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SOME REMARKS ON THE MIDPOINT RULE IN NUMERICAL INTEGRATION

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Abstract. Using some classical results from the theory of inequalities (Grüss' inequality, Hermite-Hadamard's inequality and others) we point out some quasi-midpoint quadrature formulae, for which the errors of approximation are smaller than the error given for the classical approach.

1 Introduction.

The following inequality is well known in the literature as the midpoint inequality:

$$\left|\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{\|f''\|_\infty}{24}(b-a)^3 \tag{1.1}$$

where the mapping $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is supposed to be twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), that is,
$\|f''\|_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty$. Now if we assume that
$I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we can approximate the integral $\int_a^b f(x)\,dx$ by the *midpoint quadrature* formula $M_T(f,I_h)$ having an error given by $R_T(f,I_h)$, where

$$M_T(f,I_h) = \sum_{i=0}^{n-1} f\left(\frac{x_{i+1}+x_i}{2}\right) h_i$$

and the remainder $R_T(f,I_h)$ satisfies the estimation

$$|R_T(f,I_h)| \leq \frac{\|f''\|_\infty}{24}\sum_{i=0}^{n-1} h_i^3$$

where $h_i = x_{i+1} - x_i$ for $i = 0,1,2,...,n-1$.

In this paper, by the use of some classical results from the theory of inequalities; Hölder's inequality, Grüss' inequality and the Hermite-Hadamard inequality; we provide some quasi-midpoint quadrature formulae for which the remainder terms are smaller than the classical one given above. For other results in connection with the midpoint inequality see chapter XV of the recent book by Mitrinović et al. [2].

2 Some Integral Inequalities.

The following lemma will be useful in what follows.

Lemma 2.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on (a,b). Suppose that $f'' : (a,b) \to \mathbb{R}$ is integrable on (a,b). Then we have the identity:*

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$$\int_a^b f(x)\,dx = (b-a)\,f\left(\frac{a+b}{2}\right) + \int_a^b \phi(x)\,f''(x)\,dx \tag{2.1}$$

where $\phi(x)$ *is the kernel given by*

$$\phi(x) = \begin{cases} \frac{(x-a)^2}{2} & \text{if } x \in \left[a, \frac{a+b}{2}\right] \\ \frac{(b-x)^2}{2} & \text{if } x \in \left(\frac{a+b}{2}, b\right] \end{cases}. \tag{2.2}$$

Proof. We have successively

$$\int_a^b \phi(x)\,f''(x)\,dx = \int_a^{\frac{a+b}{2}} \frac{(x-a)^2}{2} f''(x)\,dx + \int_{\frac{a+b}{2}}^b \frac{(b-x)^2}{2} f''(x)\,dx,$$

integrating by parts twice we eventually obtain

$$\begin{aligned}\int_a^b \phi(x)\,f''(x)\,dx &= \int_a^{\frac{a+b}{2}} f(x)\,dx - \frac{b-a}{2} f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b f(x)\,dx \\ &\quad + \frac{a-b}{2} f\left(\frac{a+b}{2}\right) \\ &= \int_a^b f(x)\,dx - (b-a)\,f\left(\frac{a+b}{2}\right)\end{aligned}$$

and the identity (2.1) is proved.■

The following theorem containing an integral inequality, which is known in the literature as the *midpoint inequality*, holds.

Theorem 2.2. *Let* f *be as above. Then we have the inequality*

$$\left|\int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right)\right| \le \begin{cases} \frac{(b-a)^3}{24}\|f''\|_\infty, & \text{if } f'' \in L_\infty(a,b), \\ \frac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q, & \text{if } f'' \in L_q(a,b) \text{ where } \frac{1}{p}+\frac{1}{q}=1, p>1, \\ \frac{(b-a)^2}{8}\|f''\|_1 & \text{if } f'' \in L_1(a,b). \end{cases} \tag{2.3}$$

Proof. Using the representation (2.1) we have that

$$\left|\int_a^b f(x)\,dx - (b-a)\,f\left(\frac{a+b}{2}\right)\right| \le \int_a^b |\phi(x)|\,|f''(x)|\,dx.$$

