RGMIA Research Report Collection, Vol. 1, No. 2, 1998
http://sci.vut.edu.au/∼rgmia

NEW ESTIMATION OF THE REMAINDER IN THE TRAPEZOIDAL FORMULA WITH APPLICATIONS

S.S. DRAGOMIR AND T.C. PEACHEY

Abstract. A new inequality for the trapezoidal formula in terms of p-norms is presented with applications to numerical integration and special means.

1 Introduction

Integral inequalities have been used extensively in most subjects involving mathematical analysis. They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. In this paper, by the use of an integral identity, we point out some new integral inequalities for the trapezoidal rule and apply these to special means: p-logarithmic means, logarithmic means, identric means etc., and in numerical integration.

Classically, the error bounds for the trapezoidal quadrature rule depend on the maximum norms of the second derivative of the integrand. The new upper bounds for the quadrature rules obtained in this paper have the merit that they depend on only the first derivative of the integrand and thus they are particularly useful for integrals with integrands having bounded first derivatives, but unbounded second derivatives in some norms.

2 The Results

We shall start with the following lemma which contains an interesting integral identity.

Lemma 2.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $f' \in L_1[a, b]$. Then we have the identity*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{(b-a)^2}\int_a^b\int_a^b (y-x)f'(y)\,dxdy. \tag{2.1}$$

Proof. Our proof uses the well-known relations

$$\int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\int_a^{t_1} f(t_0)\,dt_0 = \int_a^{t_n}\frac{(t_n-u)^{n-1}}{(n-1)!}f(u)\,du, \tag{2.2}$$

and

$$\int_{t_n}^b dt_{n-1}\int_{t_{n-1}}^b dt_{n-2}\cdots\int_{t_1}^b f(t_0)\,dt_0 = \int_{t_n}^b\frac{(u-t_n)^{n-1}}{(n-1)!}f(u)\,du \tag{2.3}$$

valid for $f \in L_1[a, b]$ and any positive integer n. We consider

$$\begin{aligned} I &= \int_a^b\int_a^b (y-x)f'(y)\,dxdy \\ &= \int_a^b\int_x^b (y-x)f'(y)\,dydx \;-\; \int_a^b\int_a^x (x-y)f'(y)\,dydx \;=\; T_1 - T_2. \end{aligned}$$

Applying (2.3) to the inner integral in T_1 gives

$$\begin{aligned} T_1 &= \int_a^b dx \int_x^b du \int_u^b f'(t)\,dt = \int_a^b dx \int_x^b [f(b) - f(u)]\,du \\ &= \int_a^b (v-a)[f(b) - f(v)]\,dv = \int_a^b (a-v) f(v)\,dv + \frac{1}{2}(b-a)^2 f(b). \end{aligned}$$

Similarly, applying (2.2) to T_2,

$$T_2 = \int_a^b (b-v) f(v)\,dv - \frac{1}{2}(b-a)^2 f(a).$$

Combining these yields

$$I = T_1 - T_2 = \int_a^b (a-b) f(v)\,dv + \frac{1}{2}(b-a)^2 [f(a) + f(b)]$$

and the identity (2.1) follows. ∎

The lemma may be used to prove

Theorem 2.2. *With the above assumptions, we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \le \begin{cases} \frac{b-a}{3} \|f'\|_\infty, \\ \left[\frac{2(b-a)}{(p+1)(p+2)} \right]^{\frac{1}{p}} \|f'\|_q, \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\|_1. \end{cases} \tag{2.4}$$

Proof. From (2.1)

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \le \frac{1}{(b-a)^2} \int_a^b \int_a^b |y-x| |f'(y)|\,dy\,dx = I.$$

We treat the three cases in turn.

(i) Since

$$\int_a^b \int_a^b |y-x|\,|f'(y)|\,dy\,dx \le \|f'\|_\infty \int_a^b \int_a^b |y-x|\,dy\,dx = \|f'\|_\infty \frac{(b-a)^3}{3}$$

so that $I \le \frac{1}{3}(b-a)\|f'\|_\infty$.

