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SOME INTEGRAL INEQUALITIES OF GRÜSS TYPE

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Abstract. Some classical and new integral inequalities of Grüss type are presented.

1 Grüss Integral Inequality

In 1935, G. Grüss, proved the following integral inequality which gives an estimation for the integral of a product in terms of the product of integrals (see for example [1, p. 296])

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$

$$\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma);$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$\varphi\leq f(x)\leq\Phi,\qquad\gamma\leq g(x)\leq\Gamma\tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the *best possible* and is achieved for $f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right)$.

We give here a weighted version of Grüss' inequality

Theorem 1.1. *Let f and g be two functions defined and integrable on $[a,b]$. If (1.1) holds for each $x\in[a,b]$, where $\varphi,\Phi,\gamma,\Gamma$ are given real constants, and $h:[a,b]\to[0,\infty)$ is integrable and $\int_a^b h(x)dx>0$, then*

$$\left|\int_a^b h(x)dx\cdot\int_a^b f(x)\,g(x)\,h(x)dx-\int_a^b f(x)\,h(x)dx\cdot\int_a^b g(x)\,h(x)dx\right|\tag{1.2}$$

$$\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma)\left(\int_a^b h(x)dx\right)^2$$

and the constant $\frac{1}{4}$ is the best possible.

For the sake of completeness we give here a simple proof of this fact which is similar with the classical one for unweighted case (compare with [1, p. 296]).

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Let us note that the following equality is valid:

$$\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,g(x)\,h(x)dx - \frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,h(x)dx \cdot \frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx = \frac{1}{2\left(\int_a^b h(x)dx\right)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,h(x)h(y)dxdy. \tag{1.3}$$

Applying Cauchy-Buniakowski-Schwarz's integral inequality for double integrals we have

$$\left[\frac{1}{2\left(\int_a^b h(x)dx\right)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,h(x)h(y)dxdy\right]^2 \leq \frac{1}{2\left(\int_a^b h(x)dx\right)^2}\int_a^b\int_a^b (f(x)-f(y))^2\,h(x)h(y)dxdy \times \frac{1}{2\left(\int_a^b h(x)dx\right)^2}\int_a^b\int_a^b (g(x)-g(y))^2\,h(x)h(y)dxdy$$
$$= \left[\frac{1}{\int_a^b h(x)dx}\int_a^b f^2(x)\,h(x)dx - \left(\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,h(x)dx\right)^2\right] \times \left[\frac{1}{\int_a^b h(x)dx}\int_a^b g^2(x)\,h(x)dx - \left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx\right)^2\right]. \tag{1.4}$$

The following equality also holds

$$\frac{1}{\int_a^b h(x)dx}\int_a^b f^2(x)\,h(x)dx - \left(\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,h(x)dx\right)^2$$

$$= \left(\Phi - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx - \varphi \right)$$

$$- \frac{1}{\int_a^b h(x)dx} \int_a^b (\Phi - f(x))\,(f(x) - \varphi)\, h(x)dx.$$

As, $(\Phi - f(x))\,(f(x) - \varphi) \geq 0$ for each $x \in [a, b]$, then

$$\frac{1}{\int_a^b h(x)dx} \int_a^b f^2(x)\, h(x)dx - \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx \right)^2 \tag{1.5}$$

$$\leq \left(\Phi - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx - \varphi \right).$$

Similarly, we have

$$\frac{1}{\int_a^b h(x)dx} \int_a^b g^2(x)\, h(x)dx - \left(\frac{1}{\int_a^b h(x)dx} \int_a^b g(x)\, h(x)dx \right)^2 \tag{1.6}$$

$$\leq \left(\Gamma - \frac{1}{\int_a^b h(x)dx} \int_a^b g(x)\, h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b g(x)\, h(x)dx - \gamma \right).$$

Now, by (1.3), (1.4), (1.5) and (1.6) we get

$$\left| \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, g(x)\, h(x)dx \right. \tag{1.7}$$

$$\left. - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx \cdot \frac{1}{\int_a^b h(x)dx} \int_a^b g(x)\, h(x)dx \right|$$

$$\leq \left(\Phi - \frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx \right) \cdot \left(\frac{1}{\int_a^b h(x)dx} \int_a^b f(x)\, h(x)dx - \varphi \right)$$

$$\times\left(\Gamma-\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx-\gamma\right).$$

