\sigma^2(a,b) = \frac{\int_a^b t^2\,dt}{\int_a^b dt} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

respectively. Substituting into (3.1) produces (1.2). Note that the interval mean is simply the midpoint (4.1).

4.2 Logarithm

This weight is present in many physical problems; the main body of which exhibit some axial symmetry. Special logarithmic rules are used extensively in the Boundary Element Method popularized by Brebbia (see for example [2]). Some applications of which include bubble cavitation [3] and viscous drop deformation ([8] and more recently by [9]).

With $w(t) = \ln(1/t)$, $a = 0$, $b = 1$, (2.2) and (2.3) are

$$\mu(0,1) = \frac{\int_0^1 t\ln(1/t)\,dt}{\int_0^1 \ln(1/t)\,dt} = \frac{1}{4}$$

and

$$\sigma^2(0,1) = \frac{\int_0^1 t^2\ln(1/t)\,dt}{\int_0^1 \ln(1/t)\,dt} - \left(\frac{1}{4}\right)^2 = \frac{7}{144}$$

respectively. Substituting into (3.1) gives

$$\left|\int_0^1 \ln(1/t)f(t)\,dt - f(x) + \left(x - \frac{1}{4}\right)f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{7}{144} + \left(x - \frac{1}{4}\right)^2\right).$$

The optimal point

$$x = \mu(0,1) = \frac{1}{4}$$

is closer to the origin than the midpoint (4.1) reflecting the strength of the log singularity.

4.3 Jacobi

Substituting $w(t) = 1/\sqrt{t}$, $a = 0$, $b = 1$ into (2.2) and (2.3) gives

$$\mu(0,1) = \frac{\int_0^1 \sqrt{t}\,dt}{\int_0^1 1/\sqrt{t}\,dt} = \frac{1}{3}$$

and

$$\sigma^2(0,1) = \frac{\int_0^1 t\sqrt{t}\,dt}{\int_0^1 1/\sqrt{t}\,dt} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$$

respectively. Hence, the inequality for a Jacobi weight is

$$\left|\frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}\,dt - f(x) + \left(x - \frac{1}{3}\right) f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{4}{45} + \left(x - \frac{1}{3}\right)^2\right).$$

The optimal point

$$x = \mu(0,1) = \frac{1}{3}$$

is again shifted to the left of the mid-point due to the $t^{-1/2}$ singularity at the origin.

4.4 Chebyshev

The mean and variance for the Chebyshev weight $w(t) = 1/\sqrt{1-t^2}$, $a = -1, b = 1$ are

$$\mu(-1,1) = \frac{\int_{-1}^1 t/\sqrt{1-t^2}\,dt}{\int_{-1}^1 1/\sqrt{1-t^2}\,dt} = 0$$

and

$$\sigma^2(-1,1) = \frac{\int_{-1}^1 t^2\sqrt{1-t^2}\,dt}{\int_{-1}^1 1/\sqrt{1-t^2}\,dt} - 0^2 = \frac{1}{2}$$

respectively. Hence, the inequality corresponding to the Chebyshev weight is

$$\left|\frac{1}{\pi}\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt - f(x) + xf'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{1}{2} + x^2\right).$$

The optimal point

$$x = \mu(-1,1) = 0$$

is at the mid-point of the interval reflecting the symmetry of the Chebyshev weight over its interval.

4.5 Laguerre

The conditions in Theorem 3.1 are not violated if the integral domain is infinite. The Laguerre weight $w(t) = e^{-t}$ is defined for positive values, $t \in [0, \infty)$. The mean and variance of the Laguerre weight are

$$\mu(0,\infty) = \frac{\int_0^\infty te^{-t}\,dt}{\int_0^\infty e^{-t}\,dt} = 1$$

and

$$\sigma^2(0,\infty) = \frac{\int_0^\infty t^2e^{-t}\,dt}{\int_0^\infty e^{-t}\,dt} - 1^2 = 1$$

respectively.

