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AN ESTIMATION FOR $\ln k$

S.S. DRAGOMIR AND A. SOFO

Abstract. In this paper we point out a better estimate for *ln* k than Kicey and Goel in their recent paper [1] from American Mathematical Monthly.

1 Introduction

In their recent paper Kicey and Goel [1], established the following series expansion for $\ln k$, $k = 2, 3, ...$

$$\ln k = \sum_{i=1}^{\infty} \left[1 + k \left(\left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \frac{1}{i} \tag{1.1}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Basically, Kicey and Goel proved the following inequality:

$$\left| \ln k - \sum_{i=1}^{Nk} \left[1 + k \left(\left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \frac{1}{i} \right| \leq \frac{k-1}{N} \tag{1.2}$$

for all $k \geq 2$ and $N \geq 1$. In this paper the authors shall prove that inequality (1.2) can be improved as follows.

2 The Results

The following result holds.

Theorem 2.1. *With the above assumptions, we have the inequality:*

(2.1)
$$\left| \ln k - \sum_{i=1}^{Nk} \left[1 + k \left(\left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \frac{1}{i} \right| \leq \frac{1}{Nk} \min \left\{ k-1, \left(\frac{k-1}{(q+1)\,k} \right)^{1/q} \left(\frac{k^{2p-1}-1}{2p-1} \right)^{1/p} \right\}$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *for all* $k \geq 2$ *and* $N \geq 1$.

We prove, firstly the following lemma.

Lemma 2.2. *Let* $f : [a, b] \to \mathbf{R}$ *be an absolutely continuous mapping on* $[a, b]$. *Then we have*

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the inequality:

(2.2)
$$\left|\int_a^b f(x)\,dx-(b-a)\,f(b)\right|\le\begin{cases}\dfrac{(b-a)^2}{2}\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex]\dfrac{(b-a)^{1+1/q}}{(q+1)^{1/q}}\left\|f'\right\|_p & \text{if } f'\in L_p[a,b] \text{ where } p>1, \text{ and } \frac{1}{p}+\frac{1}{q}=1;\\[2ex](b-a)\left\|f'\right\|_1.\end{cases}$$

Proof. Integrating by parts we have

$$\int_a^b (x-a)\,f'(x)\,dx=(b-a)\,f(b)-\int_a^b f(x)\,dx$$

and from this identity, we may write

$$\begin{aligned}\left|\int_a^b f(x)\,dx-(b-a)\,f(b)\right| &\le\int_a^b\left|(x-a)\,f'(x)\right|dx\\&\le\left\|f'\right\|_\infty\int_a^b (x-a)\,dx\\&=\frac{(b-a)^2}{2}\left\|f'\right\|_\infty\end{aligned}$$

and the first inequality in (2.2) is proved. Now using Hölder's inequality we obtain

$$\begin{aligned}\int_a^b (x-a)\left|f'(x)\right|dx &\le\left\|f'\right\|_p\left[\int_a^b (x-a)^q\,dx\right]^{1/q}\\&=\frac{(b-a)^{1+1/q}}{(q+1)^{1/q}}\left\|f'\right\|_p\end{aligned}$$

and the second inequality in (2.2) is proved. Finally, we may write

$$\int_a^b (x-a)\left|f'(x)\right|dx\le(b-a)\left\|f'\right\|_1$$

and therefore (2.2) is completely proved. ∎

The following Lemma also holds:

Lemma 2.3. *Let f be as above and let $I_n : a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division*

of $[a,b]$*. Then we have the inequality:*

$$\left|\int_a^b f(x)\,dx-\sum_{i=0}^{n-1}h_i f(x_{i+1})\right|\le\begin{cases}\dfrac{\left\|f'\right\|_\infty}{2}\sum\limits_{i=0}^{n-1}h_i^2;\\[2ex]\dfrac{\left\|f'\right\|_p}{(q+1)^{1/q}}\left(\sum\limits_{i=0}^{n-1}h_i^{q+1}\right)^{1/q};\\[2ex]\nu(h)\left\|f'\right\|_1;\end{cases}\tag{2.3}$$

where $h_i:=x_{i+1}-x_i, i=0,1,2,...,n-1$ *and* $\nu(h):=\max\limits_{i=(0,n-1)}h_i$.

