

AN OSTROWSKI TYPE INEQUALITY FOR DOUBLE INTEGRALS IN TERMS OF L_p - NORMS AND APPLICATIONS IN NUMERICAL INTEGRATION

S.S. DRAGOMIR, N.S. BARNETT AND P. CERONE

ABSTRACT. An inequality of the Ostrowski type for double integrals and applications in Numerical Analysis in connection with cubature formulae are given.

1 INTRODUCTION

In 1938, A. Ostrowski proved the following integral inequality [5, p. 468]

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbf{R}$ is bounded on (a, b) , i.e.,*

$\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

For some generalizations see the book [5, p. 468-484] by Mitrinović, Pečarić and Fink.

Some applications of the above results in Numerical Integration and for special means have been given in [3] by S.S. Dragomir and S. Wang.

In [4] Dragomir and Wang established the following Ostrowski type inequality for differentiable mappings whose derivatives belong to L_p -spaces.

Theorem 1.2. *Let $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_p(a, b)$ ($p > 1, \frac{1}{p} + \frac{1}{q} = 1$), then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p$$

for all $x \in [a, b]$, where $\|f'\|_p := \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}}$, is the $L_p(a, b)$ -norm.

Note that the above inequality can also be obtained from Theorem 1.1 [5, p. 471] due to A.M. Fink.

For other Ostrowski type inequalities, see the papers [1, 2 and 4].

In 1975, G.N. Milovanović generalized Theorem 1.1 where f is a function of several variables [5, p. 468].

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Theorem 1.3. Let $f : \mathbf{R}^m \rightarrow \mathbf{R}$ be a differentiable function defined on $D = \{(x_1, \dots, x_m) \mid a_i \leq x_i \leq b_i (i = 1, \dots, m)\}$ and let $\left| \frac{\partial f}{\partial x_i} \right| \leq M_i (M_i > 0, i = 1, \dots, m)$ in D . Furthermore, let function $x \mapsto p(x)$ be integrable and $p(x) > 0$ for every $x \in D$. Then for every $x \in D$, we have the inequality:

$$\left| f(x) - \frac{\int_D p(y) f(y) dy}{\int_D p(y) dy} \right| \leq \frac{\sum_{i=1}^m M_i \int_D p(y) |x_i - y_i| dy}{\int_D p(y) dy}.$$

In the present paper we point out an Ostrowski type inequality for double integrals in terms of L_p -norms and apply it in Numerical Integration obtaining a general cubature formula.

2 THE RESULTS

The following inequality of Ostrowski's type for mappings of two variables holds:

Theorem 2.1. Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be a continuous mapping on $[a, b] \times [c, d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a, b) \times (c, d)$ and is in $L_p((a, b) \times (c, d))$, i.e.,

$$\|f''_{s,t}\|_p := \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right|^p dx dy \right)^{\frac{1}{p}} < \infty, \quad p > 1$$

then we have the inequality:

$$\begin{aligned} (2.1) \quad & \left| \int_a^b \int_c^d f(s,t) ds dt - [(b-a) \int_c^d f(x,t) dt + (d-c) \int_a^b f(s,y) ds \right. \\ & \left. - (d-c)(b-a)f(x,y)] \right| \\ & \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f''_{s,t}\|_p \end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Integrating by parts successively, we have the equality:

$$\begin{aligned} (2.2) \quad & \int_a^x \int_c^y (s-a)(t-c) f''_{s,t}(s,t) dt ds \\ & = (y-c)(x-a)f(x,y) - (y-c) \int_a^x f(s,y) ds \\ & \quad - (x-a) \int_c^y f(x,t) dt + \int_a^x \int_c^y f(s,t) ds dt. \end{aligned}$$

By similar computations we have,

$$\begin{aligned}
 (2.3) \quad & \int_a^x \int_y^d (s-a)(t-d) f''_{s,t}(s,t) ds dt \\
 &= (x-a)(d-y) f(x,y) - (d-y) \int_a^x f(s,y) ds \\
 &\quad - (x-a) \int_y^d f(x,t) dt + \int_a^x \int_c^y f(s,t) ds dt.
 \end{aligned}$$

