

AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $L_1(A, B)$ AND APPLICATIONS

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ABSTRACT. An inequality of Ostrowski type for twice differentiable mappings whose derivatives belong to $L_1(a, b)$ and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

1 INTRODUCTION

In 1938, Ostrowski (see for example [3, p. 468]) proved the following integral inequality:

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$, then we have the inequality:*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In paper [2], the same authors considered another inequality of Ostrowski type for $\|\cdot\|_1$ norm as follows:

Theorem 1.2. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping in I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1[a, b]$, then we have the inequality:*

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{(b-a)} \right] \|f'\|_1$$

for all $x \in [a, b]$.

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for n -time differentiable mappings (see for example [3, p. 468]) from which we would like to mention only the case of twice differentiable mappings [3, p. 470]:

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Theorem 1.3. Let $f : [a, b] \rightarrow \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) , i.e., $\|f''\|_\infty = \sup_{t \in (a, b)} |f''(t)| < \infty$. Then we have the inequality:

$$(1.3) \quad \left| \frac{1}{2} \left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{\|f''\|_\infty}{4} (b-a)^2 \left[\frac{1}{12} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]$$

for all $x \in (a, b)$.

In this paper we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|\cdot\|_1$ -norm of the second derivative f'' and apply it in numerical integration and for some special means such as : logarithmic mean, identric mean, p -logarithmic mean etc.

2 SOME INTEGRAL INEQUALITIES

The following inequality of Ostrowski's type for mappings which are twice differentiable, holds.

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$, twice differentiable on (a, b) and $f'' \in L_1(a, b)$. Then we have the inequality:

$$(2.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ \leq \frac{1}{2(b-a)} \left(\left|x - \frac{a+b}{2}\right| + \frac{1}{2}(b-a) \right)^2 \|f''\|_1 \leq \frac{b-a}{2} \|f''\|_1$$

for all $x \in [a, b]$.

Proof. Let us define the mapping $K(\cdot, \cdot) : [a, b]^2 \rightarrow \mathbf{R}$ given by

$$K(x, t) := \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a, x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x, b] \end{cases}.$$

Integrating by parts, we have successively

$$\int_a^b K(x, t) f''(t) dt \\ = \int_a^x \frac{(t-a)^2}{2} f''(t) dt + \int_x^b \frac{(t-b)^2}{2} f''(t) dt \\ = \frac{(t-a)^2}{2} f'(t) \Big|_a^x - \int_a^x (t-a) f'(t) dt + \frac{(t-b)^2}{2} f'(t) \Big|_x^b - \int_x^b (t-b) f'(t) dt$$

$$\begin{aligned}
&= \frac{(x-a)^2}{2} f'(x) - \left[(t-a) f(t) \Big|_a^x - \int_a^x f(t) dt \right] \\
&\quad - \frac{(b-x)^2}{2} f'(x) - \left[(t-b) f(t) \Big|_x^b - \int_x^b f(t) dt \right] \\
&= \frac{1}{2} \left[(x-a)^2 - (b-x)^2 \right] f'(x) - (x-a) f(x) \\
&\quad + \int_a^x f(t) dt + (x-b) f(x) + \int_x^b f(t) dt \\
&= (b-a) \left(x - \frac{a+b}{2} \right) f'(x) - (b-a) f(x) + \int_a^b f(t) dt
\end{aligned}$$

from where we get the integral identity:

$$(2.2) \quad \int_a^b f(t) dt = (b-a) f(x) - (b-a) \left(x - \frac{a+b}{2} \right) f'(x) + \int_a^b K(x, t) f''(t) dt$$

for all $x \in [a, b]$, which is interesting in itself, too.

