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TWO PARAMETERS AND TWO POINTS REPRESENTATIONS FOR FUNCTIONS OF BOUNDED VARIATION WITH APPLICATIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish some two parameters two points representations with integral remainders for functions of locally bounded variation and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

1. Introduction

Let $f : I \to \mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$, the interior of the interval I. In the recent paper [12] we considered the problem of approximating an absolutely continuous function by using an affine combination of the values in two points $f(a)$, $f(b)$ where a, $b \in \mathring{I}$ and two free parameters δ, $\gamma \in \mathbb{C}$ as follows

$$f(x) \approx (1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a)\delta - \lambda(b-x)\gamma \tag{1.1}$$

for $\lambda \in \mathbb{C} \setminus \{0,1\}$ and $x \in \mathring{I}$.

The following representation result has been obtained in [12]:

Theorem 1. *Let $f : I \to \mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$, the interior of the interval I. Then for any x, a, $b \in \mathring{I}$ and $\lambda \in \mathbb{C} \setminus \{0,1\}$, δ, $\gamma \in \mathbb{C}$ we have*

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a)\delta - \lambda(b-x)\gamma + S_\lambda(x,a,b;\delta,\gamma), \tag{1.2}$$

where the remainder $S_\lambda(x,a,b;\delta,\gamma)$ is given by

$$S_\lambda(x,a,b;\delta,\gamma) := (1-\lambda)(x-a)\int_0^1 [f'((1-s)a+sx)-\delta]\,ds + \lambda(b-x)\int_0^1 [\gamma - f'((1-s)x+sb)]\,ds. \tag{1.3}$$

Now, for ϕ, $\Phi \in \mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [13])

$$\bar{U}_I(\phi,\Phi) := \left\{ g : I \to \mathbb{C} \mid \operatorname{Re}\left[(\Phi - g(t))\left(\overline{g(t)} - \overline{\phi}\right)\right] \geq 0 \text{ for almost every } t \in I \right\}$$

and

$$\bar{\Delta}_I(\phi,\Phi) := \left\{ g : I \to \mathbb{C} \mid \left| g(t) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2}|\Phi - \phi| \text{ for a.e. } t \in I \right\}.$$

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The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_I(\phi,\Phi)$ and $\bar{\Delta}_I(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_I(\phi,\Phi) = \bar{\Delta}_I(\phi,\Phi). \tag{1.4}$$

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi,\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned}\bar{U}_I(\phi,\Phi) = \{g : I \to \mathbb{C} \mid\ &(\operatorname{Re}\Phi - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\phi) \\ &+ (\operatorname{Im}\Phi - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\phi) \geq 0 \text{ for a.e. } t \in I\}.\end{aligned} \tag{1.5}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_I(\phi,\Phi) := \{g : I \to \mathbb{C} \mid\ &\operatorname{Re}(\Phi) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\phi) \\ &\text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in I\}.\end{aligned} \tag{1.6}$$

One can easily observe that $\bar{S}_I(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_I(\phi,\Phi) \subseteq \bar{U}_I(\phi,\Phi). \tag{1.7}$$

The following result holds:

Theorem 2. *Let $f : I \to \mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$ and with the property that there exists complex numbers ϕ, $\Phi \in \mathbb{C}$ such that the derivative $f' \in \bar{U}_I(\phi,\Phi)$. Then for any x, a, $b \in \mathring{I}$ and $\lambda \in \mathbb{C}\setminus\{0,1\}$ we have*

$$\begin{aligned}&\left| f(x) - (1-\lambda) f(a) - \lambda f(b) - \frac{\phi+\Phi}{2}[x - (1-\lambda)a - \lambda b] \right| \\ &\leq \frac{1}{2}|\Phi-\phi| \left[|1-\lambda| |x-a| + |\lambda| |b-x| \right] \\ &\leq \frac{1}{2}|\Phi-\phi| \begin{cases} \max\{|1-\lambda|, |\lambda|\}(|x-a| + |b-x|), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p}(|x-a|^q + |b-x|^q)^{1/q}, \\ \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ (|1-\lambda| + |\lambda|)\max\{|x-a|, |b-x|\}. \end{cases}\end{aligned} \tag{1.8}$$

Remark 1. *For $p = q = 2$ we have for $\lambda \in [0,1]$ and $x \in [a,b]$ with $a < b$ that*

$$\begin{aligned}&\left| f(x) - (1-\lambda) f(a) - \lambda f(b) - \frac{\phi+\Phi}{2}[x - (1-\lambda)a - \lambda b] \right| \\ &\leq \frac{1}{2}|\Phi-\phi| \left[(1-\lambda)(x-a) + \lambda(b-x) \right] \\ &\leq |\Phi-\phi| \left(\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2 \right)^{1/2} \left(\frac{1}{4} + \left(x - \frac{a+b}{2}\right)^2 \right)^{1/2}.\end{aligned} \tag{1.9}$$

