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IMPROVED HERMITE-HADAMARD TYPE INEQUALITIES BY USING THE p-CONVEXITY OF DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, a new identity is established for differentiable mappings. By using the mathematical analysis techniques, some new integral inequalities of Hermite-Hadamard type for differentiable p-convex functions are proved. A comparison of the established results with previously obtained results is demonstrated to show that the results presented in this paper are better than those already exist in literature.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, the double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is very famous in the theory of convex functions and is known as the Hermite-Hadamard inequality. The inequality (1.1) is considered as a necessary and sufficient condition for a function f to be convex over an interval I.

Not long ago, the theory of convexity has received a considerable attention by a number of researchers and as a result the classical notion of convex sets and convex functions have been extended and generalized in several directions. Zhang [10], presented the concept of p-convex sets and p-convex functions defined on an interval of the set of real numbers $\mathbb{R}$, where p is a positive odd integer or a fraction with numerator and denominator as positive odd integers. The definitions of p-convex sets and p-convex functions was modified by Iscan in [7] by restricting the domain to be the interval of the set of positive real numbers so that p can be any non-zero real number. The class of p-convex functions introduced by Iscan not only contains the class of classical convex functions but also contains the class of harmonically convex functions defined over the set of positive real numbers. Moreover, a number of new Hermite-Hadamard type inequalities were proved in [7] for the class of p-convex functions. As a consequence of the extensions and generalizations of the classical convexity, Hermite-Hadamard inequality (1.1) has been given different forms and numerous bounds related to the middle and leftmost, and middle and the rightmost terms are proved by using a variety of generalizations of the usual convexity, see for instance [1, 3, 4, 5, 7, 8, 9] and the references cited therein.

In this article, we prove some integral inequalities for differentiable p-convex functions. The results of this paper generalize few known results as well as improve

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some results given in [7] and [9]. We also provide a numerical example by using the software Matlab and Mathematica to justify our claim.

In what follows we recall some basic definitions related to p-convex sets, p-convex functions and Hermite-Hadamard type inequalities for differentiable p-convex functions.

Definition 1. [10] *An interval I is said to be p-convex if*

$$M_p(x,y;\alpha)=[\alpha x^p+(1-\alpha)y^p]^{\frac{1}{p}}\in I$$

for all $x,\ y\in I$ and $\alpha\in[0,1]$, where $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2t+1$, $k,\ r,\ t\in\mathbb{N}$.

Definition 2. [10] *Let I be a p-convex set. A function $f:I\to\mathbb{R}$ is said to be p-convex function or f is said to belong to the class $PC(I)$, if*

$$f(M_p(x,y;\alpha))\leq\alpha f(x)+(1-\alpha)f(y)$$

for all $x,\ y\in I$ and $\alpha\in[0,1]$.

Remark 1. *It is clear from the Definition 2 that the p-convex functions are the convex functions in the classical sense for $p=1$. Since $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2t+1$, $k,\ r,\ t\in\mathbb{N}$, this shows that $p\neq-1$. Hence the class $PC(I)$ does not contain the harmonic convex functions.*

Remark 2. [7] *If $I\subset(0,\infty)$ and $p\in\mathbb{R}\setminus\{0\}$, then*

$$M_p(x,y;\alpha)=[\alpha x^p+(1-\alpha)y^p]^{\frac{1}{p}}\in I$$

for all $x,\ y\in I$ and $\alpha\in[0,1]$.

Based on Remark 2, the following modification of p-convex functions was given in [7] by Iscan.

Definition 3. [7] *Let $I\subset(0,\infty)$ and $p\in\mathbb{R}\setminus\{0\}$. A function $f:I\to\mathbb{R}$ is said to be p-convex function or f is said to belong to the class $PC(I)$, if*

$$f(M_p(x,y;\alpha))\leq\alpha f(x)+(1-\alpha)f(y)\tag{1.2}$$

for all $x,\ y\in I$ and $\alpha\in[0,1]$. If the inequality (1.2) is reversed, then f is said to be p-concave.

According to Definition 3, It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I\subset(0,\infty)$, respectively.

The following is the corrected version of proposition given in [7] that describes how the convexity and p-convexity are related.

Proposition 1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a function and $p\in\mathbb{R}\setminus\{0\}$, then*

(1) *If f is convex and nondecreasing, then f is p-convex for $p\in(-\infty,0)\cup(0,1]$.*
(2) *If f is p-convex and nondecreasing for $p\geq1$, then f is convex.*
(3) *If f is p-concave and nondecreasing for $p\in(-\infty,0)\cup(0,1]$, then f is concave.*
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(5) *If f is convex and nonincreasing, then f is p-convex for $p\geq1$.*
(6) *If f is p-convex and nonincreasing for $p\in(-\infty,0)\cup(0,1]$, then f is convex.*
(7) *If f is p-concave and nonincreasing for $p\geq1$, then f is concave.*

(8) *If f is concave and nonincreasing, then f is p-concave for $p \in (-\infty, 0) \cup (0, 1]$.*

Proof. (1) Suppose that f is convex and nondecreasing. For $p \in (-\infty, 0) \cup (0, 1]$, we have

$$[tx^p + (1-t)\,y^p]^{\frac{1}{p}} \leq tx + (1-t)\,y.$$

for all x, $y \in I$ and $t \in [0, 1]$. Hence by using the convexity of f, we have

$$\begin{aligned} f\left([tx^p + (1-t)\,y^p]^{\frac{1}{p}}\right) &\leq f\,(tx + (1-t)\,y) \\ &\leq tf\,(x) + (1-t) + f\,(y)\,. \end{aligned}$$

for all x, $y \in I$ and $t \in [0, 1]$. This shows that f is p-convex.