Proof. We have

$$\left|\int_a^b f(x)\,dx-\sum_{i=0}^{n-1}h_i f(x_{i+1})\right|=\left|\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}}f(x)\,dx-h_i f(x_{i+1})\right)\right|$$
$$\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\,dx-h_i f(x_{i+1})\right|$$

and using the first inequality in (2.2) we obtain

$$\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\,dx-h_i f(x_{i+1})\right|\le\frac{\left\|f'\right\|_\infty}{2}\sum_{i=0}^{n-1}h_i^2,$$

so the first inequality in (2.3) is proved. Using the second inequality in (2.2) and Hölder's discrete inequality, we obtain

$$\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\,dx-h_i f(x_{i+1})\right|\le\frac{1}{(q+1)^{1/q}}\sum_{i=0}^{n-1}h_i^{1+1/q}\left(\int_{x_i}^{x_{i+1}}\left|f'(t)\right|^p dt\right)^{1/p}$$
$$\le\frac{1}{(q+1)^{1/q}}\left(\sum_{i=0}^{n-1}\left(h_i^{1+1/q}\right)^q\right)^{1/q}\left(\sum_{i=0}^{n-1}\left(\left(\int_{x_i}^{x_{i+1}}\left|f'(t)\right|^p dt\right)^{1/p}\right)^p\right)^{1/p}$$
$$=\frac{1}{(q+1)^{1/q}}\left\|f'\right\|_p\left(\sum_{i=0}^{n-1}h_i^{q+1}\right)^{1/q}$$

and the second inequality in (2.3) is proved. Finally we have :

$$\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)\,dx-h_i f(x_{i+1})\right|\le\sum_{i=0}^{n-1}h_i\int_{x_i}^{x_{i+1}}\left|f'(t)\right|dt$$
$$\le\nu(h)\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\left|f'(t)\right|dt$$
$$=\nu(h)\left\|f'\right\|_1$$

and the Lemma is completely proved. ∎

Corollary 2.4. *If* $I_n : x_i = a + \frac{b-a}{n} i, i = 1, 2, ..., n$, *then we have the inequality*

(2.4)
$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \right| \le \begin{cases} \dfrac{(b-a)^2}{2n} \left\| f' \right\|_\infty \\ \dfrac{(b-a)^{1+1/q}}{n(q+1)^{1/q}} \left\| f' \right\|_p & \text{where } p > 1, \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \dfrac{(b-a)}{n} \left\| f' \right\|_1 . \end{cases}$$

Proof. Using (2.4) and noting that $a = N, b = Nk$,
$n = N(k-1)$ and $f(x) = \frac{1}{x}$ we have

$$\left| \int_N^{Nk} \frac{1}{x} dx - \sum_{i=1}^{Nk-N} \frac{1}{N+i} \right| \le \begin{cases} \dfrac{N(k-1)}{2} \left\| f' \right\|_\infty ; \\ \left(\dfrac{Nk-N}{q+1} \right)^{1/q} \left\| f' \right\|_p ; \\ \left\| f' \right\|_1 . \end{cases} \tag{2.5}$$

