

ON SIMPSON'S QUADRATURE FORMULA FOR DIFFERENTIABLE MAPPINGS WHOSE DERIVATIVES BELONG TO L_p -SPACES AND APPLICATIONS

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ABSTRACT. An estimation of remainder for Simpson's quadrature formula for differentiable mappings whose derivatives belong to L_p -spaces and applications in theory of special means (logarithmic mean, identric mean etc..) are given.

1 INTRODUCTION

The following inequality is well known in the literature as the *Simpson's inequality* :

$$(1.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^5$$

where the mapping $f : [a, b] \rightarrow \mathbf{R}$ is supposed to be four time continuous differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b) , that is

$$\|f^{(4)}\|_{\infty} := \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty.$$

Now, if we assume that $I_h : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ is a partition of the interval $[a, b]$ and f is as above, then we have the *Simpson's quadrature formula*:

$$(1.2) \quad \int_a^b f(x) dx = A_S(f, I_h) + R_S(f, I_h)$$

where $A_S(f, I_h)$ is the *Simpson's rule*

$$(1.3) \quad A_S(f, I_h) =: \frac{1}{6} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] h_i + \frac{2}{3} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the *remainder term* $R_S(f, I_h)$ satisfies the estimation

$$(1.4) \quad |R_S(f, I_h)| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} \sum_{i=0}^{n-1} h_i^5$$

where $h_i := x_{i+1} - x_i$ for $i = 0, \dots, n-1$.

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When we have an equidistant partitioning of $[a, b]$ given by

$$(1.5) \quad I_n : x_i := a + \frac{b-a}{n}i, \quad i = 0, \dots, n$$

then we have the formula

$$(1.6) \quad \int_a^b f(x)dx = A_{S,n}(f) + R_{S,n}(f)$$

where

$$(1.7) \quad A_{S,n}(f) := \frac{b-a}{6n} \sum_{i=0}^{n-1} \left[f\left(a + \frac{b-a}{n}i\right) + f\left(a + \frac{b-a}{n}(i+1)\right) \right] \\ + \frac{2(b-a)}{3n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}\right)$$

and the remainder satisfies the estimation

$$(1.8) \quad |R_{S,n}(f)| \leq \frac{1}{2880} \cdot \frac{(b-a)^5}{n^4} \|f^{(4)}\|_\infty.$$

In the recent paper [1] the author proved the following result for lipschitzian mappings

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be an L -lipschitzian mapping on $[a, b]$. Then we have the inequality*

$$(1.9) \quad \left| \int_a^b f(x)dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \frac{5}{36} L(b-a)^2.$$

The following corollary is useful in practice:

Corollary 1.2. *Suppose that $f : [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , whose derivative is bounded on (a, b) , i.e.,*

$$\|f'\|_\infty := \sup_{x \in (a,b)} |f'(x)| < \infty.$$

Then we have the inequality

$$(1.10) \quad \left| \int_a^b f(x)dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \frac{5}{36} \|f'\|_\infty (b-a)^2.$$

In [2], S.S. Dragomir proved a version of Simpson's inequality for mappings with bounded variation as follows:

Theorem 1.3. *Let $f : [a, b] \rightarrow \mathbf{R}$ be a mapping with bounded variation on $[a, b]$. Then we have the inequality*

$$(1.11) \quad \left| \int_a^b f(x) dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \frac{1}{3} (b-a) \bigvee_a^b(f)$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.
The constant $\frac{1}{3}$ is the best possible.

The following corollary is useful in practice

Corollary 1.4. *Suppose that $f : [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , whose derivative is integrable on (a, b) , i.e.,*

$$\|f'\|_1 := \int_a^b |f'(x)| dx < \infty.$$

Then we have the inequality

$$(1.12) \quad \left| \int_a^b f(x) dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{1}{3} \|f'\|_1 (b-a)^2.$$

For some other integral inequalities see the recent book [3].

The main aim of this paper is to point out some bounds of the remainder in terms of p -norm of the derivative f' and apply them for composite quadrature formulae and for special means.