(ii) By Hölder's integral inequality

$$\begin{aligned} \int_a^b \int_a^b |y-x|\,|f'(y)|\,dx\,dy &\le \left(\int_a^b \int_a^b |y-x|^p\,dx\,dy \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |f'(y)|^q\,dx\,dy \right)^{\frac{1}{q}} \\ &= K^{\frac{1}{p}} (b-a)^{\frac{1}{q}} \|f'\|_q \end{aligned}$$

where

$$K = \int_a^b \int_a^b |y-x|^p\,dx\,dy = 2 \int_a^b \int_x^b (y-x)^p\,dydx = \frac{2(b-a)^{p+2}}{(p+1)(p+2)}$$

using the symmetry of the integrand. Thus

$$\int_a^b \int_a^b |y-x|\,|f'(y)|\,dx\,dy \le \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}+\frac{1}{q}} \|f'\|_q$$

so that with $\frac{1}{p}+\frac{1}{q}=1$ we obtain

$$I \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}} \|f'\|_q$$

as required.

(iii) Finally, we have that

$$\int_a^b\int_a^b |y-x|\,|f'(y)|\,dx\,dy \le \left[\max_{(x,y)\in[a,b]^2}|y-x|\right]\int_a^b\int_a^b |f'(y)|\,dx\,dy = (b-a)^2\|f'\|_1$$

showing that $I \le \|f'\|_1$. The three cases in (2.4) have now been proved. ∎

Remark 2.1. *If $p=q=2$ we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \sqrt{\frac{b-a}{6}}\,\|f'\|_2. \tag{2.5}$$

Remark 2.2. *In the paper [2], S. S. Dragomir and S. Wang have obtained the following similar result as a particular case of an Ostrowski type inequality.*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma) \le \frac{1}{2}(b-a)\|f'\|_\infty \tag{2.6}$$

where $\gamma := \inf_{t\in(a,b)} f(t) > -\infty$ and $\Gamma := \sup_{t\in(a,b)} f(t) < \infty$.

Remark 2.3. *In [1] S. S. Dragomir and S. Wang have obtained the following result*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^{\frac{1}{p}}\|f'\|_q}{(p+1)^{\frac{1}{p}}} \tag{2.7}$$

as a particular case of Ostrowski's inequality for q-norms. Since

$$\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} \le \left[\frac{1}{p+1}\right]^{\frac{1}{p}} \textit{ for } p>1,$$

then our estimate in (2.4) is better *than that embodied in (2.7).*

Remark 2.4. *In [3], S. S. Dragomir and S. Wang obtained the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \|f'\|_1 \tag{2.8}$$

as a particular case of an Ostrowski type inequality for the L_1 norm.

Remark 2.5. *In 1938, by means of geometrical considerations, K. S. K. Iyengar [4, p.471] has proved the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)\|f'\|_\infty}{4}-\frac{(f(b)-f(a))^2}{4(b-a)\|f'\|_\infty} \le \frac{(b-a)\|f'\|_\infty}{4} \tag{2.9}$$

which is a better inequality than our first inequality in (2.2).

In conclusion, Theorem 2.2 gives the following new result

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}} \|f'\|_q \tag{2.10}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, and the particular case

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left[\frac{b-a}{6}\right]^{\frac{1}{2}} \|f'\|_2. \tag{2.11}$$

All our further applications for special means and in numerical integration for the trapezoidal formula will be based on these new results.

