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A HADAMARD-JENSEN INEQUALITY AND AN APPLICATION TO THE ELASTIC TORSION PROBLEM

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Abstract. The (generalised) torsion function u of a domain $\Omega \subset \mathbb{R}^n$ is a function which is zero on the boundary of the domain and whose Laplacian is minus one at every point in the interior of the domain. Denote by $|\Omega|$ the measure of Ω, x_c its centroid. We establish, for convex Ω,

$$\frac{3}{2(n+1)^2} \le \frac{1}{(n+1)^2} \frac{|\Omega|}{\int_\Omega u} \max_\Omega u \le \frac{|\Omega| u(x_c)}{\int_\Omega u} \le \frac{|\Omega|}{\int_\Omega u} \max_\Omega u \le \frac{1}{2}(n+1)(n+2).$$

Various improvements, generalisations and possible applications are discussed.

1 Foundational materials

1.1 Introduction

The main results in this paper concern positive functions u with certain concavity properties (usually that u to some power is concave) defined on a convex domain Ω and vanishing on its boundary. We find bounds for quantities, with $k > 0$, like

$$\zeta(\Omega, k, u, x_p) = \frac{|\Omega| u(x_p)^k}{\int_\Omega u^k}, \tag{1.1}$$

where x_p is a given point in Ω. In some bounds, notably those from the Hadamard inequality, $x_p = x_c$ the centroid of Ω. In other bounds, $x_p = x_m$ the location of the maximum of u.

The Research Report version of this paper, and supplements at the same Web site, [1], contain further details and applications.

Without concavity properties for u or information concerning u such as one might have from being given that u solves some partial differential equation problem, results on ζ are limited. Here, however, is one easy result.

Theorem 1.1. *For any domain Ω and positive function u defined on Ω, the functional $\zeta(\Omega, k, u, x_p)$ defined by equation (1.1) satisfies the following.*

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(i) *For any* $x_p \in \Omega$, $\zeta(\Omega, \cdot, u, x_p)$ *is positive and logconcave on* $[0, \infty)$.

(ii) *Let* x_m *be a location of the maximum of* u. *Then* $\zeta(\Omega, \cdot, u, x_m)$ *is monotonic nondecreasing on* $[0, \infty)$. *For* $k \in [0, \infty)$, $\zeta(\Omega, 0, u, x_m) = 1 \leq \zeta(\Omega, k, u, x_m)$.

A simple application of the Hőlder inequality gives, for $0 \leq t \leq 1$ and omitting arguments which are the same in the ζ,

$$\frac{1}{\zeta((1-t)k_0 + tk_1)} \leq \frac{1}{\zeta(k_0)^{1-t}} \frac{1}{\zeta(k_1)^t},$$

which proves (i).

Write $\zeta_m(k) = \zeta(k, x_m)$. The result $1 \leq \zeta_m(k)$ is easily improved for the solutions u of our partial differential equation problems: see, for example, the leftmost inequality – on $\zeta_m(1)$ – in the abstract.

1.2 Hadamard's inequality

The Hermite-Hadamard inequality for convex functions defined on an interval of the real line is given, with some history, in [10]. The result immediately below is a generalisation of this to higher space dimensions.

Theorem 1.2. *Let* Ω *be a convex, compact subset of* $\mathbb{R}^n$ *with nonempty interior. Suppose that the mapping* $\Phi : \Omega \to \mathbb{R}$ *is differentiable and convex on* Ω. *Define, for* $x_p \in \Omega$,

$$I_1(x_p) = \int_\Omega \Phi(x)dx - |\Omega|\Phi(x_p), \qquad I_2(x_p) = \int_\Omega \langle \nabla\Phi(x), x\rangle dx - \langle x_p, \int_\Omega \nabla\Phi(x)dx\rangle,$$

where $|\Omega|$ *is the volume of* Ω. *Then one has the inequalities*

$$0 \leq I_1(x_c), \tag{1.2}$$

$$I_1(x_p) \leq I_2(x_p), \tag{1.3}$$

where x_c *is the centroid of* Ω. *If* $u = \Phi$ *is concave rather than convex, the inequalities are reversed, and, if additionally* $u \geq 0$, $\zeta(\Omega, 1, u, x_c) \geq 1$.