Now, if $f'' \in L_\infty(a,b)$, then

$$\begin{aligned}\int_a^b |\phi(x)|\,|f''(x)|\,dx &\le \|f''\|_\infty \int_a^b |\phi(x)|\,dx \\ &= \|f''\|_\infty \left[\int_a^{\frac{a+b}{2}} \frac{(x-a)^2}{2} dx + \int_{\frac{a+b}{2}}^b \frac{(b-x)^2}{2} dx\right] \\ &= \frac{(b-a)^3}{24}\|f''\|_\infty.\end{aligned}$$

If $f'' \in L_q(a,b)$ where $\frac{1}{p}+\frac{1}{q}=1,\ p>1$ then we have, by Hölder's inequality, that

$$\int_a^b |\phi(x)|\,|f''(x)|\,dx \le \|f''\|_q \left(\int_a^b |\phi(x)|^p\,dx\right)^{\frac{1}{p}}.$$

But

$$\begin{aligned}\int_a^b |\phi(x)|^p\,dx &= \int_a^{\frac{a+b}{2}} \left(\frac{(x-a)^2}{2}\right)^p dx + \int_{\frac{a+b}{2}}^b \left(\frac{(b-x)^2}{2}\right)^p dx \\ &= \frac{(b-a)^{2p+1}}{8^p(2p+1)}\end{aligned}$$

and therefore the second inequality in (2.3) holds. Finally, if $f'' \in L_1(a,b)$, then

$$\begin{aligned}\int_a^b |\phi(x)|\,|f''(x)|\,dx &\le \max_{x\in(a,b)} \phi(x)\,\|f''\|_1 \\ &= \frac{(b-a)^2}{8}\,\|f''\|_1\end{aligned}$$

and therefore the last inequality in (2.3) is also proved.

An example will now be presented to illustrate that the different norms in (2.3) provide better bounds on the error depending on the behaviour of the integrand. We may take, without loss of generality, in the right hand of (2.3), $a=0$ and $b-a=2\beta$ so that

$$T_1 = \frac{\beta^3}{3}\sup_{t\in(0,2\beta)} |f''(t)|$$

$$T_2 = \frac{\beta^2}{2}\left(\frac{2\beta}{2p+1}\right)^{\frac{1}{p}} \left(\int_0^{2\beta} |f''(t)|^q\,dt\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1, p>1 \text{ and}$$

$$T_3 = \frac{\beta^2}{2}\int_0^{2\beta} |f''(t)|\,dt.$$

Consider the example $f''(t)=e^t$, the Figure 1, shows, on the (p,β) plane, the contours, from the horizontal axis, of the ratios $\frac{T_1}{T_2}=1, \frac{T_1}{T_3}=1$ and $\frac{T_2}{T_3}=1$. The regions A, B, C and D are respectively represented by the inequalities:

$$\begin{aligned} A &: T_1 < T_2 < T_3,\ B : T_1 < T_3 < T_2, \\ C &: T_2 < T_3 < T_1,\ D : T_3 < T_2 < T_1. \end{aligned}$$

Hence, we have demonstrated that each of the bounds T_1, T_2 or T_3 are best in a particular region of (p,β).

$\frac{T_2}{T_3} = 1$

β D

C

$\frac{T_1}{T_3} = 1$ $\frac{T_1}{T_2} = 1$

B A

p

FIGURE 1: The diagram shows regions A, B, C, D of the (p, β) plane, separated by the contours of $\frac{T_1}{T_2} = 1, \frac{T_1}{T_3} = 1$ and $\frac{T_2}{T_3} = 1$.