3 Applications To Special Means

Let us recall first some special means that we will use in the sequel:

(a) The *arithmetic mean:* $A=A(a,b):=(a+b)/2, \quad a,b\ge 0,$

(b) the *geometric mean:* $G=G(a,b):=\sqrt{ab}, \quad a,b\ge 0,$

(c) the *harmonic mean:* $H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b>0,$

(d) the *logarithmic mean:*

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\ne b \end{cases} \qquad a,b>0,$$

(e) the *identric mean:*

$$I=I(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\ne b \end{cases} \qquad a,b>0,$$

(f) the *p-logarithmic mean:*

$$L_p=L_p(a,b):=\begin{cases} a & \text{if } a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\ne b \end{cases} \qquad a,b>0, \text{ and } p\in\mathbb{R}\backslash\{-1,0\}.$$

These means are often used in numerical approximation and in other areas.

The following simple relationships are known:

$$H\le G\le L\le I\le A$$

and L_p is monotonically increasing in $p\in\mathbb{R}$ with $L_0:=I$ and $L_{-1}:=L$.

1. Let us assume that $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^s$, $s\in\mathbb{R}\backslash\{-1,0\}$ and $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then obviously

$$\frac{f(a)+f(b)}{2}=\frac{a^s+b^s}{2}=A(a^s,b^s),$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{b-a}\int_a^b x^s\,dx=\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}=L_s^s(a,b).$$

Since $f'(x) = sx^{s-1}$,

$$\|f'\|_q = \left[|s|^q \int_a^b x^{q(s-1)} dx\right]^{1/q} = |s|(b-a)^{1/q}\left[\frac{1}{b-a}\int_a^b x^{q(s-1)} dx\right]^{1/q} = |s|(b-a)^{1/q} L_{q(s-1)}^{s-1}$$

so the inequality (2.10) becomes

$$|A(a^s, b^s) - L_s^s(a,b)| \le \left[\frac{2(b-a)}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1}(a,b).$$

That is, we have

$$|A(a^s, b^s) - L_s^s(a,b)| \le |s|(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1}(a,b), \tag{3.1}$$

for $0 \le a \le b < \infty$. In particular, for $p = q = 2$,

$$|A(a^s, b^s) - L_s^s(a,b)| \le \frac{|s|(b-a)}{\sqrt{6}} L_{2(s-1)}^{s-1}(a,b). \tag{3.2}$$

2. Let us assume that $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\frac{f(a)+f(b)}{2} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = H^{-1}(a,b),$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \frac{dx}{x} = \frac{\ln b - \ln a}{b-a} = L^{-1}(a,b),$$

$$f'(x) = -\frac{1}{x^2}, \qquad \|f'\|_q = \left[\int_a^b \frac{dx}{x^{2q}}\right]^{1/q} = (b-a)^{\frac{1}{q}} L_{-2q}^{-2}.$$

Then (2.10) becomes

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-2q}^{-2}(a,b)$$

This yields the inequality

$$0 \le L - H \le \frac{(b-a)LH}{L_{-2q}^2}\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{q(s-1)}^{s-1} \tag{3.3}$$

for $0 \le a \le b < \infty$.

In particular, for $p = q = 2$ we have

$$0 \le L - H \le \frac{(b-a)LH}{\sqrt{6}\,L_{-4}^2}. \tag{3.4}$$

3. Let us assume that $f : (0,\infty) \to \mathbb{R}$, $f(x) = \ln x$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\frac{f(a)+f(b)}{2} = \frac{\ln a + \ln b}{2} = \ln G$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \ln x\,dx = \frac{1}{b-a}\ln\left(\frac{b^b}{a^a}\right) - 1 = \ln I,$$

$$f'(x) = \frac{1}{x}, \qquad \|f'\|_q = \left[\int_a^b \frac{dx}{x^q}\right]^{1/q} = (b-a)^{\frac{1}{q}} L_{-q}^{-1}.$$

Then inequality (2.10) gives

$$|\ln G - \ln I| \le (b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-q}^{-1}$$

Thus

$$1 \le \frac{I}{G} \le \exp\left[(b-a)\left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} L_{-q}^{-1}\right] \tag{3.5}$$

for $0 \le a \le b < \infty$.