Corollary 2. *With the assumptions of Theorem 2 for the function f, we have for any x, a, $b \in \mathring{I}$ that*

$$\left| f(x) - \frac{1}{b-a}\left[(b-x) f(a) + (x-a) f(b)\right] \right| \leq |\Phi-\phi| \frac{|(b-x)(x-a)|}{|b-a|} \tag{1.10}$$

and

$$\left|f(x)-\frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right]-(\phi+\Phi)\left(x-\frac{a+b}{2}\right)\right| \le |\Phi-\phi|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|b-a|. \tag{1.11}$$

Proof. Follows by Theorem 2 on taking $\lambda=\frac{x-a}{b-a}$ and $\lambda=\frac{b-x}{b-a}$, respectively. □

Corollary 3. *With the assumptions of Theorem 2 for the function f, we have for any a, $b\in\mathring{I}$ and $\lambda\in[0,1]$ that*

$$|f((1-\lambda)a+\lambda b)-(1-\lambda)f(a)-\lambda f(b)|\le|\Phi-\phi|(1-\lambda)\lambda|b-a| \tag{1.12}$$

and

$$\left|f((1-\lambda)b+\lambda a)-(1-\lambda)f(a)-\lambda f(b)-(\phi+\Phi)(b-a)\left(\frac{1}{2}-\lambda\right)\right| \le|\Phi-\phi|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]|b-a|. \tag{1.13}$$

Remark 2. *If we take $\lambda=\frac{1}{2}$ in either of the inequalities from Corollary 3 we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{4}|\Phi-\phi||b-a| \tag{1.14}$$

for any a, $b\in\mathring{I}$. The constant $\frac{1}{4}$ is best possible in (1.14).

For related trapezoid type inequalities, see [1]-[8], [14]-[24] and the references therein.

Motivated by the above results, we establish in this paper some error estimates for approximating a function of locally bounded variation by the use of formula (1.1). Applications for logarithmic and exponential functions and reverse inequalities for the celebrated arithmetic mean-geometric mean inequality are given as well.

2. Some Identities

We use the following convention for the integral in the case when $b<a$,

$$\int_a^b u(s)\,d(v):=-\int_b^a u(s)\,d(v)$$

provided the second Riemann-Stieltjes integral exists in the classical sense.

We start with the following representation result:

Theorem 3. *Let $f:I\to\mathbb{C}$ be a function of locally bounded variation on $\mathring{I}$, the interior of the interval I. Then for any x, a, $b\in\mathring{I}$ and $\lambda\in\mathbb{C}\backslash\{0,1\}$, δ, $\gamma\in\mathbb{C}$ we have*

$$f(x)=(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)(x-a)\delta-\lambda(b-x)\gamma + R_\lambda(x,a,b;\delta,\gamma), \tag{2.1}$$

where the remainder $R_\lambda(x,a,b;\delta,\gamma)$ *is given by*

$$R_\lambda(x,a,b;\delta,\gamma):=(1-\lambda)\int_a^x d(f(s)-\delta\ell(s))+\lambda\int_x^b d(\gamma\ell(s)-f(s)), \tag{2.2}$$

while ℓ *is the identity function on* I, *namely* $\ell(s)=s$, $s\in I$.

Proof. Since $f:I\to\mathbb{C}$ is a function of locally bounded variation on $\mathring{I}$, then for any $x,\ a,\ b\in\mathring{I}$ the Riemann-Stieltjes integrals

$$\int_a^x d(f(s)-\delta\ell(s)) \text{ and } \int_x^b d(\gamma\ell(s)-f(s))$$

exist and we have

$$\int_a^x d(f(s)-\delta\ell(s))=f(x)-\delta\ell(x)-[f(a)-\delta\ell(a)]$$

and

$$\int_x^b d(\gamma\ell(s)-f(s))=\gamma\ell(b)-f(b)-[\gamma\ell(x)-f(x)].$$

Therefore

$$\begin{aligned}
&R_\lambda(x,a,b;\delta,\gamma) \\
&=(1-\lambda)\int_a^x d(f(s)-\delta\ell(s))+\lambda\int_x^b d(\gamma\ell(s)-f(s)) \\
&=(1-\lambda)(f(x)-\delta\ell(x)-[f(a)-\delta\ell(a)]) \\
&+\lambda(\gamma\ell(b)-f(b)-[\gamma\ell(x)-f(x)]) \\
&=(1-\lambda)(f(x)-f(a)-\delta(x-a))+\lambda(\gamma(b-x)-f(b)+f(x)) \\
&=(1-\lambda)f(x)-(1-\lambda)f(a)-(1-\lambda)(x-a)\delta \\
&-\lambda f(b)+\lambda f(x)+\lambda(b-x)\gamma \\
&=f(x)-(1-\lambda)f(a)-\lambda f(b)-(1-\lambda)(x-a)\delta+\lambda(b-x)\gamma,
\end{aligned} \tag{2.3}$$

which is clearly equivalent to (2.1). □

Corollary 4. *Let* $f:I\to\mathbb{C}$ *be a function of locally bounded variation on* $\mathring{I}$. *Then for any* $x,\ a,\ b\in\mathring{I}$ *and* $\delta,\ \gamma\in\mathbb{C}$ *we have*

$$f(x)=\frac{1}{b-a}[(b-x)f(a)+(x-a)f(b)]+\frac{(b-x)(x-a)}{b-a}(\delta-\gamma) \tag{2.4}$$

$$+R_1(x,a,b;\delta,\gamma),$$

where the remainder $R_1(x,a,b;\delta,\gamma)$ *is given by*

$$R_1(x,a,b;\delta,\gamma):=\frac{b-x}{b-a}\int_a^x d(f(s)-\delta\ell(s))+\frac{x-a}{b-a}\int_x^b d(\gamma\ell(s)-f(s)).$$