Now,

$$\begin{aligned}
 (2.4) \quad & \int_x^b \int_y^d (s-b)(t-d) f''_{s,t}(s,t) ds dt \\
 &= (d-y)(b-x) f(x,y) - (d-y) \int_x^b f(s,y) ds \\
 &\quad - (b-x) \int_y^d f(x,t) dt + \int_x^b \int_y^d f(s,t) ds dt
 \end{aligned}$$

and finally

$$\begin{aligned}
 (2.5) \quad & \int_x^b \int_c^y (s-b)(t-c) f''_{s,t}(s,t) ds dt \\
 &= (y-c)(b-x) f(x,y) - (y-c) \int_x^b f(s,y) ds \\
 &\quad - (b-x) \int_c^y f(x,t) dt + \int_x^b \int_c^y f(s,t) ds dt.
 \end{aligned}$$

If we add the equalities (2.2) – (2.5) we get, in the right hand side:

$$\begin{aligned}
 & [(y-c)(x-a) + (x-a)(d-y) \\
 & + (d-y)(b-x) + (y-c)(b-x)] f(x,y)
 \end{aligned}$$

$$\begin{aligned}
& - (d - c) \int_a^x f(s, y) ds - (d - c) \int_x^b f(s, y) ds - (b - a) \int_c^y f(x, t) dt \\
& - (b - a) \int_y^d f(x, t) dt + \int_a^x \int_c^y f(s, t) ds dt + \int_a^x \int_y^d f(s, t) ds dt \\
& + \int_x^b \int_y^d f(s, t) ds dt + \int_x^b \int_c^y f(s, t) ds dt \\
& = (d - c) (b - a) f(x, y) - (d - c) \int_a^b f(s, y) ds \\
& - (b - a) \int_c^d f(x, t) dt + \int_a^b \int_c^d f(s, t) ds dt.
\end{aligned}$$

For the first part, let us define the kernels: $p : [a, b]^2 \rightarrow \mathbf{R}$, $q : [c, d]^2 \rightarrow \mathbf{R}$ given by:

$$p(x, s) := \begin{cases} s - a & \text{if } s \in [a, x] \\ s - b & \text{if } s \in (x, b] \end{cases}$$

and

$$q(y, t) := \begin{cases} t - c & \text{if } t \in [c, y] \\ t - d & \text{if } t \in (y, d]. \end{cases}$$

Now, we deduce that the left part can be represented as :

$$\int_a^b \int_c^d p(x, s) q(y, t) f''_{s,t}(s, t) ds dt.$$

Consequently, we get the identity

$$\begin{aligned}
(2.6) \quad & \int_a^b \int_c^d p(x, s) q(y, t) f''_{s,t}(s, t) ds dt \\
& = (d - c) (b - a) f(x, y) - (d - c) \int_a^b f(s, y) ds \\
& - (b - a) \int_c^d f(x, t) dt + \int_a^b \int_c^d f(s, t) ds dt
\end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$.

Now, using the identity (2.6) we get

$$\begin{aligned} & \left| \int_a^b \int_c^d f(s, t) ds dt - [(b-a) \int_c^d f(x, t) dt + (d-c) \int_a^b f(s, y) ds \right. \\ & \quad \left. - (d-c)(b-a)f(x, y)] \right| \\ & \leq \int_a^b \int_c^d |p(x, s)q(y, t)| |f''_{s,t}(s, t)| ds dt. \end{aligned}$$