In particular, for $p = q = 2$ we have

$$1 \le \frac{I}{G} \le \exp\left[\frac{(b-a)}{\sqrt{6}\,L_{-2}}\right]. \tag{3.6}$$

4 Applications In Numerical Integration

We discuss here the application of the inequality (2.10) in Numerical Integration to obtain some new estimates of the remainder term in the classical trapezoidal rule.

Theorem 4.1. *Let $f : [a,b] \to I\!R$ be a differentiable function on (a,b) and assume that f' is q-integrable on $[a,b]$, that is that $f' \in L_q[a,b]$, $q > 1$. If $I_h : a = x_0 < x_1 < \ldots < x_n = b$ is a partition of $[a,b]$, then we have*

$$\int_a^b f(x)\,dx = T(f, I_h) + R(f, I_h) \tag{4.1}$$

where $T(f, I_h)$ is the trapezoidal quadrature rule, i.e.,

$$T(f, I_h) = \sum_{i=0}^{n-1}\left[\frac{f(x_i) + f(x_{i+1})}{2}\right] h_i \tag{4.2}$$

where $h_i = x_{i+1} - x_i$ for all $i = 0, 1, 2, \ldots, n-1$ and the remainder $R(f, I_h)$ satisfies the inequality

$$|R(f, I_h)| \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} \|f'\|_q \left[\sum_{i=1}^{n-1} h_i^{p+1}\right]^{\frac{1}{p}}. \tag{4.3}$$

Proof. Applying the inequality (2.10) on the interval $[x_i, x_{i+1}]$ where $i = 0, 1, \ldots, n-1$ we have that

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} h_i - \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \le \left[\frac{2}{(p+1)(p+2)}\right]^{\frac{1}{p}} h_i^{1+\frac{1}{p}} \left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx\right)$$

for all $i = 0, 1, 2, \ldots, n-1$. Summing these inequalities and using Hölder's discrete inequality we have that

$$\begin{aligned}
|R(f, I_h)| &\leq \sum_{i=0}^{n-1} \left| \frac{f(x_i) + f(x_{i+1})}{2} h_i - \int_{x_i}^{x_{i+1}} f(x)\, dx \right| \\
&\leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \sum_{i=0}^{n-1} h_i^{\frac{p+1}{p}} \left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx \right)^{\frac{1}{q}} \\
&\leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(h_i^{\frac{p+1}{p}} \right)^p \right)^{\frac{1}{p}} \left(\sum_{i=0}^{n-1} \left(\left(\int_{x_i}^{x_{i+1}} |f'(x)|^q dx \right)^{\frac{1}{q}} \right)^q \right)^{\frac{1}{q}} \\
&= \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} h_i^{p+1} \right)^{\frac{1}{p}} \|f'\|_q.
\end{aligned}$$

The theorem is thus proved. ∎

Corollary 4.2. *With the above assumptions, if $f' \in L_2[a, b]$ we have*

$$|R(f, I_h)| \leq \frac{\|f'\|_2}{\sqrt{6}} \left(\sum_{i=1}^{n-1} h_i^3 \right)^{\frac{1}{2}}. \tag{4.4}$$

Suppose now that I_h denotes the equidistant partitioning of $[a, b]$ given by

$$I_h : \quad x_i = a + \frac{b-a}{n}\, i, \quad i = 0, 1, \ldots, n.$$

For this partition we have the following corollary.

Corollary 4.3. *Under the assumptions of Theorem 4.1,*

$$\int_a^b f(x)\, dx = T_n(f) + R_n(f) \tag{4.5}$$

where $T_n(f)$ is the trapezoidal quadrature rule for the partition I_h, that is

$$T_n(f) = \frac{b-a}{2n} \sum_{i=0}^{n-1} \left[f\left(a + i\, \frac{b-a}{n} \right) + f\left(a + (i+1) \frac{b-a}{n} \right) \right] \tag{4.6}$$

and the remainder term $R_n(f)$ satisfies the estimate

$$|R_n(f)| \leq \left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \frac{(b-a)^{1+\frac{1}{p}} \|f'\|_q}{n} \qquad \text{for } n \geq 1. \tag{4.7}$$