Alternatively, we have

$$\begin{aligned}
f(x)&=\frac{1}{b-a}[(x-a)f(a)+(b-x)f(b)] \\
&+\frac{1}{b-a}\left[(x-a)^2\delta-(b-x)^2\gamma\right]+R_2(x,a,b;\delta,\gamma),
\end{aligned} \tag{2.5}$$

where the remainder $R_2(x,a,b;\delta,\gamma)$ *is given by*

$$R_2(x,a,b;\delta,\gamma) := \frac{x-a}{b-a}\int_a^x d(f(s)-\delta\ell(s)) + \frac{b-x}{b-a}\int_x^b d(\gamma\ell(s)-f(s)). \tag{2.6}$$

Proof. Follows by Theorem 3 on taking $\lambda = \frac{x-a}{b-a}$ and $\lambda = \frac{b-x}{b-a}$, respectively. □

The following particular case is of interest as well:

Corollary 5. *Let* $f: I \to \mathbb{C}$ *be a function of locally bounded variation on* $\mathring{I}$. *Then for any* a, $b \in \mathring{I}$, $\lambda \in [0,1]$ *and* δ, $\gamma \in \mathbb{C}$ *we have*

$$f((1-\lambda)a+\lambda b) = (1-\lambda)f(a)+\lambda f(b)+(1-\lambda)\lambda(b-a)(\delta-\gamma) + R_{1,\lambda}(a,b;\delta,\gamma), \tag{2.7}$$

where the remainder $R_{1,\lambda}(a,b;\delta,\gamma)$ *is given by*

$$R_{1,\lambda}(a,b;\delta,\gamma) := (1-\lambda)\int_a^{(1-\lambda)a+\lambda b} d(f(s)-\delta\ell(s)) + \lambda\int_{(1-\lambda)a+\lambda b}^b d(\gamma\ell(s)-f(s)). \tag{2.8}$$

Alternatively, we have

$$f(\lambda a+(1-\lambda)b) = (1-\lambda)f(a)+\lambda f(b)+(b-a)\left[(1-\lambda)^2\delta-\lambda^2\gamma\right] + R_{2,\lambda}(a,b;\delta,\gamma), \tag{2.9}$$

where the remainder $R_{2,\lambda}(a,b;\delta,\gamma)$ *is given by*

$$R_{2,\lambda}(a,b;\delta,\gamma) := (1-\lambda)\int_a^{\lambda a+(1-\lambda)b} d(f(s)-\delta\ell(s)) + \lambda\int_{\lambda a+(1-\lambda)b}^b d(\gamma\ell(s)-f(s)). \tag{2.10}$$

Remark 3. *Let* f *be as in Theorem 3, then for any* a, $b \in \mathring{I}$ *and* $\lambda \in \mathbb{C}\setminus\{0,1\}$, δ, $\gamma \in \mathbb{C}$ *we have*

$$f\left(\frac{a+b}{2}\right) = (1-\lambda)f(a)+\lambda f(b)+\frac{1}{2}(b-a)[(1-\lambda)\delta-\lambda\gamma] + R_\lambda(a,b;\delta,\gamma), \tag{2.11}$$

where the remainder $R_\lambda(a,b;\delta,\gamma)$ *is given by*

$$R_\lambda(a,b;\delta,\gamma) := (1-\lambda)\int_a^{\frac{a+b}{2}} d(f(s)-\delta\ell(s)) + \lambda\int_{\frac{a+b}{2}}^b d(\gamma\ell(s)-f(s)). \tag{2.12}$$

The case $\delta = \gamma = 0$ in (2.1) produces the following simple identities for each distinct x, a, $b \in \mathring{I}$ and $\lambda \in \mathbb{C}\setminus\{0,1\}$

$$f(x) = (1-\lambda)f(a)+\lambda f(b)+R_\lambda(x,a,b), \tag{2.13}$$

where the remainder $R_\lambda(x,a,b)$ is given by

$$R_\lambda(x,a,b) := (1-\lambda)\int_a^x df(s) - \lambda\int_x^b df(s). \tag{2.14}$$

We then have for each distinct $x,\, a,\, b\in \mathring{I}$ that

$$f(x)=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+U(x,a,b), \tag{2.15}$$

where

$$U(x,a,b):=\frac{b-x}{b-a}\int_a^x df(s)-\frac{x-a}{b-a}\int_x^b df(s). \tag{2.16}$$

and

$$f(x)=\frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right]+V(x,a,b), \tag{2.17}$$

where

$$V(x,a,b):=\frac{x-a}{b-a}\int_a^x df(s)-\frac{b-x}{b-a}\int_x^b df(s). \tag{2.18}$$