But we have that

$$\left\| f' \right\|_\infty = \frac{1}{N^2},$$

$$\left\| f' \right\|_p = \left(\int_N^{Nk} \frac{1}{x^{2p}} dx \right)^{1/p} = \left(\frac{k^{2p-1} - 1}{(2p-1)(Nk)^{2p-1}} \right)^{1/p}$$

and

$$\left\| f' \right\|_1 = \frac{k-1}{Nk},$$

hence from (2.5) we obtain

(2.6)
$$\left| \ln k - \sum_{i=1}^{Nk-N} \frac{1}{N+i} \right| \le \begin{cases} \dfrac{k-1}{2N} ; \\ \dfrac{1}{Nk} \left(\dfrac{k-1}{(q+1)k} \right)^{1/q} \left(\dfrac{k^{2p-1} - 1}{2p-1} \right)^{1/p} & \text{where } p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1; \\ \dfrac{k-1}{Nk} . \end{cases}$$

from (2.6), $\frac{k-1}{2} \ge \frac{k-1}{k}$ for $k \ge 2$, hence, by the identity, see [1],

$$\sum_{i=1}^{Nk-N} \frac{1}{N+i} = \sum_{i=1}^{Nk} \left[1 + k \left(\left\lfloor \frac{i-1}{k} \right\rfloor - \left\lfloor \frac{i}{k} \right\rfloor \right) \right] \frac{1}{i},$$

Theorem 1 is proved. ∎

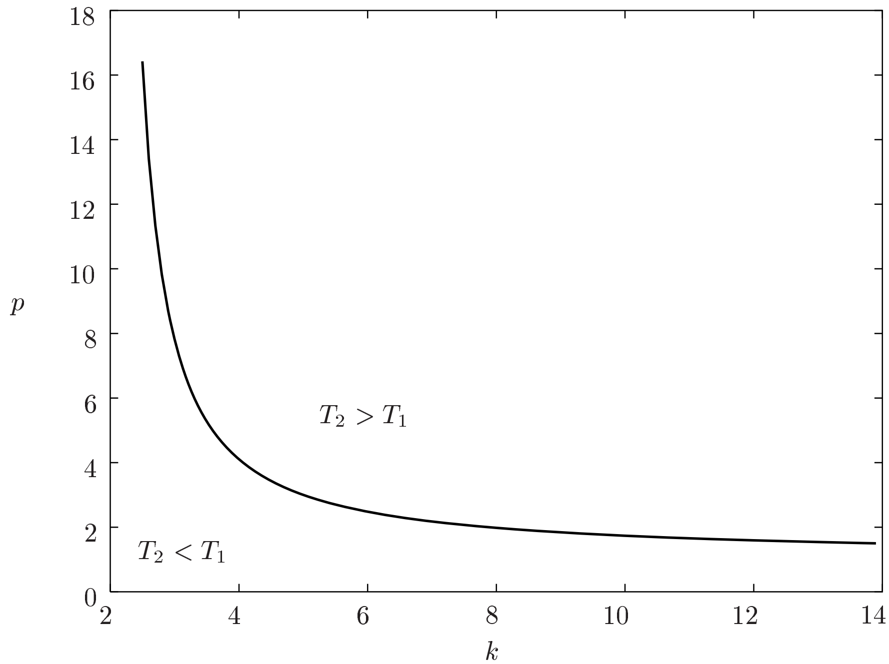

FIGURE 1: The contour of $T_2/T_1 = 1$ on the $k-p$ plane.

Remark 2.1. *Clearly, for a minimum of* (2.6) *we need only investigate the terms* $T_1 = k-1$ *and* $T_2 = \left(\frac{k-1}{(q+1)k}\right)^{1/q}\left(\frac{k^{2p-1}-1}{2p-1}\right)^{1/p}$.

Using a computer package we may obtain the contour line $\frac{T_2}{T_1} = 1$ as follows.

From figure 1, the region on the left of the contour line is described by $T_2 < T_1$ and the region on the right of the contour line is described by $T_2 > T_1$. This demonstrates, clearly, that each of the bounds T_1 or T_2 may be best under different circumstances.

References

[1] Kicey, C., Goel, S. *A Series for* $\ln k$. American Mathematical Monthly, Vol.105, June-July, pp.552-554, 1998.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, Victoria, Australia

E-mail address: {sever, sofo}@matilda.vut.edu.au