Using the elementary inequality for real numbers:

$$4pq\le(p+q)^2,\qquad p,q\in\mathbf{R}$$

we can state

$$4\left(\Phi-\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b f(x)\,h(x)dx-\varphi\right)\le(\Phi-\varphi)^2 \tag{1.8}$$

and

$$4\left(\Gamma-\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx\right)\cdot\left(\frac{1}{\int_a^b h(x)dx}\int_a^b g(x)\,h(x)dx-\gamma\right)\le(\Gamma-\gamma)^2. \tag{1.9}$$

Now, combining (1.7) with (1.8) and (1.9) we deduce the desired inequality (1.2).

To prove the sharpness of (1.2), let choose $h(x)=1$, $f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right)$ for all $x\in[a,b]$. Then

$$\frac{1}{b-a}\int_a^b f(x)\,dx=1,$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b g(x)\,dx=0,$$

$$\Phi-\varphi=\Gamma-\gamma=2$$

and the equality in (1.2) is realized.■

For other inequalities of Grüss type see the book [1], where many other references are given.

We omit the details.

2 The Case When Both Mappings Are Lipschitzian

The following inequality of Grüss' type for lipschitzian mappings holds :

Theorem 2.1. *Let $f, g : [a,b] \to \mathbf{R}$ be two lipschitzian mappings with the constants $L_1 > 0$ and $L_2 > 0$, i.e.,*

$$|f(x) - f(y)| \le L_1 |x-y|, \qquad |g(x) - g(y)| \le L_2 |x-y| \tag{2.1}$$

for all $x, y \in [a,b]$. If $p : [a,b] \to [0,\infty)$ is integrable, then

$$\left| \int_a^b p(x)\,dx \cdot \int_a^b p(x) f(x) g(x)\,dx - \int_a^b p(x) f(x)\,dx \cdot \int_a^b p(x) g(x)\,dx \right| \tag{2.2}$$

$$\le L_1 L_2 \left[\int_a^b p(x)\,dx \cdot \int_a^b p(x) x^2 dx - \left(\int_a^b p(x)\,x dx \right)^2 \right]$$

and the inequality is sharp.

Proof. By (2.1) we have that

$$|(f(x) - f(y))(g(x) - g(y))| \le L_1 L_2 (x-y)^2$$

for all $x, y \in [a,b]$.

Multiplying by $p(x)p(y) \ge 0$ and integrating on $[a,b]^2$, we get

$$\left| \int_a^b \int_a^b p(x) p(y) (f(x) - f(y))(g(x) - g(y))\,dxdy \right|$$

$$\le \int_a^b \int_a^b p(x) p(y) |(f(x) - f(y))(g(x) - g(y))|\,dxdy$$

$$\le L_1 L_2 \int_a^b \int_a^b p(x) p(y) (x-y)^2\,dxdy.$$

As it is easy to see that

$$\frac{1}{2} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))\,p(x) p(y)\,dxdy$$

$$= \int_a^b p(x)\,dx \int_a^b p(x) f(x) g(x)\,dx - \int_a^b p(x) f(x)\,dx \int_a^b p(x) g(x)\,dx$$

and

$$\frac{1}{2} \int_a^b \int_a^b p(x) p(y) (x-y)^2\,dxdy = \int_a^b p(x)\,dx \int_a^b p(x) x^2 dx - \left(\int_a^b p(x)\,x dx \right)^2$$

the inequality (2.2) is thus obtained.