Using Hölder's integral inequality for double integrals, we get

$$\begin{aligned} & \int_a^b \int_c^d |p(x, s)q(y, t)| |f''_{s,t}(s, t)| ds dt \\ & \leq \left(\int_a^b \int_c^d |p(x, s)q(y, t)|^q ds dt \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d |f''_{s,t}(s, t)|^p ds dt \right)^{\frac{1}{p}} \\ & = \left(\int_a^b |p(x, s)|^q ds \right)^{\frac{1}{q}} \left(\int_c^d |q(y, t)|^q dt \right)^{\frac{1}{q}} \|f''_{s,t}\|_p \\ & = \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f''_{s,t}\|_p \end{aligned}$$

and the theorem is proved. ■

Corollary 2.2. *Under the above assumptions, we have the inequality:*

$$\begin{aligned} (2.7) \quad & \left| \int_a^b \int_c^d f(s, t) ds dt - \left[(b-a) \int_c^d f\left(\frac{a+b}{2}, t\right) dt \right. \right. \\ & \quad \left. \left. + (d-c) \int_a^b f\left(s, \frac{c+d}{2}\right) ds - (d-c)(b-a)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \right| \\ & \leq \frac{(b-a)^{1+\frac{1}{q}}(d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \|f''_{s,t}\|_p. \end{aligned}$$

Remark 2.1. Consider the mapping $g : [\alpha, \beta] \rightarrow \mathbf{R}$, $g(t) = (t - \alpha)^m + (\beta - t)^m$, ($m \geq 1$). Taking into account the fact that one has the properties

$$\inf_{t \in [\alpha, \beta]} g(t) = g\left(\frac{\alpha + \beta}{2}\right) = \frac{(\beta - \alpha)^m}{2^{m-1}}$$

and

$$\sup_{t \in [\alpha, \beta]} g(t) = g(\alpha) = g(\beta) = (\beta - \alpha)^m$$

then, the above inequality (2.7) is the best that can be obtained from (2.1).

Remark 2.2. Now, if we assume that $f(s, t) = h(s)h(t)$, $h : [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and suppose that $\|h'\|_p < \infty$, then from (2.1) we get (for $x = y$)

$$\begin{aligned} & \left| \int_a^b h(s) ds \int_a^b h(s) ds - h(x)(b-a) \int_a^b h(s) ds \right. \\ & \quad \left. - h(x)(b-a) \int_a^b h(s) ds + (b-a)^2 h^2(x) \right| \\ & \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{2}{q}} \|h'\|_p^2 \end{aligned}$$

i.e.

$$\left[\int_a^b h(s) ds - h(x)(b-a) \right]^2 \leq \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{2}{q}} \|h'\|_p^2$$

which is clearly equivalent to Ostrowski's inequality. Consequently (2.1) can be also regarded as a generalization for double integrals of the result embodied in Theorem 1.2.

3 APPLICATIONS FOR CUBATURE FORMULAE

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < \dots < y_{m-1} < y_m = b$ and $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$), $\eta_j \in [y_j, y_{j+1}]$ ($j = 0, \dots, m-1$) be intermediate points. Consider the sum

$$\begin{aligned} C(f, I_n, J_m, \xi, \eta) & := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t) dt \\ & + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j) ds - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \end{aligned}$$

for which we assume that the involved integrals can more easily be computed than the original double integral

$$D := \int_a^b \int_c^d f(s, t) ds dt,$$

and

$$h_i := x_{i+1} - x_i \quad (i = 0, \dots, n-1), \quad l_j := y_{j+1} - y_j \quad (j = 0, \dots, m-1).$$

With this assumption, we can state the following cubature formula:

Theorem 3.1. *Let $f : [a, b] \times [c, d] \rightarrow \mathbf{R}$ be as in Theorem 2.1 and I_n, J_m, ξ and η be as above. Then we have the cubature formula:*

$$\int_a^b \int_c^d f(s, t) \, ds dt = C(f, I_n, J_m, \xi, \eta) + R(f, I_n, J_m, \xi, \eta)$$

where the remainder term $R(f, I_n, J_m, \xi, \eta)$ satisfies the estimation:

$$(3.1) \quad |R(f, I_n, J_m, \xi, \eta)|$$

$$\leq \|f''_{s,t}\|_p \left[\sum_{i=0}^{n-1} \left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\times \left[\sum_{j=0}^{m-1} \left(\frac{(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\leq \frac{\|f''_{s,t}\|_p}{(q+1)^{\frac{2}{q}}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