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The following theorem, regarding an integral inequality also holds.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping on (a, b). If $f'' : (a, b) \to \mathbb{R}$ satisfies the condition*

$$\gamma \leq f''(x) \leq \Gamma \text{ for all } x \in (a, b), \tag{2.4}$$

then the following inequality is satisfied:

$$\left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}\left(f'(b) - f'(a)\right) \right| \leq \frac{(b-a)^3 (\Gamma - \gamma)}{32}. \tag{2.5}$$

Proof. Applying Grüss' integral inequality [[1], p.296], we may state that

$$\left| \frac{1}{b-a}\int_a^b \phi(x) f''(x)\,dx - \frac{1}{b-a}\int_a^b \phi(x)\,dx \frac{1}{b-a}\int_a^b f''(x)\,dx \right| \leq \frac{(b-a)^2 (\Gamma - \gamma)}{32} \tag{2.6}$$

as $0 \leq \phi(x) \leq \frac{(b-a)^2}{8}$ for all $x \in [a, b]$. It may be easily seen that $\frac{1}{b-a}\int_a^b \phi(x)\,dx = \frac{(b-a)^3}{24}$ and

$\frac{1}{b-a}\int_a^b f''(x)\,dx = \frac{f'(b)-f'(a)}{b-a}$ and hence from (2.6) we may write

$$\left|\int_a^b \phi(x) f''(x)\,dx - \frac{(b-a)^2}{24}\left(f'(b)-f'(a)\right)\right| \le \frac{(b-a)^3(\Gamma-\gamma)}{32},$$

which, by identity (2.1), is clearly equivalent to inequality (2.5).■

Now, using the celebrated Hermite-Hadamard integral inequality for convex functions, $g : [a,b] \to \mathbb{R}$, which may be written as

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(x)\,dx \le \frac{g(a)+g(b)}{2} \tag{2.7}$$

we obtain the following theorem.

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be as in the above theorem; then we have the following double inequality:*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24} \tag{2.8}$$

and the estimation

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) - \frac{(\gamma+\Gamma)(b-a)^2}{48}\right| \le \frac{(\Gamma-\gamma)(b-a)^3}{48}. \tag{2.9}$$

Proof. Let us choose in (2.7) $g(x) = f(x) - \frac{\gamma x^2}{2}$, then $g(x)$ is a convex function in x, since $g''(x) \ge 0$, and hence

$$f\left(\frac{a+b}{2}\right) - \frac{\gamma(a+b)^2}{8} \le \frac{1}{b-a}\left(\int_a^b f(x)\,dx - \frac{\gamma(b^3-a^3)}{6}\right)$$

which is equivalent to

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) &\ge \frac{\gamma}{2}\left(\frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2\right) \\ &= \frac{\gamma(b-a)^2}{24}, \end{aligned}$$

and the first part of (2.8) is therefore obtained. For the second part, let $g(x) = \frac{x^2\Gamma}{2} - f(x)$, and similar manipulations, as previous lead to the second part of (2.8). The inequality (2.9) is now obvious by (2.8), the details have been omitted.■

3 Composite Rules.

We now consider applications of the integral inequalities developed in the previous section, to obtain some midpoint composite rules.

Theorem 3.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on (a,b). If $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$, then we have*

$$\int_a^b f(x)\,dx = A_M(f, I_h) + R_M(f, I_h) \tag{3.1}$$

where

$$A_M(f, I_h) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

is the midpoint quadrature rule and the remainder $R_M(f, I_h)$ *satisfies the inequality*

$$(3.2)\qquad |R_M(f, I_h)| \le \begin{cases} \frac{1}{24}\|f''\|_\infty \sum_{i=0}^{n-1} h_i^3 \\ \frac{1}{8(2p+1)^{\frac{1}{p}}}\|f''\|_q \left(\sum_{i=0}^{n-1} h_i^{2p+1}\right)^{\frac{1}{p}}, \quad \text{where } \frac{1}{p}+\frac{1}{q}=1, \quad p>1 \text{ and} \\ \frac{1}{8}\|f''\|_1 \nu^2(I_h) \end{cases}$$

where $h_i := x_{i+1} - x_i, i = 0, 1, 2, ..., n-1$ *and* $\nu(I_h) = \max_{i=0,...,n-1} h_i$.