Now, if we chose $f(x) = L_1 x$, $g(x) = L_2 x$, then f is L_1-lipschitzian , g is L_2-lipschitzian and the equality in (2.2) is realized for any p as above. ∎

Corollary 2.2. *Under the above assumptions, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{2.3}$$

$$\leq \frac{L_1L_2(b-a)^2}{12}.$$

The constant $\frac{1}{12}$ *is the best possible.*

We note that the above corollary is a natural generalization of a well-known result by Čebyšev (see for example [1, p. 297]) :

Corollary 2.3. *Let* $f,g:[a,b]\to\mathbf{R}$ *be two differentiable mappings whose derivatives are bounded on* (a,b). *Denote* $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty$. *Then we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{2.4}$$

$$\leq \frac{\|f'\|_\infty\|g'\|_\infty}{12}(b-a)^2.$$

The constant $\frac{1}{12}$ *is the best possible.*

3 The Case When f Is Lipschitzian

We are able now to prove another inequality of Grüss type assuming that only one mapping is lipschitzian as follows:

Theorem 3.1. *Let* $f:[a,b]\to\mathbf{R}$ *be a* $M-$*lipschitzian mapping on* $[a,b]$. *Then we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{3.1}$$

$$\leq\begin{cases} M\|g\|_1 & \text{provided that } g\in L_1[a,b]\\[2mm] M\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|g\|_q & \text{provided that } g\in L_q[a,b]\\ \qquad p>1 & \text{and } \frac{1}{p}+\frac{1}{q}=1;\\[2mm] M\frac{(b-a)^3}{3}\|g\|_\infty & \text{provided that } g\in L_\infty[a,b].\end{cases}$$

Proof. We have that

$$|f(x)\,g(y)-f(y)\,g(y)|\leq M|x-y|\,|g(y)|$$

for all $x,y\in[a,b]$, from where, by integration on $[a,b]^2$, we get that

$$\left|\int_a^b\int_a^b (f(x)\,g(y)-f(y)\,g(y))\,dxdy\right|\leq M\int_a^b\int_a^b|x-y|\,|g(y)|\,dxdy.$$

But

$$\int_a^b\int_a^b (f(x)g(y)-f(y)g(y))\,dxdy=\int_a^b f(x)\,dx\int_a^b g(x)\,dx-(b-a)\int_a^b f(x)g(x)\,dx.$$

Now, if $g\in L_1[a,b]$, then

$$\int_a^b\int_a^b |x-y|\,|g(y)|\,dxdy\le (b-a)\max_{(x,y)\in[a,b]^2}|x-y|\int_a^b |g(y)|\,dy=(b-a)^2\|g\|_1.$$

Now, assume that $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, $g\in L_q[a,b]$. Then by Hölder's integral inequality we have:

$$\int_a^b\int_a^b |x-y|\,|g(y)|\,dxdy$$

$$\le\left(\int_a^b\int_a^b |x-y|^p\,dxdy\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b |g(y)|^q\,dxdy\right)^{\frac{1}{q}}=K^{\frac{1}{p}}(b-a)^{\frac{1}{q}}\|g\|_q$$

where

$$\begin{aligned}K&:=\int_a^b\int_a^b |x-y|^p\,dxdy=\int_a^b\left(\int_a^b |y-x|^p\,dy\right)dx\\&=\int_a^b\left(\int_a^x |x-y|^p\,dy+\int_x^b |y-x|^p\,dy\right)dx\\&=\int_a^b\left[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{p+1}\right]dx=\frac{2(b-a)^{p+2}}{(p+1)(p+2)}\end{aligned}$$

and then we get

$$\int_a^b\int_a^b |x-y|\,|g(y)|\,dxdy\le\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}}(b-a)^{2+\frac{1}{p}}\|g\|_q.$$

Finally, assuming that $g\in L_\infty[a,b]$, we have that

$$\int_a^b\int_a^b |x-y|\,|g(y)|\,dxdy\le\|g\|_\infty\int_a^b\int_a^b |x-y|\,dxdy=\frac{(b-a)^3}{3}\|g\|_\infty.$$

The theorem is thus proved. ∎

The following corollary is important in applications.

Corollary 3.2. *Let $f:[a,b]\to\mathbf{R}$ be a differentiable mapping whose derivative is bounded on (a,b). Then we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\tag{3.2}$$

$$\leq \begin{cases} \|f'\|_\infty \|g\|_1 & \text{provided that } g \in L_1[a,b] \\ \left[\dfrac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|f'\|_\infty \|g\|_q & \text{provided that } g \in L_q[a,b] \\ \qquad p>1, \frac{1}{p}+\frac{1}{q}=1; & \\ \dfrac{(b-a)}{3} \|f'\|_\infty \|g\|_\infty & \text{provided that } g \in L_\infty[a,b]. \end{cases}$$

4 The Case When f Is $M-g$-Lipschitzian

Another generalization of Grüss' integral inequality is embodied in the following theorem:

Theorem 4.1. *Let $f,g:[a,b]\to\mathbf{R}$ be two integrable mappings on $[a,b]$ such that*

$$|f(x)-f(y)| \leq M|g(x)-g(y)| \text{ for all } x,y\in[a,b]. \tag{4.1}$$

Then we have the inequality:

$$\left|\int_a^b p(x)\,dx \cdot \int_a^b p(x) f(x) g(x)\,dx - \int_a^b p(x) f(x)\,dx \cdot \int_a^b p(x) g(x)\,dx\right| \tag{4.2}$$

$$\leq M\left[\int_a^b p(x)\,dx \int_a^b p(x) g^2(x)\,dx - \left(\int_a^b p(x) g(x)\,dx\right)^2\right]$$

where $p:[a,b]\to[0,\infty)$ is an arbitrary integrable function on $[a,b]$. The inequality (4.2) is sharp.

Proof. By condition (4.1) we have

$$|(f(x)-f(y))(g(x)-g(y))| \leq M(g(x)-g(y))^2 \quad \text{for all } x,y\in[a,b].$$

Multiplying by $p(x)p(y)\geq 0$ and integrating on $[a,b]^2$ we get

$$\left|\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy\right|$$

$$\leq \int_a^b\int_a^b p(x)p(y)|(f(x)-f(y))(g(x)-g(y))|\,dxdy$$

$$\leq M\int_a^b\int_a^b p(x)p(y)(g(x)-g(y))^2\,dxdy$$

which is clearly equivalent to (4.2).

Now, if we choose $f(x)=Mx$, $g(x)=x$, then the equality in the above inequality is realized for any p as above . ∎

The following corollary is important for applications.

Corollary 4.2. *Let $f, g : [a,b] \to \mathbf{R}$ be two differentiable mappings with $g'(x) \neq 0$ on (a,b) and there exists a constant $M > 0$ so that:*

$$\left|\frac{f'(x)}{g'(x)}\right| \leq M \quad \text{for all } x \in (a,b) . \tag{4.3}$$

Then we have the inequality (4.2)*.The inequality is sharp.*

Proof. Use the Cauchy's mean value theorem, i.e., for every $x, y \in [a,b]$ with $x \neq y$, there exists a c between x and y so that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}.$$

Consequently, for each $x, y \in [a,b]$ we have

$$|f(x) - f(y)| \leq M |g(x) - g(y)|$$

i.e., (4.1) holds. Applying Theorem 4.1, we get (4.3) . ∎

Remark 4.1. *Under the assumption of Corollary 4.2 we can choose*

$$M = \sup_{x \in (a,b)} \left|\frac{f'(x)}{g'(x)}\right| = \left\|\frac{f'}{g'}\right\|_{\infty},$$

assuming that the norm is finite.

Remark 4.2. *If f, g are as in the above theorem, then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x) dx - \frac{1}{b-a}\int_a^b f(x) dx \cdot \frac{1}{b-a}\int_a^b g(x) dx\right| \tag{4.4}$$

$$\leq M \left[\frac{1}{b-a}\int_a^b g^2(x) dx - \left(\frac{1}{b-a}\int_a^b g(x) dx\right)^2\right]$$

and the inequality is sharp.

2. If f, g are as in Corollary 4.2, then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x) dx - \frac{1}{b-a}\int_a^b f(x) dx \cdot \frac{1}{b-a}\int_a^b g(x) dx\right|$$

$$\leq \left\|\frac{f'}{g'}\right\|_{\infty} \left[\frac{1}{b-a}\int_a^b g^2(x) dx - \left(\frac{1}{b-a}\int_a^b g(x) dx\right)^2\right]$$

and the inequality is sharp.