We also have

$$f((1-\lambda)a+\lambda b)=(1-\lambda)f(a)+\lambda f(b)+U_\lambda(a,b), \tag{2.19}$$

where the remainder $U_\lambda(a,b)$ is given by

$$U_\lambda(a,b):=(1-\lambda)\int_a^{(1-\lambda)a+\lambda b} df(s)-\lambda\int_{(1-\lambda)a+\lambda b}^b df(s) \tag{2.20}$$

and

$$f((1-\lambda)b+\lambda a)=(1-\lambda)f(a)+\lambda f(b)+V_\lambda(a,b), \tag{2.21}$$

where the remainder $V_\lambda(a,b)$ is given by

$$V_\lambda(a,b):=(1-\lambda)\int_a^{(1-\lambda)b+\lambda a} df(s)-\lambda\int_{(1-\lambda)b+\lambda a}^b df(s). \tag{2.22}$$

Moreover, if we take in (2.13) $x=\frac{a+b}{2}$ for each distinct $a,\, b\in \mathring{I}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$, then we have

$$f\left(\frac{a+b}{2}\right)=(1-\lambda)f(a)+\lambda f(b)+S_\lambda(a,b), \tag{2.23}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b):=(1-\lambda)\int_a^{\frac{a+b}{2}} df(s)-\lambda\int_{\frac{a+b}{2}}^b df(s). \tag{2.24}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right)=\frac{f(a)+f(b)}{2}+S(a,b), \tag{2.25}$$

where

$$S(a,b):=\frac{1}{2}\left(\int_a^{\frac{a+b}{2}} df(s)-\int_{\frac{a+b}{2}}^b df(s)\right). \tag{2.26}$$

3. Inequalities for Functions of Locally Bounded Variation

We use the following convention for the total variation of a function in the case when $b < a$,

$$\bigvee_a^b (f) := -\bigvee_b^a (f) \tag{3.1}$$

provided the function f is of bounded variation in the classical sense.

Theorem 4. *Let $f : I \to \mathbb{C}$ be a function of locally bounded variation on $\mathring{I}$, the interior of the interval I. Then for any $x,\ a,\ b \in \mathring{I}$ and $\lambda \in \mathbb{C}\backslash \{0,1\}$, $\delta,\ \gamma \in \mathbb{C}$ we have*

$$\begin{aligned} &|(1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a)\delta - \lambda(b-x)\gamma - f(x)| \\ &\le |1-\lambda| \left|\bigvee_a^x (f-\delta\ell)\right| + |\lambda| \left|\bigvee_x^b (\gamma\ell - f)\right| \\ &\le \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\left|\bigvee_a^x (f-\delta\ell)\right| + \left|\bigvee_x^b (\gamma\ell - f)\right| \right), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} \left(\left|\bigvee_a^x (f-\delta\ell)\right|^q + \left|\bigvee_x^b (\gamma\ell - f)\right|^q \right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ (|1-\lambda| + |\lambda|) \max\left\{ \left|\bigvee_a^x (f-\delta\ell)\right|, \left|\bigvee_x^b (\gamma\ell - f)\right| \right\}. \end{cases} \end{aligned} \tag{3.2}$$

Proof. We use the fact that for $p : [\alpha, \beta] \to \mathbb{R}$ continuous and $v : [\alpha, \beta] \to \mathbb{R}$ of bounded variation the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and

$$\left| \int_\alpha^\beta p(t)\, dv(t) \right| \le \sup_{t\in[\alpha,\beta]} |p(t)| \bigvee_\alpha^\beta (v). \tag{3.3}$$

Using the identity (2.1) the convention (3.1) and the property (3.3) we have

$$\begin{aligned} &|(1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a)\delta - \lambda(b-x)\gamma - f(x)| \\ &= \left| (1-\lambda) \int_a^x d(f(s) - \delta\ell(s)) + \lambda \int_x^b d(\gamma\ell(s) - f(s)) \right| \end{aligned}$$

$$\leq |1-\lambda| \left| \int_a^x d(f(s) - \delta\ell(s)) \right| + |\lambda| \left| \int_x^b d(\gamma\ell(s) - f(s)) \right|$$

$$\leq |1-\lambda| \left| \bigvee_a^x (f - \delta\ell) \right| + |\lambda| \left| \bigvee_a^x (\gamma\ell - f) \right|$$

$$\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\left| \bigvee_a^x (f-\delta\ell) \right| + \left| \bigvee_a^x (\gamma\ell - f) \right| \right), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} \left(\left| \bigvee_a^x (f-\delta\ell) \right|^q + \left| \bigvee_a^x (\gamma\ell - f) \right|^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ (|1-\lambda| + |\lambda|) \max\left\{ \left| \bigvee_a^x (f-\delta\ell) \right|, \left| \bigvee_a^x (\gamma\ell - f) \right| \right\}. \end{cases}$$

The last part is obvious by Hölder's inequality

$$cd + uv \leq \begin{cases} \max\{c, u\} (d+v) \\ \\ (c^p + u^p)^{1/p} (d^q + v^q)^{1/q}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