Proof. As Φ is differentiable in the interior of Ω we know that

$$\Phi(x) - \Phi(y) \geq \langle (\nabla\Phi)(y), x - y\rangle \qquad \forall\, x, y \in \Omega, \tag{1.4}$$

where $\langle a, b\rangle = a \cdot b$ is the usual inner product in $\mathbb{R}^n$. Applying this with $y = x_c$ and integrating over x gives

$$\begin{aligned} I_1(x_c) &= \int_\Omega \Phi(x)dx - |\Omega|\Phi(x_c) \geq \int_\Omega \langle \nabla\Phi(x_c), x - x_c\rangle dx, \\ &= \langle \nabla\Phi(x_c), \int_\Omega (x - x_c)\, dx\rangle = 0, \end{aligned}$$

which establishes inequality (1.2).

Let x_p be any point in the interior of Ω. By inequality (1.4) we also deduce that

$$\Phi(x_p) - \Phi(x) \geq \langle (\nabla\Phi)(x), x_p - x \rangle \qquad \forall\, x \in \Omega\ .$$

Integrating this inequality with respect to x over Ω gives

$$\begin{aligned} -I_1(x_p) &= |\Omega|\Phi(x_p) - \int_\Omega \Phi(x)dx \\ &\geq \int_\Omega \langle \nabla\Phi(x), x_p \rangle dx - \int_\Omega \langle \nabla\Phi(x), x \rangle dx = -I_2(x_p), \end{aligned}$$

from which we get inequality (1.3). ∎

2 Positive power-concave functions vanishing on $\partial\Omega$

2.1 Notation

We apologise for the switch from convex functions to concave ones. However, this seems to be forced on us because of applications to partial differential equations coming in Section 3.

In this section we are concerned with classes of positive functions defined on the convex set Ω. Specifically, for $1 \leq \alpha$, define

$$\begin{aligned} \mathcal{U}_\alpha &= \{u \geq 0 \mid\ u^{1/\alpha} \text{ is concave}\}, \\ \mathcal{U}_{\alpha,0} &= \{u \mid\ u \in \mathcal{U}_\alpha \text{ which are zero on } \partial\Omega\}. \end{aligned}$$

The centroid x_c is introduced in Section 1.2. Define also

$$u_m = \max_\Omega u, \qquad u(x_m) = u_m.$$

The results stated in the rest of this subsection, subsection 2.1, are for $u \geq 0$ and $1 \leq \xi$. We have

$$(2.1) \qquad \Big(\frac{1}{|\Omega|}\int_\Omega u^{1/\xi}\Big) \leq \Big(\frac{1}{|\Omega|}\int_\Omega u\Big)^{1/\xi}, \qquad \text{or} \qquad \zeta(1)^{1/\xi} \leq \zeta\Big(\frac{1}{\xi}\Big).$$

This inequality is simply Hőlder's inequality, and is recorded here because of our concern – in our main application, the torsion problem – with $\int_\Omega u$. (It is also follows from the logconcavity of ζ using $\zeta(0) = 1$.) Another trivial inequality is

$$(2.2) \qquad \frac{1}{|\Omega|}\int_\Omega u \leq u_m^{(\xi-1)/\xi}\frac{1}{|\Omega|}\int_\Omega u^{1/\xi}, \qquad \text{or} \qquad \zeta_m\Big(\frac{1}{\xi}\Big) \leq \zeta_m(1),$$

consistent with Theorem 1.1 (ii).

If $u \geq 0$ and $1 \leq \xi$, and $u = 0$ on the boundary of Ω, an application of the divergence theorem gives

$$(2.3) \qquad -\int_\Omega x \cdot \nabla\big(u^{1/\xi}\big) = -\int_\Omega \operatorname{div}\big(xu^{1/\xi}\big) + n\int_\Omega u^{1/\xi} = n\int_\Omega u^{1/\xi}.$$

2.2 Concave function results

Theorem 2.1. *Let* $\alpha \geq 1$. $\forall u \in \mathcal{U}_{\alpha,0}$

$$\frac{1}{n+1} u_m^{1/\alpha} \leq \frac{1}{|\Omega|} \int_\Omega u^{1/\alpha} \leq u(x_c)^{1/\alpha}, \tag{2.4}$$

or

$$\frac{1}{(n+1)} \zeta_m\big(\frac{1}{\alpha}\big) \leq 1 \leq \zeta\big(\frac{1}{\alpha}, x_c\big) \leq \zeta_m\big(\frac{1}{\alpha}\big) \leq (n+1).$$

From this:

$$u_m \leq c_m(n,\alpha) u_c \qquad \textit{where} \qquad c_m(n,\alpha) = (n+1)^\alpha; \tag{2.5}$$

$$\zeta_m(1) = \frac{|\Omega| u_m}{\int_\Omega u} \leq c_{r,0}(n,\alpha) \qquad \textit{where} \qquad c_{r,0}(n,\alpha) = (n+1)^\alpha. \tag{2.6}$$

Proof. The right-hand part of the first inequalities (2.4-r) follows on applying the generalisation of Hadamard's inequality (1.2) given in Section 1.2 to the function $\Phi = -u^{1/\alpha}$. The left-hand part (2.4-l) follows from the fact that $u^{1/\alpha}$ lies above a 'cone' base Ω and height $u_m^{1/\alpha}$. The next inequality is a combination of both parts of inequalities (2.4).

Inequality (2.6) follows from inequalities (2.1), with $\xi = \alpha$, and the left-hand part (2.4-l). ∎

Here is an alternative proof of the left-hand inequality (2.4-l). We start with applying inequality (1.3) at any point $x_p \in \Omega$:

$$-\int_\Omega u^{1/\alpha} + |\Omega| \big(u(x_p)\big)^{1/\alpha} \leq -\int_\Omega x \cdot \nabla\big(u^{1/\alpha}\big).$$

Next apply (2.3), with $\xi = \alpha$. This gives

$$|\Omega| u(x_p)^{1/\alpha} \leq (n+1) \int_\Omega u^{1/\alpha}.$$

This inequality is best when $x_p = x_m$ which gives the left-hand inequality (2.4-l) as stated in the theorem.

Combining the next two Theorems improves on inequality (2.6).

Theorem 2.2. *Let* $\Omega^* = \{x \in \mathbb{R}^n \mid \ |x| < \rho\}$. *Let* $U = U_m(1 - |x|/\rho)$ *define a conical graph over* Ω^*. *Let* $\xi \geq 0$.

$$\frac{|\Omega^*| U_m^\xi}{\int_\Omega U^\xi} = c_r(n,\xi), \qquad \textit{where} \qquad c_r(n,\xi) = \frac{\Gamma(n+\xi+1)}{\Gamma(n+1)\Gamma(\xi+1)}. \tag{2.7}$$

Proof. Using spherical polar coordinates, we find

$$\frac{\int_\Omega U^\xi}{\int_\Omega 1} = U_m^\xi \frac{\int_0^1 (1-t)^\xi t^{n-1}\, dt}{\int_0^1 t^{n-1}\, dt} = U_m^\xi \frac{\Gamma(n+1)\Gamma(\xi+1)}{\Gamma(n+\xi+1)}.$$

Refer to [11, 2] for the definitions and properties of Schwarz symmetrisation. ∎

The following result is in the standard books on convexity for Steiner symmetrisation, and is also true for Schwarz symmetrisation. If v is concave over Ω, its Schwarz symmetrisation v^* is concave over the symmetrised domain Ω^*. (The result for Schwarz symmetrisation is proved in some books by combining the Blaschke Selection Principle with the result for Steiner symmetrisation.) This fact is used in the following proof.

Theorem 2.3. *Let* $\alpha \geq 1$, $\xi \geq 0$. $\forall u \in \mathcal{U}_{\alpha,0}$

$$\zeta_m(\xi) = \frac{|\Omega| u_m^\xi}{\int_\Omega u^\xi} \leq c_r(n, \alpha\xi), \tag{2.8}$$

where c_r *is defined in the preceding lemma.*

Proof. Translate the origin so that it is at x_m, $x_m = 0$. Define, for $x_{\partial\Omega} \in \partial\Omega$, the cone

$$U_c(t x_{\partial\Omega}) = u_m^{1/\alpha}(1-t).$$

$U_c \in \mathcal{U}_{1,0}$ and $u \geq U_c^\alpha$.

Next, consider the Schwarz symmetrisation U_c^* of U_c.

$$\int_\Omega u^\xi \geq \int_\Omega U_c^{\alpha\xi} = \int_{\Omega^*} \left(U_c^*\right)^{\alpha\xi}.$$

The level curve $U_c = u_m^{1/\alpha}(1-t)$ is geometrically similar to $\partial\Omega$: it is $\{t x_{\partial\Omega} \mid x_{\partial\Omega} \in \partial\Omega\}$.

$$|\{x \mid U_c(x) > u_m^{1/\alpha}(1-t)\}| = t^n |\Omega|.$$

From this U_c^* is a 'circular cone', so the preceding lemma can be applied and gives the result. ∎

When $\xi = 1$ and $\alpha > 1$ inequality (2.8) improves on inequality (2.6): in the case $\alpha = 1$ they are equal. For the application to the torsion problem, we note $c_r(n, 2) = (n+1)(n+2)/2$, the right-most expression in the inequality given in the Abstract.

3 The elastic torsion problem

3.1 Preliminaries

In their 1951 book [11], and subsequent papers, Pólya and Szegő were concerned with bounding various 'physical' domain functionals in terms of 'geometrical' ones. The geometric functionals include (when $n = 2$) the area of Ω (or the measure of Ω for general n), $|\Omega|$, its centroid x_c, its polar moment of inertia I_c about the centroid, etc.. The physical domain functionals arise from various partial differential equation problems. For ease of exposition we consider, in this Section, just one problem, the elastic torsion problem. In the actual physical problem concerning elastic torsion, $n = 2$: see [11].

Given a domain Ω, the problem of finding a u, twice continuously differentiable in Ω and continuous on the closure of Ω satisfying, for some given positive constant μ,

$$\begin{aligned} -\Delta u &= \mu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on the boundary of} \quad \Omega, \end{aligned}$$

is called the (St Venant elastic) torsion problem. There is no loss of generality in taking $\mu = 1$.

A functional of interest in some applications is the maximum of the torsion function, u_m, and its location x_m, $u(x_m) = u_m = \max_\Omega u$. Other functionals of significance in elasticity include the following. The *torsional rigidity* is

$$S = \int_\Omega u = \int_\Omega |\nabla u|^2 = -\frac{1}{n}\int_\Omega x \cdot \nabla u.$$

Another functional studied here is $u_c = u(x_c)$, the torsion function evaluated at the centroid, x_c.

In [11] there is some concern with nondimensional combinations of domain functionals. Quantities like $S|\Omega|^{-2}$, $SI_c|\Omega|^{-4}$ – appropriate when $n = 2$ – and similar appear in tables at the back of the book, and others are scattered throughout the text and elsewhere in the literature. Amongst various ways of unifying the bounds on domain functionals, say for some non-dimensional combination Q is to find positive lower bounds Q_{LB} and finite upper bounds Q_{UB} so that the bounds have the form

$$Q_{LB} \le Q(\Omega) \le Q_{UB} \qquad \text{for } \Omega \text{ in some class of domains.}$$

A favorite class of domains is the bounded convex domains.

For a survey of the elastic torsion problem in convex domains, for the case $n = 2$, see [5].

Theorem 3.1. *Let u be the torsion function of a convex domain Ω. Then the square root of u, $\sqrt{u}$ is concave.*

For $n = 2$ this was proved in Makar-Limanov, [7]. The result for higher space dimensions was first proved in [6].

Theorem 3.2. (Sperb [12]). *Let u be the torsion function of a bounded convex domain Ω. Then,*

$$\frac{\beta+2}{2}\int_\Omega u^{1/\beta} \le u_m \int_\Omega u^{(1-\beta)/\beta}, \qquad \beta > 0, \tag{3.1}$$

$$\frac{3}{2} \le \frac{|\Omega| u_m}{\int_\Omega u} = \zeta_m(1). \tag{3.2}$$

Proof. Define the quantities P_k,

$$P_k = |\nabla u|^2 + k\mu u.$$

Henceforth $\mu = 1$. In this paragraph we repeat proofs by Payne and by Sperb [12] that, for convex Ω,

$$P_2 \le 2u_m. \tag{3.3}$$

P_2 satisfies an elliptic differential inequality (or equation when $n = 2$),

$$\Delta P_2 + \frac{\ell \cdot \nabla P_2}{|\nabla u|^2} = R(n) \ge 0, \qquad \ell = 2\nabla u - \frac{1}{2}\nabla P_2,$$

and $R(2) = |\nabla P_2|^2/(2|\nabla u|^2)$. The coefficients in this differential inequality become singular at points where $|\nabla u| = 0$, and only at these points. An application of the maximum principle establishes that the maximum of P_2 occurs either at a point where $|\nabla u| = 0$ or on the boundary of Ω. A calculation given on p.76 of [12] shows that at any point on $\partial\Omega$,

$$\frac{\partial P_2}{\partial n} = (n-1)|\nabla u|^2 M,$$

where M is the mean curvature of $\partial\Omega$. When Ω is convex, $M \ge 0$, and the Hopf form of the maximum principle shows that it is impossible for the maximum of P_2 to be attained on $\partial\Omega$. The details are given at pp.76-77 of [12]. Thus inequality (3.3) is established. (The result was first established, for $n = 2$, by Payne.)

An application of the divergence theorem (to $\mathrm{div}(u^{1/\beta}\nabla u)$) shows that

$$\int_\Omega u^{1/\beta} = \frac{1}{\beta}\int_\Omega u^{(1-\beta)/\beta}|\nabla u|^2.$$

Applying inequality (3.3), $P_2 \le 2u_m$, in the preceding gives inequality (3.1) which at $\beta = 1$ is

$$3\int_\Omega u = \int_\Omega P_2 \le 2u_m|\Omega|,$$

hence inequality (3.2). ∎

3.2 The new results

Theorem 3.3. *The torsion function u of a bounded convex domain Ω satisfies*

$$\frac{3}{2(n+1)^2} \leq \frac{1}{(n+1)^2}\frac{|\Omega|u_m}{\int_\Omega u} \leq \frac{|\Omega|u_c}{\int_\Omega u} \ , \tag{3.4}$$

$$c_r(n,2)^{-1} \leq \frac{|\Omega|u_c}{\int_\Omega u} \leq \frac{|\Omega|u_m}{\int_\Omega u} \leq c_r(n,2). \tag{3.5}$$

If, in addition, $u \in \mathcal{U}_{0,\alpha}$ for $1 \leq \alpha \leq 2$,

$$\frac{3}{2(n+1)^\alpha} \leq \frac{1}{(n+1)^\alpha}\frac{|\Omega|u_m}{\int_\Omega u} \leq \frac{|\Omega|u_c}{\int_\Omega u}, \tag{3.6}$$

$$c_r(n,\alpha)^{1-\alpha} \leq \frac{|\Omega|u_c}{\int_\Omega u} \leq \frac{|\Omega|u_m}{\int_\Omega u} \leq c_r(n,\alpha). \tag{3.7}$$

Proof. Inequalities (3.4) and (3.5) follow on using Theorem 3.1 and inequalities (3.6) and (3.7).

The right-most parts of inequalities (3.7) follow from (2.6). From inequality (2.2) with $\xi = \alpha$ and the Hadamard inequality (2.4)

$$\frac{1}{|\Omega|}\int_\Omega u \leq u_m^{(\alpha-1)/\alpha}\frac{1}{|\Omega|}\int_\Omega u^{1/\alpha} \leq u_m^{(\alpha-1)/\alpha}u_c^{1/\alpha} \ ,$$

from which

$$1 \leq \Big(\frac{|\Omega|u_m}{\int_\Omega u}\Big)^(\alpha-1)\ \Big(\frac{|\Omega|u_c}{\int_\Omega u}\Big) \ .$$

Using the rightmost inequality of (3.7) to control the first term, the u_m-term, of the last inequality yields the leftmost inequality of (3.7).

From Theorem 3.2 and inequalities (2.5) (for $u \in \mathcal{U}_{0,\alpha}$, $1 \leq \alpha \leq 2$), we have

$$\frac{3}{2} \leq \frac{|\Omega|u_m}{\int_\Omega u} \qquad \text{and} \qquad u_m \leq (n+1)^\alpha u_c.$$

These yield inequalities (3.6). This completes the proof. ∎

3.3 Exact solutions, possible best constants, related items

3.3.1 $n = 1$.

The case $n = 1$ is trivial: $|\Omega|u_c/\int_\Omega u = \frac{3}{2}$.

3.3.2 $n = 2$ and some closed-form torsion functions in some simple domains.

Simple exact solutions are convenient for checking against inequalities, and helping in the formulation of conjectures. The solutions given in this section are polynomials. (Numerous other exact solutions are available. There is some interest in these when the domain Ω is easy to describe, and the solutions are elementary transcendental functions. See references in [5, 11].)

Considering quadratic and cubic polynomials for u one can solve the elastic torsion problem in an ellipse and in an equilateral triangle. One finds the following.

Ω	$\{\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$	$\{y + \frac{a}{\sqrt{3}} > 0, \sqrt{3}\lvert x\rvert < y - \frac{2a}{\sqrt{3}}\}$
$u(x,y)$	$\frac{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{1}{4\sqrt{3}a}\left(y + \frac{a}{\sqrt{3}}\right)\left((y - \frac{2a}{\sqrt{3}})^2 - 3x^2\right)$
$S = \int_\Omega u$	$\frac{\pi}{4}\frac{ab}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{\sqrt{3}a^4}{20}$
u_m	$\frac{1}{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}$	$\frac{a^2}{9}$
Ω_1	Ω	$\{x^2 + y^2 \le \frac{a^2}{3}\}$

The concavity set Ω_1 is defined in [5]. For both the ellipse and the equilateral triangle, x_m coincides with the centroid.

For the ellipse, u is concave. For the equilateral triangle, u is not concave (and, in fact, $u^{1/\alpha}$ is not concave for any $\alpha < 2$). For the ellipse, Ω_e,

$$\frac{|\Omega_e| u_c}{S} = 2.$$

We expect that for a family of rectangles tending to an infinite strip,

$$\frac{|\Omega_s| u_c}{S} \to \frac{3}{2}.$$

To date the extreme values we have found are
$|\Omega| u_c / S = \frac{20}{9}$ for an equilateral triangle, and
$|\Omega| u_c / S \sim \frac{4}{3}$ for a thin isosceles triangle or a thin sector.
We have no numerical evidence to contradict a conjecture that, at least amongst circular-arc triangles, these are the extreme values.

Using a pair of well-known inequalities for the $n = 2$ torsion problem, the inequalities (3.5) can be improved replacing $c(2,2) = 6$ by 4. Let ρ denote the inradius of Ω. The first of these inequalities, proved in [11], is

$$\frac{1}{8}\rho^2 |\Omega| \le S,$$

an inequality which becomes an equality when Ω is a disk. The second, proved in [12] is

$$u_m \le \frac{1}{2}\rho^2, \qquad \text{for convex } \Omega,$$

an inequality which becomes an equality when Ω is a strip. Combining these last two inequalities gives

$$\frac{|\Omega|u_m}{S} \leq 4,$$

as previously asserted. (See [1] for a reference which can be used to establish an upper bound for $|\Omega|u_m/S$ for any simply connected $\Omega \subset \mathbb{R}^2$.)

4 Conclusion

In this paper we have proved inequalities involving u_c the torsion function evaluated at the centroid of a convex domain Ω.

In [1] this is generalised to positive solutions of the semilinear equation $-\Delta u = u^\gamma$, $0 \leq \gamma < 1$, vanishing on $\partial\Omega$. ($\gamma = 0$ is the torsion problem.) The starting point for our application of the Hadamard inequality, to solutions of this semilinear problem, is the result in [6, 4, 2] that $u^{(1-\gamma)/2}$ is concave. [12] also provides results generalising those we used in Section 3.

The second author has a possible application area, discussed in supplements to [1]. This concerns a problem from steady plane inviscid hydrodynamics: vortex pairs. In a popular formulation of this, [3], the area and centroid of the vortex core region is prescribed, and a modified stream function solves the elastic torsion problem in the core. (The main weakness in the suggestion for this application area is that, though there is numerical evidence favouring the convexity of the vortex cores, there is, as yet, no simple proof of their convexity.)

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