2 SIMPSON'S INEQUALITY IN TERMS OF p -NORMS

The following result holds:

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative belongs to $L_p(a, b)$. Then we have the inequality*

$$(2.1) \quad \left| \int_a^b f(x) dx - \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{1}{6} \left[\frac{2^{q+1}+1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \|f'\|_p$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

Proof. Using the integration by parts formula we have:

$$(2.2) \quad \int_a^b s(x) f'(x) dx = \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x) dx$$

where

$$s(x) := \begin{cases} x - \frac{5a+b}{6}, & x \in [a, \frac{a+b}{2}) \\ x - \frac{a+5b}{6}, & x \in [\frac{a+b}{2}, b]. \end{cases}$$

Indeed,

$$\begin{aligned}
 \int_a^b s(x)f'(x)dx &= \int_a^{\frac{a+b}{2}} \left(x - \frac{5a+b}{6}\right) f'(x)dx + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+5b}{6}\right) f'(x)dx \\
 &= \left(x - \frac{5a+b}{6}\right) f(x) \Big|_a^{\frac{a+b}{2}} + \left(x - \frac{a+5b}{6}\right) f(x) \Big|_{\frac{a+b}{2}}^b - \int_a^b f(x)dx \\
 &= \frac{b-a}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(x)dx
 \end{aligned}$$

and the identity is proved.

Applying Hölder's integral inequality we get

$$(2.3) \quad \left| \int_a^b s(x)f'(x)dx \right| \leq \left(\int_a^b |s(x)|^q dx \right)^{\frac{1}{q}} \|f'\|_p.$$

Let us compute

$$\begin{aligned}
 \int_a^b |s(x)|^q dx &= \int_a^{\frac{a+b}{2}} \left|x - \frac{5a+b}{6}\right|^q dx + \int_{\frac{a+b}{2}}^b \left|x - \frac{a+5b}{6}\right|^q dx \\
 &= \int_a^{\frac{5a+b}{6}} \left(\frac{5a+b}{6} - x\right)^q dx + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left(x - \frac{5a+b}{6}\right)^q dx \\
 &\quad + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left(\frac{a+5b}{6} - x\right)^q dx + \int_{\frac{a+5b}{6}}^b \left(x - \frac{a+5b}{6}\right)^q dx \\
 &= \frac{1}{q+1} \left[- \left(\frac{5a+b}{6} - x\right)^{q+1} \Big|_a^{\frac{5a+b}{6}} + \left(x - \frac{5a+b}{6}\right)^{q+1} \Big|_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \right. \\
 &\quad \left. - \left(\frac{a+5b}{6} - x\right)^{q+1} \Big|_{\frac{a+b}{2}}^{\frac{a+5b}{6}} + \left(x - \frac{a+5b}{6}\right)^{q+1} \Big|_{\frac{a+5b}{6}}^b \right] \\
 &= \frac{1}{q+1} \left[\left(\frac{5a+b}{6} - a\right)^{q+1} + \left(\frac{a+b}{2} - \frac{5a+b}{6}\right)^{q+1} \right. \\
 &\quad \left. - \left(\frac{a+5b}{6} - \frac{a+b}{2}\right)^{q+1} + \left(b - \frac{a+5b}{6}\right)^{q+1} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{a+5b}{6} - \frac{a+b}{2} \right)^{q+1} + \left(b - \frac{a+5b}{6} \right)^{q+1} \Big] \\
& = \frac{(2^{q+1} + 1)(b-a)^{q+1}}{3(q+1)6^q}.
\end{aligned}$$

Now, using the inequality (2.3) and the identity (2.2) we deduce the desired result (2.1). ■

The following corollary for Simpson's composite formula holds:

Corollary 2.2. *Let f and I_h be as above. Then we have Simpson's rule (1.2) and the remainder $R_S(f, I_h)$ satisfies the estimation*

$$(2.4) \quad |R_S(f, I_h)| \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{1+q} \right)^{\frac{1}{q}}.$$

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) to get

$$\begin{aligned}
& \left| \int_{x_i}^{x_{i+1}} f(x) dx - \frac{h_i}{3} \left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(\frac{x_i + x_{i+1}}{2}\right) \right] \right| \\
& \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} \|f'\|_p h_i^{1+\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Summing the above inequalities over i from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we get

$$\begin{aligned}
|R_S(f, I_h)| & \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) dx - \frac{h_i}{3} \left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(\frac{x_i + x_{i+1}}{2}\right) \right] \right| \\
& \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} \left(\int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \left(h_i^{1+\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \times \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} |f'(t)|^p dt \right)^{\frac{1}{p}} \right)^p \right)^{\frac{1}{p}} \\
& = \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{1+q} \right)^{\frac{1}{q}}
\end{aligned}$$

and the corollary is proved. ■

The case of equidistant partitioning is embodied in the following corollary:

Corollary 2.3. *Let f be as above and if I_n is an equidistant partitioning of $[a, b]$, then we have the estimation*

$$|R_{S,n}(f)| \leq \frac{1}{6n} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \|f'\|_p.$$

Remark 2.1. If we want to approximate the integral $\int_a^b f(x)dx$ by Simpson's formula $A_{S,n}(f)$ with an accuracy less than $\varepsilon > 0$, we need at least $n_\varepsilon \in \mathbf{N}$ points for the division I_n , where

$$n_\varepsilon := \left\lceil \frac{1}{6\varepsilon} \left(\frac{2^{q+1} + 1}{3(q+1)} \right)^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \|f'\|_p \right\rceil + 1$$

and $[r]$ denotes the integer part of $r \in \mathbf{R}$.

Remark 2.2. If the mapping $f : [a, b] \rightarrow \mathbf{R}$ is neither four time differentiable nor the fourth derivative is bounded on (a, b) , then we can not apply the classical estimation in Simpson's formula using the fourth derivative. But if we assume that $f' \in L_p(a, b)$, then we can use instead of (1.4), the formula (2.4).

We give here a class of mappings whose first derivatives belong to $L_p(a, b)$ but having the fourth derivatives unbounded on the given interval.

Let $f_s : [a, b] \rightarrow \mathbf{R}$, $f_s(x) := (x-a)^s$ where $s \in (3, 4)$. Then obviously

$$f'_s(x) := s(x-a)^{s-1}, \quad x \in (a, b)$$

and

$$f_s^{(4)}(x) = \frac{s(s-1)(s-2)(s-3)}{(x-a)^{4-s}}, \quad x \in (a, b).$$

It is clear that $\lim_{x \rightarrow a^+} f_s^{(4)}(x) = +\infty$ but $\|f'_s\|_p = s \frac{(b-a)^{s-1+\frac{1}{p}}}{((s-1)p+1)^{\frac{1}{p}}} < \infty$.

3 APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means:

1. *Arithmetic mean*

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0;$$

2. *Geometric mean*

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0;$$

3. *Harmonic mean*

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0;$$

4. *Logarithmic mean*

$$L = L(a, b) := \frac{b-a}{\ln b - \ln a}, \quad a, b > 0, a \neq b;$$

5. *Identric mean*

$$I = I(a, b) := \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \quad a, b > 0, a \neq b;$$

6. *p-Logarithmic mean*

$$S_p = S_p(a, b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, \quad p \in \mathbf{R} \setminus \{-1, 0\}, a, b > 0, a \neq b.$$

It is well known that S_p is monotonous nondecreasing over $p \in \mathbf{R}$ with $S_{-1} := L$ and $S_0 := I$. In particular, we have the following inequalities

$$(3.1) \quad H \leq G \leq L \leq I \leq A.$$

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means.

1. Let $f : [a, b] \rightarrow \mathbf{R}$ ($0 < a < b$), $f(x) = x^s$, $s \in \mathbf{R} \setminus \{-1, 0\}$.
Then

$$\frac{1}{b-a} \int_a^b f(x) dx = S_s^s(a, b), \quad f\left(\frac{a+b}{2}\right) = A^s(a, b),$$

$$\frac{f(a) + f(b)}{2} = A(a^s, b^s) \quad \text{and} \quad \|f'\|_p = |s| S_{(s-1)p}^{s-1}(b-a)^{\frac{1}{p}}.$$

Using the inequality (2.1) we get

$$(3.2) \quad \left| S_s^s(a, b) - \frac{1}{3}A(a^s, b^s) - \frac{2}{3}A^s(a, b) \right| \\ \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} |s| S_{(s-1)p}^{s-1}(a, b)(b-a).$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

2. Let $f : [a, b] \rightarrow \mathbf{R}$ ($0 < a < b$), $f(x) = \frac{1}{x}$.
Then

$$\frac{1}{b-a} \int_a^b f(x) dx = L^{-1}(a, b), \quad f\left(\frac{a+b}{2}\right) = A^{-1}(a, b),$$

$$\frac{f(a) + f(b)}{2} = H^{-1}(a, b) \quad \text{and} \quad \|f'\|_p = S_{-2p}^{-2}(a, b)(b-a)^{\frac{1}{p}}$$

Using the inequality (2.1) we get

$$(3.3) \quad |3HA - LA - 2LH| \leq \frac{1}{2}AHL \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} S_{-2p}^{-2}(b-a)$$

3. Let $f : [a, b] \rightarrow \mathbf{R}$ ($0 < a < b$), $f(x) = \ln x$.
Then

$$\frac{1}{b-a} \int_a^b f(x) dx = \ln I(a, b), \quad f\left(\frac{a+b}{2}\right) = \ln A(a, b),$$

$$\frac{f(a) + f(b)}{2} = \ln A(a, b) \quad \text{and} \quad \|f'\|_p = S_{-p}^{-1}(a, b)(b-a)^{\frac{1}{p}}.$$

Using the inequality (2.1) we get

$$(3.4) \quad \left| \ln \left[\frac{I}{G^{1/3}A^{2/3}} \right] \right| \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} S_{-p}^{-1}(a, b)(b-a).$$

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