□

For any $x, a, b \in \mathring{I}$ and $\lambda \in \mathbb{C} \backslash \{0, 1\}$, $\delta \in \mathbb{C}$ we have

$$|(1-\lambda) f(a) + \lambda f(b) + [x - (1-\lambda) a - \lambda b] \delta - f(x)| \tag{3.4}$$

$$\leq |1-\lambda| \left| \bigvee_a^x (f - \delta\ell) \right| + |\lambda| \left| \bigvee_x^b (f - \delta\ell) \right|$$

$$\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\left| \bigvee_a^x (f-\delta\ell) \right| + \left| \bigvee_x^b (\delta\ell - f) \right| \right), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} \left(\left| \bigvee_a^x (f-\delta\ell) \right|^q + \left| \bigvee_x^b (\delta\ell - f) \right|^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ (|1-\lambda| + |\lambda|) \max\left\{ \left| \bigvee_a^x (f-\delta\ell) \right|, \left| \bigvee_x^b (\delta\ell - f) \right| \right\}. \end{cases}$$

If we assume in (3.4) that $a < b$ and $x \in [a, b]$, then

$$|(1-\lambda) f(a) + \lambda f(b) + [x - (1-\lambda) a - \lambda b] \delta - f(x)| \tag{3.5}$$

$$\leq |1-\lambda| \bigvee_a^x (f - \delta\ell) + |\lambda| \bigvee_x^b (f - \delta\ell)$$

$$\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\bigvee_a^b (f-\delta\ell)\right), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} \left(\left(\bigvee_a^x (f-\delta\ell)\right)^q + \left(\bigvee_x^b (f-\delta\ell)\right)^q\right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}(|1-\lambda| + |\lambda|) \left[\bigvee_a^b (f-\delta\ell) + \left|\bigvee_a^x (f-\delta\ell) - \bigvee_x^b (f-\delta\ell)\right|\right]. \end{cases}$$

Taking into (3.5) $\lambda \in [0,1]$ then we get

$$\begin{aligned} (3.6) \qquad & |(1-\lambda) f(a) + \lambda f(b) + [x - (1-\lambda) a - \lambda b] \delta - f(x)| \\ & \leq (1-\lambda) \bigvee_a^x (f-\delta\ell) + \lambda \bigvee_x^b (f-\delta\ell) \\ & \leq \begin{cases} \left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right) \left(\bigvee_a^b (f-\delta\ell)\right), \\ \\ ((1-\lambda)^p + \lambda^p)^{1/p} \left(\left(\bigvee_a^x (f-\delta\ell)\right)^q + \left(\bigvee_x^b (f-\delta\ell)\right)^q\right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2} \left[\bigvee_a^b (f-\delta\ell) + \left|\bigvee_a^x (f-\delta\ell) - \bigvee_x^b (f-\delta\ell)\right|\right]. \end{cases} \end{aligned}$$

Moreover, if we take in (3.6) $\delta = 0$, then we get

$$\begin{aligned} (3.7) \qquad & |(1-\lambda) f(a) + \lambda f(b) - f(x)| \\ & \leq (1-\lambda) \bigvee_a^x (f) + \lambda \bigvee_x^b (f) \\ & \leq \begin{cases} \left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right) \left(\bigvee_a^b (f)\right), \\ \\ ((1-\lambda)^p + \lambda^p)^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2} \left[\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]. \end{cases} \end{aligned}$$

If $x \in [a,b]$ with $a < b$ and if we take $\lambda = \frac{x-a}{b-a}$ in (3.7), then we get

$$\left| \left(\frac{b-x}{b-a} \right) f(a) + \left(\frac{x-a}{b-a} \right) f(b) - f(x) \right| \le \left(\frac{b-x}{b-a} \right) \bigvee_a^x (f) + \left(\frac{x-a}{b-a} \right) \bigvee_x^b (f) \le \begin{cases} \left(\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right) \left(\bigvee_a^b (f) \right), \\ \left(\left(\frac{b-x}{b-a} \right)^p + \left(\frac{x-a}{b-a} \right)^p \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left[\bigvee_a^b (f) + \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right], \end{cases} \tag{3.8}$$

which was obtained in [10, Theorem 3.2].

For other related results, see [10] and [11].

We have:

Corollary 6. *With the assumptions of Theorem 4 for the function f, we have for any a, $b \in \mathring{I}$ and $\lambda \in [0,1]$ that*

$$|(1-\lambda) f(a) + \lambda f(b) + (1-\lambda)\lambda(b-a)(\delta - \gamma) - f((1-\lambda)a + \lambda b)| \le (1-\lambda) \left| \bigvee_a^{(1-\lambda)a+\lambda b} (f - \delta\ell) \right| + \lambda \left| \bigvee_{(1-\lambda)a+\lambda b}^b (\gamma\ell - f) \right| \le \begin{cases} \left(\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right) \left(\left| \bigvee_a^{(1-\lambda)a+\lambda b} (f - \delta\ell) \right| + \left| \bigvee_{(1-\lambda)a+\lambda b}^b (\gamma\ell - f) \right| \right), \\ ((1-\lambda)^p + \lambda^p)^{1/p} \\ \times \left(\left| \bigvee_a^{(1-\lambda)a+\lambda b} (f - \delta\ell) \right|^q + \left| \bigvee_{(1-\lambda)a+\lambda b}^b (\gamma\ell - f) \right|^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \max \left\{ \left| \bigvee_a^{(1-\lambda)a+\lambda b} (f - \delta\ell) \right|, \left| \bigvee_{(1-\lambda)a+\lambda b}^b (\gamma\ell - f) \right| \right\}. \end{cases} \tag{3.9}$$

for any δ, $\gamma \in \mathbb{C}$.