for all ξ and η as above.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ ($i = 0, \dots, n-1; j = 0, \dots, m-1$) to get:

$$\left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s, t) \, ds dt - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, t) \, dt + l_j \int_{x_i}^{x_{i+1}} f(s, \eta_j) \, ds - h_i l_j f(\xi_i, \eta_j) \right] \right|$$

$$\leq \left[\left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \left(\frac{(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}}$$

$$\times \left(\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f(s, t)|^p \, ds dt \right)^{\frac{1}{p}}$$

for all $i = 0, \dots, n-1; j = 0, \dots, m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality and Hölder's discrete inequality for double sums, we deduce

$$|R(f, I_n, J_m, \xi, \eta)|$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \right. \\
&\times \left. \left(\frac{(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \times \left(\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f(s, t)|^p ds dt \right)^{\frac{1}{p}} \\
&\leq \left[\sum_{i=0}^{n-1} \left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \\
&\times \left[\sum_{j=0}^{m-1} \left(\frac{(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \\
&\times \left[\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f(s, t)|^p ds dt \right]^{\frac{1}{p}} \\
&= \left[\sum_{i=0}^{n-1} \left(\frac{(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1}}{q+1} \right) \right. \\
&\times \left. \sum_{j=0}^{m-1} \left(\frac{(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1}}{q+1} \right) \right]^{\frac{1}{q}} \times \|f''_{s,t}\|_p.
\end{aligned}$$

To prove the second part, we observe that

$$(x_{i+1} - \xi_i)^{q+1} + (\xi_i - x_i)^{q+1} \leq (x_{i+1} - x_i)^{q+1}$$

and

$$(y_{j+1} - \eta_j)^{q+1} + (\eta_j - y_j)^{q+1} \leq (y_{j+1} - y_j)^{q+1}$$

for all i, j as above and the intermediate points ξ_i and η_j .

We omit the details. ■

Remark 3.1. As

$$\sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \leq [\nu(h)]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i = (b-a) [\nu(h)]^{\frac{1}{q}}$$

and

$$\sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}} \leq [\mu(l)]^{\frac{1}{q}} \sum_{j=0}^{m-1} l_j = (d-c) [\mu(l)]^{\frac{1}{q}}$$

where

$$\nu(h) = \max \{h_i : i = 0, \dots, n-1\},$$

and

$$\mu(l) = \max \{l_j : j = 0, \dots, m-1\},$$

the right hand side of (3.1) can be bounded by

$$\frac{1}{(q+1)^{\frac{2}{q}}} \|f''_{s,t}\|_p (b-a)(d-c) [\nu(h) \mu(l)]^{\frac{1}{q}}.$$

Now, define the sum

$$C_M(f, I_n, J_m) := \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f\left(\frac{x_i + x_{i+1}}{2}, t\right) dt \\ + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f\left(s, \frac{y_j + y_{j+1}}{2}\right) ds - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} h_i l_j f\left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}\right).$$

Then we have the best cubature formula we can get from Theorem 3.1.

Corollary 3.2. *Under the above assumptions we have*

$$\int_a^b \int_c^d f(s, t) ds dt = C_M(f, I_n, J_m) + R(f, I_n, J_m),$$

where the remainder $R(f, I_n, J_m)$ satisfies the estimation:

$$|R(f, I_n, J_m)| \leq \frac{1}{4(q+1)^{\frac{2}{q}}} \|f''_{s,t}\|_p \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \sum_{j=0}^{m-1} l_j^{1+\frac{1}{q}}.$$

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SCHOOL OF COMMUNICATIONS AND INFORMATICS, VICTORIA UNIVERSITY OF TECHNOLOGY, PO Box 14428, MELBOURNE CITY MC, VICTORIA 8001, AUSTRALIA.

E-mail address: {sever, neil, pc}@matilda.vut.edu.au