The appropriate inequality is

$$\left|\int_0^\infty e^{-t}f(t)\,dt - f(x) + (x-1)f'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(1 + (x-1)^2\right),$$

from which the optimal sample point of $x = 1$ may be deduced.

4.6 Hermite

Finally, the Hermite weight is $w(t) = e^{-t^2}$ defined over the entire real line. The mean and variance for this weight are

$$\mu(-\infty,\infty) = \frac{\int_{-\infty}^{\infty} te^{-t^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\,dt} = 0$$

and

$$\sigma^2(-\infty,\infty) = \frac{\int_{-\infty}^{\infty} t^2e^{-t^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\,dt} - 0^2 = \frac{1}{2}$$

respectively.

The inequality from Theorem 3.1 with the Hermite weight function is thus

$$\left|\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} f(t)\,dt - f(x) + xf'(x)\right| \le \frac{\|f''\|_\infty}{2}\left(\frac{1}{2} + x^2\right),$$

which results in an optimal sampling point of $x = 0$.

5 Application in Numerical Integration

Define a grid $I_n : a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = b$ on the interval [a,b], with $x_i \in [\xi_i, \xi_{i+1}]$ for $i = 0, 1, \ldots, n-1$. The following quadrature formulae for weighted integrals are obtained.

Theorem 5.1. *Let the conditions in Theorem 3.1 hold. The following weighted quadrature rule holds*

$$\int_a^b w(t)f(t)\,dt = A(f,\boldsymbol{\xi},\boldsymbol{x}) + R(f,\boldsymbol{\xi},\boldsymbol{x}) \tag{5.1}$$

where

$$A(f,\boldsymbol{\xi},\boldsymbol{x}) = \sum_{i=0}^{n-1}\big(h_i f(x_i) - h_i(x_i - \mu_i)f'(x_i)\big)$$

and

$$|R(f,\boldsymbol{\xi},\boldsymbol{x})| \le \frac{\|f''\|_\infty}{2}\sum_{i=0}^{n-1}\left[(x_i-\mu_i)^2 + \sigma_i^2\right]h_i. \tag{5.2}$$

The parameters h_i, μ_i and σ_i^2 are given by

$$h_i = m_0(\xi_i, \xi_{i+1}), \qquad \mu_i = \mu(\xi_i,\xi_{i+1}), \qquad \text{and} \qquad \sigma_i^2 = \sigma^2(\xi_i,\xi_{i+1})$$

respectively.

Proof. Apply Theorem 3.1 over the interval $[\xi_i, \xi_{i+1}]$ with $x = x_i$ to obtain

$$\left|\int_{\xi_i}^{\xi_{i+1}} w(t)f(t)\,dt - h_i f(x_i) + h_i(x_i-\mu_i)f'(x_i)\right| \le \frac{\|f''\|_\infty}{2} h_i\left((x_i-\mu_i)^2 + \sigma_i^2\right).$$

Summing over i from 0 to $n-1$ and using the triangle inequality produces the desired result. ■

Corollary 5.2. *The optimal location of the points x_i, $i = 0, 1, 2, \ldots, n-1$, and grid distribution I_n satisfy*

$$x_i = \mu_i, \qquad i = 0, 1, \ldots, n-1 \qquad \textit{and} \tag{5.3}$$

$$\xi_i = \frac{\mu_{i-1} + \mu_i}{2}, \qquad i = 1, 2, \ldots, n-1, \tag{5.4}$$

producing the composite generalized mid-point rule for weighted integrals

$$\int_a^b w(t) f(t)\, dt = \sum_{i=0}^{n-1} h_i f(x_i) + R(f, \boldsymbol{\xi}, n) \tag{5.5}$$

where the remainder is bounded by

$$|R(f, \boldsymbol{\xi}, n)| \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} h_i \sigma_i^2 \tag{5.6}$$