Using the identity (2.2) we have

$$\begin{aligned}
(2.3) \quad &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) f'(x) \right| \\
&= \frac{1}{b-a} \left| \int_a^b K(x, t) f''(t) dt \right| \\
&= \frac{1}{b-a} \left| \int_a^x \frac{(t-a)^2}{2} f''(t) dt + \int_x^b \frac{(t-b)^2}{2} f''(t) dt \right| \\
&\leq \frac{1}{b-a} \left[\int_a^x \frac{(t-a)^2}{2} |f''(t)| dt + \int_x^b \frac{(t-b)^2}{2} |f''(t)| dt \right] \\
&\leq \frac{1}{b-a} \left[\frac{(x-a)^2}{2} \int_a^x |f''(t)| dt + \frac{(b-x)^2}{2} \int_x^b |f''(t)| dt \right] \\
&\leq \frac{1}{b-a} \max \left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\} \left[\int_a^x |f''(t)| dt + \int_x^b |f''(t)| dt \right].
\end{aligned}$$

Now, let observe that

$$\max \left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{(x-a)^2 + (b-x)^2 + |(b-x)^2 - (x-a)^2|}{2} \\
&= \frac{1}{2} \left[\frac{(x-a)^2 + (b-x)^2}{2} + (b-a) \left| x - \frac{a+b}{2} \right| \right] \\
&= \frac{1}{2} \left[\frac{1}{4} (b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 + (b-a) \left| x - \frac{a+b}{2} \right| \right] \\
&= \frac{1}{2} \left(\left| x - \frac{a+b}{2} \right| + \frac{1}{2} (b-a) \right)^2.
\end{aligned}$$

Using (2.3) we deduce the desired inequality (2.1). ■

Corollary 2.2. *Let f be as above. Then we have the mid-point inequality:*

$$(2.4) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{8} (b-a) \|f''\|_1.$$

The following trapezoid inequality also holds:

Corollary 2.3. *Under the above assumptions we have:*

$$(2.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{4} (f'(b) - f'(a)) \right| \leq \frac{1}{2} (b-a) \|f''\|_1.$$

Proof. Choose in (2.1) $x = a$ and $x = b$ to get:

$$\left| f(a) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{b-a}{2} f'(a) \right| \leq \frac{b-a}{2} \|f''\|_1$$

and

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{b-a}{2} f'(b) \right| \leq \frac{b-a}{2} \|f''\|_1.$$

Adding the above two inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (2.5). ■

3 APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$). We have the following quadrature formula:

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and twice differentiable on (a, b) , whose second derivative $f'' : (a, b) \rightarrow \mathbf{R}$ belongs to $L_1(a, b)$, i.e.,*

$\|f''\|_1 := \int_a^b |f''(t)| dt < \infty$. *Then the following perturbed Riemann's type quadrature formula holds:*

$$(3.1) \quad \int_a^b f(x) dx = A(f, f', \xi, I_n) + R(f, f', \xi, I_n)$$

where

$$A(f, f', \xi, I_n) = \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i) \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) h_i$$

and the remainder satisfies the estimation:

$$(3.2) \quad |R(f, f', \xi, I_n)| \leq \frac{1}{2} \left[\frac{1}{2} \nu(h) + \sup_{i=0, \dots, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2 \|f''\|_1$$

$$\leq \frac{\nu^2(h)}{2} \|f''\|_1$$

for all ξ_i as above, where $\nu(h) = \max\{x_{i+1} - x_i | i = 0, \dots, n-1\}$.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) to get

$$\left| \int_{x_i}^{x_{i+1}} f(t) dt - h_i f(\xi_i) + \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) f'(\xi_i) \right|$$

$$\leq \frac{1}{2} \left(\left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| + \frac{1}{2} (x_{i+1} - x_i) \right)^2 \int_{x_i}^{x_{i+1}} |f''(t)| dt.$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce :

$$|R(f, f', \xi, I_n)| \leq \frac{1}{2} \sum_{i=0}^{n-1} \left[\frac{1}{2} (x_{i+1} - x_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2 \int_{x_i}^{x_{i+1}} |f''(t)| dt$$

$$\leq \frac{1}{2} \sup_{i=0, \dots, n-1} \left[\frac{1}{2} (x_{i+1} - x_i) + \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)| dt$$

$$\leq \frac{1}{2} \left[\frac{1}{2} \nu(h) + \sup_{i=0, \dots, n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right]^2 \|f''\|_1$$