In particular,

$$
\begin{aligned}
&\left|(1-\lambda) f(a)+\lambda f(b)-f((1-\lambda) a+\lambda b)\right| \\
&\leq (1-\lambda)\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f-\delta\ell)\right|+\lambda\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f-\delta\ell)\right| \\
&\leq \begin{cases}
\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left|\bigvee_{a}^{b}(f-\delta\ell)\right|, \\
\\
\left((1-\lambda)^{p}+\lambda^{p}\right)^{1/p} \\
\times\left(\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f-\delta\ell)\right|^{q}+\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f-\delta\ell)\right|^{q}\right)^{1/q}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\\
\max\left\{\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f-\delta\ell)\right|,\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f-\delta\ell)\right|\right\},
\end{cases}
\end{aligned} \tag{3.10}
$$

for any $\delta\in\mathbb{C}$.

We observe that, with the assumptions of Corollary 6 we have from (3.11) that

$$
\begin{aligned}
&\left|(1-\lambda) f(a)+\lambda f(b)-f((1-\lambda) a+\lambda b)\right| \\
&\leq (1-\lambda)\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f)\right|+\lambda\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f)\right| \\
&\leq \begin{cases}
\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left|\bigvee_{a}^{b}(f)\right|, \\
\\
\left((1-\lambda)^{p}+\lambda^{p}\right)^{1/p}\left(\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f)\right|^{q}+\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f)\right|^{q}\right)^{1/q}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\\
\max\left\{\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f)\right|,\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f)\right|\right\}.
\end{cases}
\end{aligned} \tag{3.11}
$$

If f is convex on I, then from (3.11) we get

$$
\begin{aligned}
0 &\leq (1-\lambda) f(a)+\lambda f(b)-f((1-\lambda) a+\lambda b) \\
&\leq (1-\lambda)\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(f)\right|+\lambda\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(f)\right| \\
&\leq \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left|\bigvee_{a}^{b}(f)\right|
\end{aligned} \tag{3.12}
$$

for any a, $b\in\mathring{I}$ and $\lambda\in[0,1]$.

Remark 4. *If $a<b$ and there exists the constants ϕ, $\Phi\in\mathbb{C}$ such that*

$$\bigvee_a^b\left(f-\frac{\phi+\Phi}{2}\ell\right)\le\frac{1}{2}\left|\Phi-\phi\right|(b-a),\tag{3.13}$$

then by (3.4) for $x\in[a,b]$

$$\begin{aligned}&\left|(1-\lambda)f(a)+\lambda f(b)+[x-(1-\lambda)a-\lambda b]\frac{\phi+\Phi}{2}-f(x)\right|\\&\le|1-\lambda|\bigvee_a^x\left(f-\frac{\phi+\Phi}{2}\ell\right)+|\lambda|\bigvee_x^b\left(f-\frac{\phi+\Phi}{2}\ell\right)\\&\le\frac{1}{2}\left|\Phi-\phi\right|\max\left\{|1-\lambda|,|\lambda|\right\}(b-a).\end{aligned}\tag{3.14}$$

In particular, if $\lambda\in[0,1]$, then

$$\begin{aligned}&\left|(1-\lambda)f(a)+\lambda f(b)-f((1-\lambda)a+\lambda b)\right|\\&\le\frac{1}{2}\left|\Phi-\phi\right|\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)(b-a),\end{aligned}\tag{3.15}$$

giving that

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}\left|\Phi-\phi\right|(b-a).\tag{3.16}$$

We observe that if $f:I\to\mathbb{C}$ is a locally absolutely continuous function on $\mathring{I}$ and with the property that there exists complex numbers ϕ, $\Phi\in\mathbb{C}$ such that the derivative $f'\in\bar{U}_I(\phi,\Phi)$, then for $a<b$ we have

$$\bigvee_a^b\left(f-\frac{\phi+\Phi}{2}\ell\right)\le\int_a^b\left|f'(s)-\frac{\phi+\Phi}{2}\right|ds\le\frac{1}{2}\left|\Phi-\phi\right|(b-a)$$

and the condition (3.13) is satisfied. This provides many examples, since for real valued function satisfying the condition $k\le f'(s)\le K$, for a.e. $t\in I$ and for some real constants k, K, then we have that