In particular, for $p = 2$, we have

$$|R_n(f)| \leq \frac{(b-a)^{\frac{3}{2}}}{\sqrt{6}} \frac{\|f'\|_2}{n}. \tag{4.8}$$

Given any $\epsilon > 0$, we are able using (4.7), to establish the minimum number of nodes such that the error in the numerical integration based on the equidistant trapezoidal rule is smaller than ϵ. This is contained in the following corollary.

Corollary 4.4. *Given any constant $\epsilon > 0$, if $n \geq n_\epsilon$, where*

$$n_\epsilon = \left[\left[\frac{2}{(p+1)(p+2)} \right]^{\frac{1}{p}} \frac{(b-a)^{1+\frac{1}{p}} \|f'\|_q}{\epsilon} \right] + 1$$

then $|R_n(f)| \leq \epsilon$.

Example 4.5. We give an example where the bound on R_n provided by (4.8) is better than those previously known. The equivalent bound imposed by (2.7) with $p = 2$ is

$$|R_n(f)| \leq \frac{(b-a)^{\frac{3}{2}}}{\sqrt{3}} \frac{\|f'\|_2}{n}, \tag{4.9}$$

that imposed by (2.8) is

$$|R_n| \leq \frac{(b-a)\|f'\|_1}{n}, \tag{4.10}$$

while that implied by (2.9) is

$$|R_n| \leq \frac{(b-a)^2}{4} \frac{\|f'\|_\infty}{n}. \tag{4.11}$$

As the example, we take $a = 0$, $b = 1$ and $f(x) = x^{2/3}e^{-2x/3}$ so that $f'(x) = \frac{2}{3}x^{-1/3}(1-x)e^{-2x/3}$. In this case $\|f'\|_\infty$ is infinite so (4.11) yields nothing useful. Since $f'(x)$ is positive on $(0,1)$, we have $\int_0^1 |f'(x)|\,dx = f(1) - f(0) = e^{-2/3}$. Thus (4.10) is

$$|R_n| \leq \frac{e^{-2/3}}{n} \approx \frac{0.513}{n}.$$

Also

$$\|f'\|_2^2 = \int_0^1 \frac{4}{9} e^{-2x/3} \left(\frac{1-x}{x^{1/3}}\right)^2 dx \leq \frac{4}{9}\int_0^1 (1-x)^2 x^{-2/3} dx = \frac{4}{9} B(3, \frac{1}{3}) = \frac{6}{7}.$$

Inserting this into (4.9) gives

$$|R_n| \leq \sqrt{\frac{2}{7}}\,\frac{1}{n} \approx \frac{0.535}{n}$$

while (4.8) becomes

$$|R_n| \leq \sqrt{\frac{1}{7}}\,\frac{1}{n} \approx \frac{0.378}{n}.$$

Thus in this example the new bound is superior.

References

[1] S. S. Dragomir and S. Wang, "A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules", submitted.

[2] S. S. Dragomir and S. Wang, "An inequality of Ostrowski-Grüss' type and its application to the estimation of error bounds for some special means and for some numerical quadrature rules", Computers Math. Applic., Vol 33(11), 1997, pp. 15-20.

[3] S. S. Dragomir and S. Wang, "A new inequality of Ostrowski's type in L_1 norms and applications to some special means and to some numerical quadrature rules", Tamkang J. of Math., to appear.

[4] D. S. Mitronović, J. E. Pecărić and A. M. Fink, "Inequalities Involving Functions and Their Integrals and Derivatives", Kluwer, Dordrecht, 1991.

School of Communications and Informatics,
Victoria University of Technology,
PO Box 14428,
MCMC, Melbourne 8001,
Australia
E-mail address: {sever,tom}@matilda.vut.edu.au