5 The Case When Both Mappings Are of Hölder Type

In this section we point out a Grüss' type inequality for mappings satisfying the condition of Hölder as follows :

Theorem 5.1. *Suppose that f is of $r-$Hölder type and g is of $s-$Hölder, i.e.,*

$$|f(x)-f(y)| \le H_1 |x-y|^r \quad \text{and} \quad |g(x)-g(y)| \le H_2 |x-y|^s \tag{5.1}$$

for all $x,y\in[a,b]$, where $H_1,H_2>0$ and $r,s\in(0,1]$ are fixed. Then we have the inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{5.2}$$

$$\le \frac{H_1H_2(b-a)^{r+s}}{(r+s+1)(r+s+2)}.$$

Proof. By the assumption (5.1) we have

$$|(f(x)-f(y))(g(x)-g(y))| \le H_1H_2|x-y|^{r+s}$$

for all $x,y\in[a,b]$.

Integrating on $[a,b]^2$ we get

$$\left|\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy\right|$$

$$\le \int_a^b\int_a^b |(f(x)-f(y))(g(x)-g(y))|\,dxdy \le H_1H_2\int_a^b\int_a^b |x-y|^{r+s}\,dxdy.$$

Now, we observe that :

$$\begin{aligned}\int_a^b\int_a^b |x-y|^{r+s}\,dxdy &= \int_a^b\left(\int_a^b |y-x|^{r+s}\,dy\right)dx \\ &= \int_a^b\left(\int_a^x (x-y)^{r+s}dy + \int_x^b (y-x)^{r+s}\,dy\right)dx \\ &= \int_a^b\left[\frac{(x-a)^{r+s+1}+(b-x)^{r+s+1}}{r+s+1}\right]dx \\ &= \frac{2(b-a)^{r+s+1}}{(r+s+1)(r+s+2)}\end{aligned}$$

and as

$$\frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy$$

$$=(b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx$$

we get the desired inequality (5.2). ∎

6 The Case When f' and g' Belong to Some L_p-Spaces

In this section we point out some inequalities of Grüss' type for differentiable mappings whose derivatives belong firstly to $L_\infty(a,b)$, then to $L_p(a,b)$ $(p>1)$ and finally to $L_1(a,b)$.

Theorem 6.1. *Let $f,g:[a,b]\to\mathbf{R}$ be two differentiable mappings on (a,b) and $p:[a,b]\to[0,\infty)$ is integrable on $[a,b]$. If $f',g'\in L_\infty(a,b)$, then we have the inequality*

$$\left|\int_a^b p(x)\,dx\cdot\int_a^b p(x)f(x)g(x)\,dx-\int_a^b p(x)f(x)\,dx\cdot\int_a^b p(x)g(x)\,dx\right| \tag{6.1}$$

$$\le\frac{1}{2}\int_a^b\int_a^b p(x)p(y)\left|\int_x^y|f'(t)|\,dt\right|\left|\int_x^y|g'(z)|\,dz\right|dxdy$$

$$\le\|f'\|_\infty\|g'\|_\infty\left[\int_a^b p(x)\,dx\int_a^b p(x)x^2dx-\left(\int_a^b p(x)\,xdx\right)^2\right].$$

Moreover, the inequality (6.1)*is sharp.*

Proof. Let observe that for any $x,y\in[a,b]$ we have that

$$(f(x)-f(y))(g(x)-g(y))=\int_x^y\int_x^y f'(t)g'(z)\,dtdz.$$

As $f',g'\in L_\infty(a,b)$, then we have

$$p(x)p(y)|(f(x)-f(y))(g(x)-g(y))|$$

$$\le\left|\int_x^y|f'(t)|\,dt\right|\left|\int_x^y|g'(z)|\,dz\right|p(x)p(y)\le\|f'\|_\infty\|g'\|_\infty(x-y)^2p(x)p(y)$$

for all $x,y\in[a,b]$.

By the properties of the modulus, we have

$$\left|\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy\right| \tag{6.2}$$

$$\le\int_a^b\int_a^b p(x)p(y)\left|\int_x^y|f'(t)|\,dt\right|\left|\int_x^y|g'(z)|\,dz\right|dxdy$$

$$\le\|f'\|_\infty\|g'\|_\infty\int_a^b\int_a^b(x-y)^2p(x)p(y)\,dxdy,$$

from where we get the desired inequality (6.1) .

To prove the sharpness of (6.1), let consider the mappings $f(x)=\alpha x+\beta$, $g(x)=\gamma x+\delta$ $(\alpha,\gamma>0,\ \beta,\delta\in\mathbf{R})$ on $[a,b]$. A simple calculation gives

$$\int_a^b p(x)\,dx\cdot\int_a^b p(x)f(x)g(x)\,dx-\int_a^b p(x)f(x)\,dx\cdot\int_a^b p(x)g(x)\,dx$$

$$=\frac{1}{2}\int_a^b\int_a^b p(x)p(y)\left|\int_x^y |f'(t)|\,dt\right|\left|\int_x^y |g'(z)|\,dz\right|dxdy$$

$$=\|f'\|_\infty\|g'\|_\infty\left[\int_a^b p(x)\,dx\int_a^b p(x)x^2dx-\left(\int_a^b p(x)\,xdx\right)^2\right]$$

$$=\frac{\alpha\gamma}{2}\int_a^b\int_a^b (x-y)^2\,p(x)\,p(y)\,dxdy$$

which proves that we can have equality in all inequalities in (6.1) . ∎

The following corollary holds.

Corollary 6.2. *With the above assumptions on the mappings f,g, we have :*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{6.3}$$

$$\le\frac{1}{2}\int_a^b\int_a^b\left|\int_x^y |\ f'(t)\ |\ dt\right|\left|\int_x^y |\ g'(z)\ |\ dz\right|dxdy\le\frac{\|f'\|_\infty\|g'\|_\infty}{12}(b-a)^2.$$

The constants $\frac{1}{2}$ and $\frac{1}{12}$, respectively, are the best possible.

Remark 6.1. *We shall show that some time the estimation given by classical Grüss' inequality for the difference*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$

is better than the estimation (6.3) *and some other time the other way around.*

Let $f,g:[0,1]\to[0,\infty)$ given by $f(x)=x^p,\quad g(x)=x^q,\quad p,q>1$. Then

$$\varphi=\inf_{x\in[0,1]}f(x)=0,\qquad \Phi=\sup_{x\in[0,1]}f(x)=1;$$

$$\gamma=\inf_{x\in[0,1]}g(x)=0,\qquad \Gamma=\sup_{x\in[0,1]}g(x)=1.$$

Also we have

$$f'(x)=px^{p-1},\quad g'(x)=qx^{q-1},\quad x\in[0,1]$$

and obviously $\|f'\|_\infty = p, \|g'\|_\infty = q.$

Now, we observe that

$$\frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) = \frac{1}{4}$$

and

$$\frac{\|f'\|_\infty \|g'\|_\infty}{12}(b-a)^2 = \frac{pq}{12}.$$

Consequently, if $pq > 3$, then the bound provided by Grüss' inequality is better than the bound provided by (6.3). If $pq < 3\,(p, q > 1)$ then (6.3) is better than (1.1).

Remark 6.2. *The inequality* (6.3) *is also a refinement of Čebyšev's inequality embodied in Corollary 2.2.*

The following theorem also holds

Theorem 6.3. *Let $f, g : [a, b] \to \mathbf{R}$ be two differentiable mappings on (a, b) and $p : [a, b] \to [0, \infty)$ is integrable on $[a, b]$. If $f' \in L_\alpha(a, b)$, $g' \in L_\beta(a, b)$ with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we have the inequality*

$$\left|\int_a^b p(x)\,dx \int_a^b p(x) f(x) g(x)\,dx - \int_a^b p(x) f(x)\,dx \cdot \int_a^b p(x) g(x)\,dx\right| \tag{6.4}$$

$$\leq \frac{1}{2}\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\left|\int_x^y |f'(t)|^\alpha\,dt\right| dxdy\right)^{\frac{1}{\alpha}}$$

$$\times \left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\left|\int_x^y |g'(t)|^\alpha\,dt\right| dxdy\right)^{\frac{1}{\beta}}$$

$$\leq \frac{1}{2}\|f'\|_\alpha \|g'\|_\beta \int_a^b\int_a^b |x-y|\,p(x)\,p(y)\,dxdy.$$