Proof. Applying the first inequality in (2.3) on the interval $[x_i, x_{i+1}]$ we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \le \frac{1}{24}\|f''\|_\infty h_i^3$$

for all $i = 0, 1, 2, ..., n-1$. Summing over i from 0 to $n-1$ we obtain the first part of inequality (3.2). The second inequality in (2.3) gives us

$$\left| \int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \le \frac{h_i^{2+\frac{1}{p}}}{4(2p+1)^{\frac{1}{p}}} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}}$$

for all $i = 0, 1, 2, ..., n-1$. Summing over all i and using Hölder's discrete inequality, we obtain

$$\begin{aligned} \left| \int_a^b f(x)\,dx - A_M(f, I_h) \right| &\le \frac{1}{8(2p+1)^{\frac{1}{p}}} \sum_{i=0}^{n-1} h_i^{\frac{2p+1}{p}} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}} \\ &\le \frac{1}{8(2p+1)^{\frac{1}{p}}} \left(\sum_{i=0}^{n-1} \left(h_i^{\frac{2p+1}{p}} \right)^p \right)^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right|^q dt \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\ &= \frac{1}{8(2p+1)^{\frac{1}{p}}} \left(\sum_{i=0}^{n-1} h_i^{2p+1} \right)^{\frac{1}{p}} \left(\int_a^b \left|f''(t)\right|^q dt \right)^{\frac{1}{q}} \end{aligned}$$

and the second inequality in (3.2) is proved. In the last part, we have by (2.3) that

$$\begin{aligned} |R_M(f, I_h)| &\le \frac{1}{8} \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} \left|f''(t)\right| dt \right) h_i^2 \\ &\le \frac{1}{8} \max_{i=0,...,n-1} h_i^2 \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left|f''(t)\right| dt \\ &= \frac{1}{8}\|f''\|_1 \nu^2(I_h) \end{aligned}$$

and the theorem is proved.■

Remark 3.1. *It is of some interest to note that in every book on numerical integration, encountered by the authors, only the first estimate in (3.2) is used. Sometimes, where* $\left\|f''\right\|_q$ $(q>1)$ *or* $\left\|f''\right\|_1$ *are easier to compute, it would perhaps be more appropiate to use the second or third estimates.*

We shall now investigate the case where we have an equidistant partitioning of $[a,b]$ given by: $I_h : x_i = a + \left(\frac{b-a}{n}\right) i, i = 0,1,2,...,n-1$. The following result is a consequence of theorem 3.1.

Corollary 3.2. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping on* (a,b). *Then we have*

$$\int_a^b f(x)\,dx = A_{M,n}(f) + R_{M,n}(f)$$

where

$$A_{M,n}(f) := \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a+\frac{2i+1}{2n}(b-a)\right)$$

and the remainder $R_{M,n}(f)$ *satisfies the estimate:*

$$|R_{M,n}(f)| \le \begin{cases} \frac{(b-a)^3}{12n^2}\|f''\|_\infty , \\ \frac{(b-a)^{2+\frac{1}{p}}}{8(2p+1)^{\frac{1}{p}}n^2}\|f''\|_q , \\ \frac{(b-a)^2}{8n^2}\|f''\|_1 . \end{cases}$$

The following theorem gives a quasi-midpoint formula which is sometimes more appropriate.

Theorem 3.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping on* (a,b). *If* $f'':(a,b)\to\mathbb{R}$ *satisfies the condition (2.4) and* I_h *is an arbitrary partition of* $[a,b]$ *as above, then we have*

$$\int_a^b f(x)\,dx = A_M\left(f,f',I_h\right) + \widetilde{R}_M\left(f,f',I_h\right)$$

where

$$\begin{aligned} A_M\left(f,f',I_h\right) &= \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i \\ &\quad +\frac{1}{24}\sum_{i=0}^{n-1}\left(f'(x_{i+1})-f'(x_i)\right)h_i^2 \end{aligned}$$

is a perturbed midpoint rule and the remainder term, $\widetilde{R}_M\left(f,f',I_h\right)$, *satisfies the estimation*

$$\left|\widetilde{R}_M\left(f,f',I_h\right)\right| \le \frac{\Gamma-\gamma}{32}\sum_{i=0}^{n-1}h_i^3 \tag{3.3}$$

where h_i *is as defined above.*

Proof. Writing the inequality (2.5) on the interval $[x_i,x_{i+1}]$, $i=0,1,2,...,n-1$ we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - f\left(\frac{x_i+x_{i+1}}{2}\right)h_i - \frac{1}{24}\left(f'(x_{i+1})-f'(x_i)\right)h_i^2\right| \le \frac{\Gamma-\gamma}{32}h_i^3,$$

and summing over i from 0 to $n-1$ we easily deduce the desired estimation (3.3).■

Remark 3.2. *If we consider a mapping* $f : [a, b] \to \mathbb{R}$ *so that (2.4) is satisfied and* $\frac{\Gamma - \gamma}{32} \leq \frac{\|f''\|_\infty}{24} = \frac{1}{24} \max\{|\gamma|, |\Gamma|\}$*, that is,*

$$\Gamma - \gamma \leq \frac{4}{3} \max\{|\gamma|, |\Gamma|\} \tag{3.4}$$

then the estimation provided by (3.3) is better than the first estimation in (3.2). Also notice that if $\gamma \geq 0$*, then the condition (3.4) holds.*

The following corollary is also valid.

Corollary 3.4. *Let* f *be as defined in the previous theorem, then we have*

$$\int_a^b f(x)\, dx = A_{M,n}\left(f, f'\right) + \widetilde{R}_{M,n}\left(f, f'\right) \tag{3.5}$$

where

$$\begin{aligned} A_{M,n}\left(f, f'\right) &= \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right) \\ &\quad + \frac{(b-a)^2}{24n^2}\left(f'(b) - f'(a)\right) \end{aligned}$$

and the remainder, $\widetilde{R}_{M,n}\left(f, f'\right)$*, satisfies the estimation*

$$\left|\widetilde{R}_{M,n}\left(f, f'\right)\right| \leq \frac{(M-m)(b-a)^3}{32n^2},$$

for all $n \geq 1$*, where* $m := \inf_{x \in (a,b)} f'(x) > -\infty$ *and* $M := \sup_{x \in (a,b)} f'(x) < \infty$.

Now, if we apply Theorem 2.3, we can state the following quadrature formula which is a quasi-midpoint formula.

Theorem 3.5. *Let* f *be as in Theorem 3.2. If* I_h *is a partition of the interval* $[a, b]$ *then we have*

$$\int_a^b f(x)\, dx = A_M(f, \gamma, \Gamma) + R_M(f, \gamma, \Gamma)$$

where

$$A_M(f, \gamma, \Gamma) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i + \frac{\Gamma + \gamma}{48} \sum_{i=0}^{n-1} h_i^2$$

and

$$|R_M(f, \gamma, \Gamma)| \leq \frac{\Gamma - \gamma}{48} \sum_{i=0}^{n-1} h_i^3.$$

Proof. Applying the inequality (2.9) in $[x_i, x_{i+1}]$ we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x)\, dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i - \frac{\Gamma + \gamma}{48} h_i^2 \right| \leq \frac{\Gamma - \gamma}{48} h_i^3,$$

and summing over i from 0 to $n-1$ we have the desired estimation.■

Corollary 3.6. *Let f be as above. Then we have*

$$\int_a^b f(x)\,dx = A_{M,n}(f,\gamma,\Gamma) + R_{M,n}(f,\gamma,\Gamma)$$

where

$$A_{M,n}(f,\gamma,\Gamma) = \frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a + \frac{2i+1}{2n}(b-a)\right) + \frac{(\Gamma+\gamma)(b-a)^2}{48n}$$

and the remainder satisfies the estimation

$$|R_{M,n}(f,\gamma,\Gamma)| \leq \frac{(\Gamma-\gamma)(b-a)^3}{48n^2}.$$

References

[1] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink. *Classical and New Inequalities in Analysis.* Kluwer Academic, Dordrecht, 1993.

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