$$\bigvee_a^b\left(f-\frac{k+K}{2}\ell\right)\le\frac{1}{2}(K-k)(b-a).$$

4. Some Examples

For a, $b\in(0,\infty)$ and $\lambda\in[0,1]$, consider $A_\lambda(a,b):=(1-\lambda)a+\lambda b$ the *weighted arithmetic mean* and $G_\lambda(a,b):=a^{1-\lambda}b^\lambda$ the *weighted geometric mean*. The following inequality is well known in the literature as the *arithmetic mean-geometric mean inequality*:

$$G_\lambda(a,b)\le A_\lambda(a,b).$$

If we write, for instance, the inequality (3.11) for the function $f(t) := \ln t$, then we get for any a, $b \in (0,\infty)$ and $\lambda \in [0,1]$

$$
\begin{aligned}
&|(1-\lambda)\ln a + \lambda \ln b - \ln((1-\lambda)a + \lambda b)| \\
&\leq (1-\lambda)\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(\ln)\right| + \lambda\left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(\ln)\right| \\
&\leq \begin{cases}
\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left|\bigvee_{a}^{b}(\ln)\right|, \\
((1-\lambda)^p+\lambda^p)^{1/p}\left(\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(\ln)\right|^q + \left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(\ln)\right|^q\right)^{1/q}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\max\left\{\left|\bigvee_{a}^{(1-\lambda)a+\lambda b}(\ln)\right|, \left|\bigvee_{(1-\lambda)a+\lambda b}^{b}(\ln)\right|\right\},
\end{cases}
\end{aligned}
$$

namely

$$
\begin{aligned}
(4.1)\qquad & 0 \leq \ln\left(\frac{A_\lambda(a,b)}{G_\lambda(a,b)}\right) \\
&\leq (1-\lambda)\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right| + \lambda\left|\ln\left(\frac{b}{A_\lambda(a,b)}\right)\right| \\
&\leq \begin{cases}
\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)|\ln b - \ln a|, \\
((1-\lambda)^p+\lambda^p)^{1/p}\left(\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right|^q + \left|\ln\left(\frac{b}{A_\lambda(a,b)}\right)\right|^q\right)^{1/q}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\max\left\{\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right|, \left|\frac{b}{A_\lambda(a,b)}\right|\right\},
\end{cases}
\end{aligned}
$$

for any a, $b \in (0,\infty)$ and $\lambda \in [0,1]$. This is equivalent to

$$
\begin{aligned}
(4.2)\qquad & 1 \leq \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \\
&\leq \exp\left[(1-\lambda)\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right| + \lambda\left|\ln\left(\frac{b}{A_\lambda(a,b)}\right)\right|\right] \\
&\leq \begin{cases}
\exp\left[\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)|\ln b - \ln a|\right], \\
\exp\left\{((1-\lambda)^p+\lambda^p)^{1/p}\left(\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right|^q + \left|\ln\left(\frac{b}{A_\lambda(a,b)}\right)\right|^q\right)^{1/q}\right\}, \\
p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\
\exp\left(\max\left\{\left|\ln\left(\frac{A_\lambda(a,b)}{a}\right)\right|, \left|\frac{b}{A_\lambda(a,b)}\right|\right\}\right),
\end{cases}
\end{aligned}
$$

for any a, $b \in (0,\infty)$ and $\lambda \in [0,1]$.

Let $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, then by writing the inequality (3.11) for the function $f(t) := \exp(t)$ we get

$$
\begin{aligned}
&\left|(1-\lambda)\exp x + \lambda \exp y - \exp((1-\lambda)x + \lambda y)\right| \\
&\leq (1-\lambda)\left|\bigvee_{x}^{(1-\lambda)x+\lambda y}(\exp)\right| + \lambda\left|\bigvee_{(1-\lambda)x+\lambda y}^{y}(\exp)\right|
\end{aligned}
$$

$$
\leq \begin{cases}
\left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right)\left|\bigvee_{x}^{y}(\exp)\right|, \\
\left((1-\lambda)^p + \lambda^p\right)^{1/p}\left(\left|\bigvee_{x}^{(1-\lambda)x+\lambda y}(\exp)\right|^q + \left|\bigvee_{(1-\lambda)x+\lambda y}^{y}(\exp)\right|^q\right)^{1/q}, \\
p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\max\left\{\left|\bigvee_{x}^{(1-\lambda)x+\lambda y}(\exp)\right|, \left|\bigvee_{(1-\lambda)x+\lambda y}^{y}(\exp)\right|\right\},
\end{cases}
$$

namely

$$
\begin{aligned}
0 &\leq (1-\lambda)\exp x + \lambda \exp y - (\exp x)^{1-\lambda}(\exp y)^{\lambda} \\
&\leq (1-\lambda)\left|(\exp x)^{1-\lambda}(\exp y)^{\lambda} - \exp x\right| + \lambda\left|\exp y - (\exp x)^{1-\lambda}(\exp y)^{\lambda}\right|
\end{aligned}
$$

$$
\leq \begin{cases}
\left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right)\left|\bigvee_{x}^{y}(\exp)\right|, \\
\left((1-\lambda)^p + \lambda^p\right)^{1/p} \\
\times\left(\left|(\exp x)^{1-\lambda}(\exp y)^{\lambda} - \exp x\right|^q + \left|\exp y - (\exp x)^{1-\lambda}(\exp y)^{\lambda}\right|^q\right)^{1/q}, \\
p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\max\left\{\left|(\exp x)^{1-\lambda}(\exp y)^{\lambda} - \exp x\right|, \left|\exp y - (\exp x)^{1-\lambda}(\exp y)^{\lambda}\right|\right\}.
\end{cases}
$$