Note that, the first inequality in (6.4) *is sharp.*

Proof. Using Hölder's inequality for double integrals, we have

$$\left|\int_x^y\int_x^y |f'(t)\,g'(z)|\,dtdz\right|$$

$$\leq \left|\int_x^y\int_x^y |f'(t)|^\alpha\,dtdz\right|^{\frac{1}{\alpha}}\left|\int_x^y\int_x^y |g'(z)|^\beta\,dtdz\right|^{\frac{1}{\beta}}$$

$$= |x-y|^{\frac{1}{\alpha}}\left|\int_x^y |f'(t)|^\alpha\,dt\right|^{\frac{1}{\alpha}} |x-y|^{\frac{1}{\beta}}\left|\int_x^y |g'(z)|^\alpha\,dz\right|^{\frac{1}{\beta}}$$

$$= |x-y|\left|\int_x^y |f'(t)|^\alpha\,dt\right|^{\frac{1}{\alpha}}\left|\int_x^y |g'(t)|^\beta\,dt\right|^{\frac{1}{\beta}}.$$

Now, as in the proof of Theorem 6.1, we have :

$$\left|\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy\right|$$

$$\leq \int_a^b\int_a^b p(x)p(y)\left|\int_x^y\int_x^y |f'(t)g'(z)|\,dtdz\right|dxdy$$

$$\leq \int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^{\alpha}dt\right|^{\frac{1}{\alpha}}\left|\int_x^y |g'(z)|^{\beta}dz\right|^{\frac{1}{\beta}}dxdy.$$

Using again Hölder's inequality for double integrals, we have

$$\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^{\alpha}dt\right|^{\frac{1}{\alpha}}\left|\int_x^y |g'(z)|^{\beta}dz\right|^{\frac{1}{\beta}}dxdy \tag{6.5}$$

$$\leq \left(\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^{\alpha}dt\right|dxdy\right)^{\frac{1}{\alpha}}$$

$$\times\left(\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |g'(z)|^{\beta}dz\right|dxdy\right)^{\frac{1}{\beta}}$$

and, as

$$\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy \tag{6.6}$$

$$=2\left[\int_a^b p(x)\,dx\int_a^b p(x)f(x)g(x)\,dx-\int_a^b p(x)f(x)\,dx\int_a^b p(x)g(x)\,dx\right]$$

the inequality (6.5) and (6.6) provide the first inequality in (6.4).
Now, let observe that

$$\left|\int_x^y |f'(t)|^{\alpha}dt\right|\leq \|f'\|_{\alpha}^{\alpha},\qquad \left|\int_x^y |g'(z)|^{\beta}dz\right|\leq \|g'\|_{\beta}^{\beta}$$

for all $x,y\in[a,b]$, and then

$$\left(\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^{\alpha}dt\right|dxdy\right)^{\frac{1}{\alpha}}$$

$$\times\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\left|\int_x^y |g'(z)|^\beta\,dt\right|dxdy\right)^{\frac{1}{\beta}}$$

$$\leq \|f'\|_\alpha\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\,dxdy\right)^{\frac{1}{\alpha}}\times\|g'\|_\beta\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\,dxdy\right)^{\frac{1}{\beta}}$$

$$= \|f'\|_\alpha\,\|g'\|_\beta\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\,dxdy$$

and the second inequality in (6.4) is also proved.

For the sharpness of the first inequality in (6.4), let consider the mappings $f,g:[a,b]\to\mathbf{R}$, $f(x)=mx+n, g(x)=sx+z$ with $m,t>0$. Then, obviously

$$\int_a^b p(x)\,dx\int_a^b p(x)\,f(x)\,g(x)\,dx-\int_a^b p(x)\,f(x)\,dx\cdot\int_a^b p(x)\,g(x)\,dx$$

$$=\frac{1}{2}ms\int_a^b\int_a^b p(x)\,p(y)\,(x-y)^2\,dxdy$$

and

$$\left|\int_x^y |f'(t)|^\alpha\,dt\right| = m^\alpha\,|x-y|\,,\left|\int_x^y |g'(z)|^\beta\,dz\right| = s^\beta\,|x-y|$$