Proof. The proof follows that of Corollary 3.4 where it is observed that the minimum bound (5.2) will occur at $x_i = \mu_i$. Differentiating the right hand side of (5.2) gives

$$\frac{d}{d\xi_i} \sum_{j=0}^{n-1} \left[(x_j - \mu_j)^2 + \sigma_j^2 \right] h_j = 2w(\xi_i)(x_i - x_{i-1}) \left(\xi_i - \frac{x_{i-1} + x_i}{2} \right).$$

Inspection of the second derivative at the root reveals that the stationary point is a minimum and hence the result is proved. ■

6 Numerical Results

In this section, for illustratration, the quadrature rule of Section 5 is used on the integral

$$\int_0^1 100t \ln(1/t) \cos(4\pi t)\, dt = -1.972189325199166 \tag{6.1}$$

This is evaluated using the following three rules:

(1) the composite mid-point rule, where the grid has a uniform step-size and the node is simply the mid-point of each sub-interval,

(2) the composite generalized mid-point rule (5.1). The grid, I_n, is uniform and the nodes are the mean point of each sub-interval (5.3),

(3) equation (5.5) where the grid is distributed according to (5.4) and the nodes are the sub-interval means (5.3).

Table 1 shows the numerical error of each method for an increasing number of sample points.

For a uniform grid, it can be seen that changing the location of the sampling point from the midpoint [method (1)] to the mean point [method (2)] roughly doubles the accuracy. Changing the grid distribution as well as the node point

n	Error (1)	Error (2)	Error (3)	Error ratio (3)	Bound ratio (3)
4	1.97(0)	2.38(0)	2.48(0)	–	–
8	3.41(-1)	2.93(-1)	2.35(-1)	10.56	3.90
16	8.63(-2)	5.68(-2)	2.62(-2)	8.97	3.95
32	2.37(-2)	1.31(-2)	4.34(-3)	6.04	3.97
64	6.58(-3)	3.20 (-3)	9.34(-4)	4.65	3.99
128	1.82(-3)	7.94(-4)	2.23(-4)	4.18	3.99
256	4.98(-4)	1.98(-4)	5.51(-5)	4.05	4.00

Table 1: The error in evaluating (6.1) under different quadrature rules. The parameter n is the number of sample points.

[method (3)] from the composite mid-point rule [method (1)] increases the accuracy by approximately an order of magnitude. It is important to note that the nodes and weights for method (3) can be easily calculated numerically using an iterative scheme. For example on a Pentium-90 personal computer, with $n = 64$, calculating (5.3) and (5.4) took close to 37 seconds.

Note that equations (5.3) and (5.4) are quite general in nature and only rely on the weight insofar as knowledge of the first two moments is required. This contrasts with Gaussian quadrature where for an n point rule, the first $n + 1$ moments are needed (or equivalently the $2n+1$ coefficients of the continued fraction expansion [11, 10]) to construct the appropriate orthogonal polynomial and then a root-finding procedure is called to find the abscissae [1]. This procedure, of course, can be greatly simplified for the more well known weight functions [6].

The second last column of Table 1 shows the ratio of the numerical errors for method (3) and the last column the ratio of the theoretical error bound (5.5)

$$\text{Bound ratio (3)} = \frac{|R(f, \boldsymbol{\xi}, n/2)|}{|R(f, \boldsymbol{\xi}, n)|}. \tag{6.2}$$

As n increases the numerical ratio approaches the theoretical one. The theoretical ratio is consistently close to 4. This value suggests an asymptotic form of the error bound

$$|R(f, \boldsymbol{\xi}, n)| \sim O\left(\frac{1}{n^2}\right) \tag{6.3}$$

for the log weight. Similiar results have been obtained for the other weights of Section 4. This is consistent with mid-point type rules and it is anticipated that developing other product rules, for example a generalized trapezoidal or Simpsons rule, will yield more accurate results.

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