and the estimation (3.2) is obtained. ■

Remark 3.1. If we choose above $\xi_i = \frac{x_i + x_{i+1}}{2}$, we recapture the midpoint quadrature formula

$$\int_a^b f(x) dx = A_M(f, I_n) + R_M(f, I_n)$$

where the remainder $R_M(f, I_n)$ satisfies the estimation

$$|R_M(f, I_n)| \leq \frac{1}{8} \nu^2(h) \|f''\|_1.$$

4 APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means :

(a) The arithmetic mean

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;$$

(b) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

(c) The harmonic mean:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0;$$

(d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(e) The identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \quad a, b > 0;$$

(f) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where, $p \in \mathbf{R} \setminus \{-1, 0\}$, $a, b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

Consider the mapping $f : (0, \infty) \rightarrow \mathbf{R}$, $f(x) = x^r$, $r \in \mathbf{R} \setminus \{-1, 0\}$.

Then we have for $0 < a < b$:

$$\frac{1}{b-a} \int_a^b f(x) dx = L_r^r(a, b)$$

and

$$\|f''\|_1 = |r(r-1)| (b-a) L_{r-1}^{r-1}(a, b).$$

Using the inequality (2.1) we get :

$$(4.1) \quad \begin{aligned} & \left| x^r - L_r^r - r(x-A)x^{r-1} \right| \\ & \leq \frac{1}{2} \left[|x-A| + \frac{1}{2}(b-a) \right]^2 |r(r-1)| L_{r-1}^{r-1} \end{aligned}$$

for all $x \in [a, b]$.

If in (4.1) we choose $x = A$, we get

$$(4.2) \quad |A^r - L_r^r| \leq \frac{|r(r-1)|(b-a)^2}{8} L_{r-1}^{r-1}.$$

Consider the mapping $f : (0, \infty) \rightarrow \mathbf{R}, f(x) = \frac{1}{x}$.

Then we have for $0 < a < b$:

$$\frac{1}{b-a} \int_a^b f(x) dx = L^{-1}(a, b)$$

and

$$\|f''\|_1 = 2(b-a)L_{-3}^{-3}(a, b).$$

Using the inequality (2.1), we get :

$$\left| \frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2} \right| \leq L_{-3}^{-3} \left[|x-A| + \frac{1}{2}(b-a) \right]^2$$

which is equivalent to

$$(4.3) \quad |x(L-x) + L(x-A)| \leq x^2 L L_{-3}^{-3} \left[|x-A| + \frac{1}{2}(b-a) \right]^2$$

for all $x \in [a, b]$.

Now, if we choose in (4.3), $x = A$, we get

$$(4.4) \quad 0 \leq A - L \leq \frac{1}{4} A L L_{-3}^{-3} (b-a)^2.$$

If in (4.3) we choose $x = L$, we get

$$(4.5) \quad 0 \leq A - L \leq L^2 L_{-3}^{-3} \left[L - A + \frac{1}{2}(b-a) \right]^2.$$

Let us consider the mapping $f(x) = \ln x$, $x \in [a, b] \subset (0, \infty)$.

Then we have :

$$\frac{1}{b-a} \int_a^b f(x) dx = \ln I(a, b),$$

and

$$\|f''\|_1 = (b-a)L_{-2}^{-2}(a, b).$$

Then the inequality (2.1) gives us

$$(4.6) \quad \left| \ln x - \ln I - \frac{x - A}{x} \right| \\ \leq \frac{1}{2} \left[|x - A| + \frac{1}{2} (b - a) \right]^2 L_{-2}^{-2}$$

for all $x \in [a, b]$.

Now, if in (4.6) we choose $x = A$, we get

$$(4.7) \quad 1 \leq \frac{A}{I} \leq \exp \left[\frac{1}{8} (b - a)^2 L_{-2}^{-2} \right].$$

If in (4.6) we choose $x = I$, we get

$$(4.8) \quad 0 \leq A - I \leq \frac{I}{2} \left[A - I + \frac{1}{2} (b - a) \right]^2 L_{-2}^{-2}.$$

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