then

$$\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\left|\int_x^y |f'(t)|^\alpha\,dt\right|dxdy\right)^{\frac{1}{\alpha}}$$

$$\times\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|\left|\int_x^y |g'(z)|^\beta\,dz\right|dxdy\right)^{\frac{1}{\beta}}$$

$$= ms\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|^2\,dxdy\right)^{\frac{1}{\alpha}}\times\left(\int_a^b\int_a^b p(x)\,p(y)\,|x-y|^2\,dxdy\right)^{\frac{1}{\beta}}$$

$$= ms\int_a^b\int_a^b p(x)\,p(y)\,(x-y)^2\,dxdy$$

and the equality is realized in the first inequality in (6.4). ∎

The following corollary holds.

Corollary 6.4. *Let f,g be as above. Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{6.7}$$

$$\leq\frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b|x-y|\left|\int_x^y|f'(t)|^{\alpha}dt\right|dxdy\right)^{\frac{1}{\alpha}}$$

$$\times\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b|x-y|\left|\int_x^y|g'(t)|^{\beta}dt\right|dxdy\right)^{\frac{1}{\beta}}$$

$$\leq\frac{1}{6}\|f'\|_{\alpha}\|g'\|_{\beta}(b-a).$$

The first inequality in (6.7) *is sharp.*

In a similar way we can prove the following theorem:

Theorem 6.5. *Let $f,g:[a,b]\to\mathbf{R}$ be two differentiable mappings on (a,b). If $f'\in L_\infty(a,b)$ and $g'\in L_1(a,b)$ then we have the inequalities:*

$$\left|\int_a^b p(x)\,dx\int_a^b p(x)f(x)g(x)\,dx-\int_a^b p(x)f(x)\,dx\cdot\int_a^b p(x)g(x)\,dx\right| \tag{6.8}$$

$$\leq\frac{1}{2}\int_a^b\int_a^b p(x)p(y)|x-y|\sup_{t\in[x,y]}|f'(t)|\left|\int_x^y|g'(z)|\,dz\right|dxdy$$

$$\leq\frac{1}{2}\|f'\|_{\infty}\|g'\|_{1}\int_a^b\int_a^b p(x)p(y)|x-y|\,dxdy.$$

The first inequality in (6.8) *is sharp.*

The following corollary also holds.

Corollary 6.6. *Under the above assumptions for the mappings f and g, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \tag{6.9}$$

$$\leq\frac{1}{2(b-a)^2}\int_a^b\int_a^b p(x)p(y)|x-y|\sup_{t\in[x,y]}|f'(t)|\left|\int_x^y|g'(z)|\,dz\right|dxdy$$

$$\leq\frac{1}{6}\|f'\|_{\infty}\|g'\|_{1}(b-a).$$

The first inequality in (6.9) *is sharp.*

Remark 6.3. *We note that some time the upper bound provided by* (6.4) *is better than the upper bound given by* (6.8) *and other time, the other way around.*

Indeed, choosing $f, g : [0,1] \to \mathbf{R}$, $f(x) = x^p$, $g(x) = x^q$ $(p, q > 1)$ we have

$$f'(x) = px^{p-1}, \quad g'(x) = qx^{q-1}, \quad \|f'\|_\infty = p, \quad \|g'\|_1 = 1,$$

$$\|f'\|_\alpha = \frac{p}{[\alpha(p-1)+1]^{\frac{1}{\alpha}}}$$

and

$$\|g'\|_q = \frac{q}{[\beta(q-1)+1]^{\frac{1}{\beta}}}$$

where $\alpha, \beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Also, let

$$A := \frac{1}{6} \|f'\|_\infty \|g'\|_1 (b-a) = \frac{p}{6}$$

and

$$B := \frac{1}{6} \|f'\|_\alpha \|g'\|_\beta (b-a) = \frac{pq}{6[\alpha(p-1)+1]^{\frac{1}{\alpha}} [\beta(q-1)+1]^{\frac{1}{\beta}}}.$$

If we choose $\alpha = \beta = 2$, we get

$$\frac{A}{B} = \frac{[(2p+1)(2q+1)]^{\frac{1}{2}}}{q}$$

which can be greater or less than 1 for different values of $p, q > 1$.

References

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