If in this inequality we take $a = \exp x$ and $b = \exp y$, then we get

$$
\begin{aligned}
0 &\leq A_{\lambda}(a,b) - G_{\lambda}(a,b) \\
&\leq (1-\lambda)\left|G_{\lambda}(a,b) - a\right| + \lambda\left|b - G_{\lambda}(a,b)\right| \\
&\leq \begin{cases}
\left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right)|b-a|, \\
\left((1-\lambda)^p + \lambda^p\right)^{1/p}\left(\left|G_{\lambda}(a,b) - a\right|^q + \left|b - G_{\lambda}(a,b)\right|^q\right)^{1/q}, \\
p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\max\left\{\left|G_{\lambda}(a,b) - a\right|, \left|b - G_{\lambda}(a,b)\right|\right\}.
\end{cases}
\end{aligned} \tag{4.3}
$$

References

[1] M. W. Alomari, A companion of the generalized trapezoid inequality and applications. *J. Math. Appl.* **36** (2013), 5–15.

[2] M. W. Alomari and S. S. Dragomir, Various error estimations for several Newton-Cotes quadrature formulae in terms of at most first derivative and applications in numerical integration. *Jordan J. Math. Stat.* **7** (2014), no. 2, 89–108.

[3] M. W. Alomari, M. E. Özdemir and H. Kavurmac, On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications. *Miskolc Math. Notes* **13** (2012), no. 2, 233–248.

[4] P. Cerone and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[5] P. Cerone, S. S. Dragomir and C. E. M. Pearce, A generalized trapezoid inequality for functions of bounded variation. *Turkish J. Math.* **24** (2000), no. 2, 147–163.
[6] X.-L. Cheng and J. Sun, A note on the perturbed trapezoid inequality. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 29, 7 pp
[7] S. S. Dragomir, Refinements of the generalized trapezoid inequality in terms of the cumulative variation and applications. *Cubo* **17** (2015), no. 2, 31–48.
[8] S. S. Dragomir, A functional generalization of trapezoid inequality. *Vietnam J. Math.* **43** (2015), no. 4, 663–675.
[9] S. S. Dragomir, Bounds for the deviation of a function from the chord generated by its extremities. *Bull. Aust. Math. Soc.* **78** (2008), no. 2, 225–248.
[10] S. S. Dragomir, Bounds for the deviation of a function from a generalised chord generated by its extremities with applications. *J. Inequal. Spec. Funct.* **3** (2012), no. 4, 67–76.
[11] S. S. Dragomir, Inequalities for the Riemann-Stieltjes integral of under the chord functions with applications. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **53** (2014), no. 1, 45–64.
[12] S. S. Dragomir, Two parameters and two points representation of absolutely continuous functions with integral remainder, Ppreprint RGMIA Res. Rep. Coll. **19** (2016), Art. 19. [Online http://rgmia.org/papers/v19/v19a19.pdf].
[13] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 84. [Online http://rgmia.org/papers/v14/v14a84.pdf].
[14] S. S. Dragomir and A. McAndrew, On trapezoid inequality via a Grüss type result and applications. *Tamkang J. Math.* **31** (2000), no. 3, 193–201.
[15] H. Gunawan, A note on Dragomir-McAndrew's trapezoid inequalities. *Tamkang J. Math.* **33** (2002), no. 3, 241–244.
[16] A. I Kechriniotis and N. D. Assimakis, Generalizations of the trapezoid inequalities based on a new mean value theorem for the remainder in Taylor's formula. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 3, Article 90, 13 pp.
[17] Z. Liu, Some inequalities of perturbed trapezoid type. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 47, 9 pp
[18] A. McD. Mercer, On perturbed trapezoid inequalities. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 118, 7 pp
[19] P. R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral. *J. Math. Anal. Appl.* **344** (2008), no. 2, 921–926.
[20] B. G. Pachpatte, A note on a trapezoid type integral inequality. *Bull. Greek Math. Soc.* **49** (2004), 85–90.
[21] N. Ujević, Error inequalities for a generalized trapezoid rule. *Appl. Math. Lett.* **19** (2006), no. 1, 32–37.
[22] N. Ujević, On perturbed mid-point and trapezoid inequalities and applications. *Kyungpook Math. J.* **43** (2003), no. 3, 327–334.
[23] N. Ujević, Perturbed trapezoid and mid-point inequalities and applications. *Soochow J. Math.* **29** (2003), no. 3, 249–257.
[24] W. Yang, A companion for the generalized Ostrowski and the generalized trapezoid type inequalities. *Tamsui Oxf. J. Inf. Math. Sci.* **29** (2013), no. 2, 113–127.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa