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# SOME ESTIMATIONS OF KRAFT NUMBERS AND RELATED RESULTS

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Abstract. Some inequalities for Kraft numbers which are important in coding theory [2, 3], for they lead to a simple criterion to determine whether or not there is an instantaneous code with given codeword lengths, are pointed out.

## 1 Introduction

The following remarkable theorem, published by L.G. Kraft in 1949 gives a simple criterion to determine whether or not there is an instantaneous code [1, p. 43] with given code word lengths [1, p. 44].

**Theorem 1.1.** *(Kraft's Theorem) We have*

1. *If $C$ is an $r$-ary instantaneous code with code word lengths $l_1, ..., l_n$, then these lengths must satisfy Kraft's inequality*

$$\sum_{k=1}^{n} \frac{1}{r^{l_k}} \leq 1. \tag{1.1}$$

2. *If the numbers $l_1, l_2, ..., l_n$ and $r$ satisfy Kraft's inequality* (1.1), *then there is an instantaneous $r$-ary code with codeword lengths $l_1, ..., l_n$.*

It is interesting to observe that Kraft's inequality is also necessary and sufficient for the existence of a uniquely decipherable code. Of course, Kraft's inequality is sufficient since any instantaneous code is also uniquely decipherable. The necessity of Kraft's inequality was proved by McMillan in 1956 [1, p. 47]:

**Theorem 1.2.** *(McMillan's Theorem). If $C = \{c_1, ..., c_n\}$ is a uniquely decipherable $r$-ary code, then its code word lengths must satisfy Kraft's inequality* (1.1).

Define now for an $r$-ary code $C$ having the code word lengths $l_1, ..., l_n$ the Kraft numbers

$$K_r(l_1, ..., l_n) = \sum_{k=1}^{n} \frac{1}{r^{l_k}}.$$

In what follows we shall point out some new inequalities for Kraft numbers which are closely connected with the inequalities (1.1). Some related results with Kraft's theorem are also given.

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## 2 The Results

We shall start with the following lemma which is of interest in itself.

**Lemma 2.1.** *Let $r, l_i$ $(i = 1, ..., n)$ be real numbers with $r > 1$. Then we have the double inequality*

$$\ln r \sum_{i=1}^{n} \frac{\log_r \left(r^{l_i}\right)}{r^{l_i}} \leq 1 - \sum_{i=1}^{n} \frac{1}{r^{l_i}} \leq \ln r \left[\frac{1}{n}\sum_{i=1}^{n} l_i - \log_r n\right]. \tag{2.1}$$

*The equality holds iff $l_i = \log_r n$ for all $i \in \{1, ..., n\}$.*

*Proof.* The exponential map $f : \mathbf{R} \to (0, \infty), f(x) = r^x$ is strictly convex on $\mathbf{R}$.

Recall that for a convex mapping $f$ which is differentiable on its domain, we have the double inequality:

$$f^{'}(y)(x-y) \leq f(x) - f(y) \leq f^{'}(x)(x-y) \tag{2.2}$$

for all $x, y$ in the domain of $f$.

As $f^{'}(x) = r^x \ln r$, then by (2.2) we get

$$r^y (x-y) \ln r \leq r^x - r^y \leq r^x (x-y) \ln r, \qquad x, y \in \mathbf{R}. \tag{2.3}$$

Now if we choose into the inequality (2.3) $x = -l_i, y = \log_r \left(\frac{1}{n}\right)$ we deduce

$$r^{-l_i}\left[-l_i - \log_r \left(\frac{1}{n}\right)\right] \ln r \geq r^{-l_i} - \frac{1}{n} \geq \frac{1}{n}\left[-l_i - \log_r \left(\frac{1}{n}\right)\right] \ln r$$

for all $i \in \{1, ..., n\}$, which is equivalent to:

$$(l_i - \log_r n)\, r^{-l_i} \ln r \leq \frac{1}{n} - \frac{1}{r^{l_i}} \leq \frac{1}{n}(l_i - \log_r n) \ln r \tag{2.4}$$

for all $i \in \{1, ..., n\}$.

Summing in (2.4) over $i$ from 1 to $n$, we deduce (2.1). The case of equality follows by the strict convexity of the mapping $f(x) = r^x$ $(r > 1, x \in \mathbf{R})$. We shall omit the details. ∎

**Theorem 2.2.** *Let $C = (c_1, ..., c_n)$ be an $r$-ary code having the codeword lengths $l_1, ..., l_n$. Then we have the estimation for the Kraft's number:*

$$\frac{1}{n \ln r}\sum_{i=1}^{n}\left[\ln (nr) - l_i [\ln r]^2\right] \leq K_r (l_1, ..., l_n) \tag{2.5}$$
$$\leq \frac{1}{n \ln r}\sum_{i=1}^{n}\left[\frac{r^{l_i}\ln r + n \ln n - n l_i [\ln r]^2}{r^{l_i}}\right].$$

*The equality holds iff $l_i = \log_r n$.*

*Proof.* By Lemma 2.1, we have

$$K_r (l_1, ..., l_n) \geq 1 - \ln r \left[\frac{1}{n}\sum_{i=1}^{n} l_i - \log_r n\right]$$
$$= \frac{1}{n \ln r}\sum_{i=1}^{n}\left[\ln(nr) - l_i (\ln r)^2\right]$$

and

$$K_r(l_1,...,l_n) \le 1 - \ln r \sum_{i=1}^{n} \frac{\log_r\left(\frac{r^{l_i}}{n}\right)}{r^{l_i}}$$

$$= \frac{1}{n \ln r} \sum_{i=1}^{n} \left[ \frac{r^{l_i} \ln r + n \ln r - n l_i [\ln r]^2}{r^{l_i}} \right].$$

The case of equality is obvious by the same lemma. ∎

**Corollary 2.3.** *Let $C = (c_1, ..., c_n)$ be an $r$-ary code having the codeword lengths $l_1, ..., l_n$. If*

$$\frac{1}{n}(l_1 + ... + l_n) < \log_r n, \tag{2.6}$$

*then $C$ is not uniquely decipherable.*

**Corollary 2.4.** *If the real numbers $r, l_i (i = 1, ..., n)$ satisfy the inequality:*

$$\frac{\sum_{i=1}^{n} \frac{l_i}{r^{l_i}}}{\sum_{i=1}^{n} \frac{1}{r^{l_i}}} \ge \log_r n \tag{2.7}$$

*then there is an instantaneous $r$-ary code with codeword lengths $l_1, ..., l_n$.*

*Proof.* Note that the inequality (2.7) is clearly equivalent to

$$\sum_{i=1}^{n} \frac{l_i - \log_r n}{r^{l_i}} \ge 0$$

but by the inequality (2.1) we have

$$0 \le \ln r \sum_{i=1}^{n} \frac{l_i - \log_r n}{r^{l_i}} \le 1 - K_r(l_1, ..., l_n)$$

and, then

$$K_r(l_1, ..., l_n) \le 1.$$

Applying Kraft's theorem we deduce the desired conclusion. ∎

**Lemma 2.5.** *Let $r, l_i \ge 1 \,(i = 1, ..., n)$ be real numbers. Then we have the inequality:*

$$\frac{1}{n}\sum_{i=1}^{n} l_i \left(1 - \frac{n^{\frac{1}{l_i}}}{r}\right) \ge 1 - \sum_{i=1}^{n} \frac{1}{r^{l_i}} \ge r \sum_{i=1}^{n} \frac{l_i}{r^{l_i} n^{\frac{1}{l_i}}} \left(1 - \frac{n^{\frac{1}{l_i}}}{r}\right). \tag{2.8}$$

*The equality holds iff $l_i = \log_r n, i = 1, ..., n$.*

*Proof.* The mapping $g(x) = x^p, p \ge 1$ is strictly convex on $(0, \infty)$ so by the inequality $(2.2)$, we have the inequality

$$pb^{p-1}(a - b) \le a^p - b^p \le pa^{p-1}(a - b) \tag{2.9}$$

for all $a, b \in [0, \infty)$.

Let choose in (2.9)

$$p = l_i \geq 1, \quad a = \frac{1}{r} \quad , b = \left(\frac{1}{r}\right)^{\frac{1}{l_i}}$$

to get for all $i \in \{1, ..., n\}$

$$l_i \left(\frac{1}{n}\right)^{\frac{l_i - 1}{l_i}} \left(\frac{1}{r} - \left(\frac{1}{n}\right)^{\frac{1}{l_i}}\right) \leq r^{-l_i} - \frac{1}{n} \leq l_i \left(\frac{1}{r}\right)^{l_i - 1} \left(\frac{1}{r} - \left(\frac{1}{n}\right)^{\frac{1}{l_i}}\right)$$

which is equivalent to

$$\frac{1}{rn} l_i n^{\frac{1}{l_i}} - \frac{l_i}{n} \leq r^{-l_i} - \frac{1}{n} \leq l_i \left(\frac{1}{r}\right)^{l_i} - l_i \left(\frac{1}{r}\right)^{l_i - 1} \left(\frac{1}{n}\right)^{\frac{1}{l_i}} \tag{2.10}$$

for all $i \in \{1, ..., n\}$.

Summing into the inequality (2.10) over $i$ from 1 to $n$, we derive

$$\frac{1}{rn} \sum_{i=1}^{n} l_i n^{\frac{1}{l_i}} - \frac{1}{n} \sum_{i=1}^{n} l_i \leq \sum_{i=1}^{n} \frac{1}{r^{l_i}} - 1 \leq \sum_{i=1}^{n} l_i \left(\frac{1}{r}\right)^{l_i} - \sum_{i=1}^{n} \frac{l_i}{r^{l_i - 1}} \frac{1}{n^{\frac{1}{l_i}}}$$

which is equivalent to (2.8).

The case of equality holds from the strict convexity of $g$ and taking into account that $\frac{1}{r} = \left(\frac{1}{n}\right)^{\frac{1}{l_i}}$ iff $\frac{1}{l_i} \log_r \frac{1}{n} = -1$, i.e., $l_i = \log_r n, i = 1, ..., n$. ∎

In the following theorem we give an estimation of Kraft numbers $K_r(l_1, ..., l_n)$ holds.

**Theorem 2.6.** *Let $C = (c_1, ..., c_n)$ be an $r$-ary code with the codeword lengths $l_1, ..., l_n$. Then we have the estimation*

$$\begin{aligned} \frac{1}{nr} \sum_{i=1}^{n} \left[n^{\frac{1}{l_i}+1} - r(l_i - 1)\right] &\leq K_r(l_1, ..., l_n) \\ &\leq \frac{1}{nr} \sum_{i=1}^{n} \left[\frac{r^{l_i+1} n^{\frac{1}{l_i}} \left(n^{\frac{1}{l_i}+1} + 1\right) - nr^2 l_i}{r^{l_i} n^{\frac{1}{l_i}}}\right] \end{aligned} \tag{2.11}$$

*The equality holds in* (2.11) *iff* $l_i = \log_r n$.

*Proof.* By Lemma 2.5 we have

$$\begin{aligned} K_r(l_1, ..., l_n) \geq 1 - \frac{1}{n} \sum_{i=1}^{n} \left(l_i - \frac{n^{\frac{1}{l_i}}}{r}\right) &= \frac{1}{nr} \sum_{i=1}^{n} \left[r(1 - l_i) + n^{\frac{1}{l_i}+1}\right] \\ &= \frac{1}{nr} \sum_{i=1}^{n} \left[n^{\frac{1}{l_i}+1} - r(l_i - 1)\right] \end{aligned}$$

and

$$\begin{aligned} K_r(l_1, ..., l_n) &\leq 1 - \sum_{i=1}^{n} \left[\frac{r l_i}{r^{l_i} n^{\frac{1}{l_i}}} - n^{\frac{1}{l_i}}\right] \\ &= \frac{1}{nr} \sum_{i=1}^{n} \left[\frac{r^{l_i+1} n^{\frac{1}{l_i}} \left(1 + n^{\frac{1}{l_i}+1}\right)}{r^{l_i} n^{\frac{1}{l_i}}}\right] \end{aligned}$$

and the inequality (2.11) is proved. The case of equality follows by Lemma 2.5, too. ∎

**Proposition 2.7.** *Let $C=(c_1,...,c_n)$ be an $r$-ary code with the codeword lengths $l_1,...,l_n$. If*

$$\frac{\sum_{i=1}^{n} l_i n^{\frac{1}{l_i}}}{\sum_{i=1}^{n} l_i} > r \tag{2.12}$$

*then $C$ is not uniquely decipherable.*

*Proof.* If we would assume that $C$ is uniquely decipherable, then by McMillan's theorem we have that $K_r(l_1,...,l_n) \geq 1$ which implies

$$0 \leq 1 - K_r(l_1,...,l_n) \leq \sum_{i=1}^{n} l_i \left(1 - \frac{n^{\frac{1}{l_i}}}{r}\right)$$

and then $\sum_{i=1}^{n} l_i \geq \frac{1}{r} \sum_{i=1}^{n} n^{\frac{1}{l_i}} l_i$ which contradicts (2.12). ∎

Finally, we obtain the following sufficient condition for the existence of an instantaneous code having a given the non-negative integers $r$ and the lengths $l_1,...,l_n$.

**Theorem 2.8.** *If the non-negative integers $r, l_i$ $(i=1,...,n)$ satisfy the inequality:*

$$r \geq \frac{\sum_{i=1}^{n} \frac{l_i}{r^{l_i}}}{\sum_{i=1}^{n} \frac{l_i}{r^{l_i} \ n^{\frac{1}{l_i}}}}$$

*then there is one instantaneous $r$-ary code with codeword lengths $l_1,...,l_n$.*

The proof follows by Lemma 2.5 and Kraft's theorem. We shall omit the details.

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