(2) Suppose that f is p-convex and nondecreasing for $p \geq 1$. For $p \geq 1$, we have

$$tx + (1-t)\,y \leq [tx^p + (1-t)\,y^p]^{\frac{1}{p}} \text{ for all } t \in [0, 1]\,.$$

for all x, $y \in I$ and $t \in [0, 1]$. Hence by using the p-convexity of f, we have

$$\begin{aligned} f\,(tx + (1-t)\,y) &\leq f\left([tx^p + (1-t)\,y^p]^{\frac{1}{p}}\right) \\ &\leq tf\,(x) + (1-t) + f\,(y)\,. \end{aligned}$$

The results (3), (4), (5), (6), (7) and (8) can be proved similarly. □

According to Proposition 1, the following p-convex and p-concave functions can be constructed.

Example 1. [7] *Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$, then f is p-convex function for $p \in (-\infty, 0) \cup (0, 1]$ and f is p-concave function for $p \geq 1$.*

Example 2. [7] *Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{-p}$, $p \geq 1$, then f is p-convex function.*

Example 3. [7] *Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$ and $p \geq 1$, then f is p-convex function.*

Example 4. [7] *Let $f : (0, \infty) \to \mathbb{R}$,$f(x) = \ln x$ and $p \geq 1$, then f is p-concave function.*

The following Hermite-Hadamard type inequalities were obtained in [7].

Theorem 1. [7] *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus \{0\}$, and a, $b \in I$ with $a < b$. If $f \in L[a, b]$, then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p} \int_a^b \frac{f\,(x)}{x^{1-p}} dx \leq \frac{f\,(a) + f\,(b)}{2}. \tag{1.3}$$

The inequalities (1.3) are sharp.

Here we recall Gamma, Beta and Hyeprgeomtric functions. These special functions are used to get estimates between the middle and the rightmost terms in (1.3).

The Gamma function is defined as

$$\Gamma\,(x) = \int_0^\infty e^{-x} t^{x-1} dt,$$

The Beta function, also known as the Euler integral of the first kind, is defined as

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\, x>0, y>0$$

and the integral form of the hypergeometric function is defined as follows

$${}_2F_1(x,y;c;z)=\frac{1}{B(y,y-c)}\int_0^1 t^{x-1}(1-t)^{c-y-1}(1-zt)^{-a}\,dt,$$

where $|z|<1$ and $c>y>0$.

The following error bounds of the difference between the middle and the rightmost terms in (1.3) were proved by using the modified definition of p-convex functions.

Theorem 2. [7] *Let Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $f'\in L[a,b]$. If $\left|f'\right|^q$ is p-convex on $[a,b]$ for $q\geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \tag{1.4}$$
$$\leq\left(\frac{b^p-a^p}{2p}\right)C_1^{1-\frac{1}{q}}\left[C_2\left|f'(a)\right|^q+C_3\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$C_1=C_1(a,b;p)=\frac{1}{4}\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}$$
$$\times\left[{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)+{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)\right],$$
$$C_2=C_2(a,b;p)=\frac{1}{24}\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}\left[{}_2F_1\left(1-\frac{1}{p},2;4;\frac{a^p-b^p}{a^p+b^p}\right)\right.$$
$$\left.+6\cdot{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)+{}_2F_1\left(1-\frac{1}{p},2;4;\frac{b^p-a^p}{a^p+b^p}\right)\right]$$

and

$$C_3=C_1-C_2.$$

Theorem 3. [7] *Let Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $f'\in L[a,b]$. If $\left|f'\right|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \tag{1.5}$$
$$\leq\frac{b^p-a^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left[C_4\left|f'(a)\right|^q+C_5\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$C_4=C_4(a,b;p;q)=\begin{cases}\frac{1}{2a^{pq-q}}\cdot{}_2F_1\left(q-\frac{q}{p},1,3;1-\left(\frac{b}{a}\right)^p\right) & p<0\\ \frac{1}{2b^{pq-q}}\cdot{}_2F_1\left(q-\frac{q}{p},2,3;1-\left(\frac{a}{b}\right)^p\right) & p>0\end{cases},$$

$$C_5 = C_4(a,b;p;q) = \begin{cases} \frac{1}{2a^{pq-q}} \cdot {}_2F_1\left(q-\frac{q}{p}, 2, 3; 1-\left(\frac{b}{a}\right)^p\right) & p<0 \\ \frac{1}{2b^{pq-q}} \cdot {}_2F_1\left(q-\frac{q}{p}, 1, 3; 1-\left(\frac{a}{b}\right)^p\right) & p>0 \end{cases}.$$

Theorem 4. [7] *Let Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° , $a, b \in I^\circ$ with $a < b$, $p \in \mathbb{R}\backslash\{0\}$ and $f' \in L[a,b]$. If $\left|f'\right|^q$ is p-convex on $[a,b]$ for $q > 1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \le \frac{b^p-a^p}{2p} C_6^{\frac{1}{r}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{1.6}$$

where

$$C_6 = C_6(a,b;p;r) = \begin{cases} \frac{1}{2a^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p}, 1, 2; 1-\left(\frac{b}{a}\right)^p\right) & p<0 \\ \frac{1}{2b^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p}, 1, 2; 1-\left(\frac{a}{b}\right)^p\right) & p>0 \end{cases}.$$

In the next section, we will prove some improved integral inequalities of Hermite-Hadamard type by using the modified definition of p-convexity. The software Mathematica is used to demonstrate that the results of this paper are better than those proved in [7].

2. Hermite-Hadamard type using p-convex functions

In order to prove our results of this paper we first prove the following auxilary result.

Lemma 1. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° (the interior of the interval I) and let a, $b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, the following equality holds*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \\ &= \left(\frac{b^p-a^p}{4p}\right)\left[\int_0^1 M_{p-1}\left(a,b;\frac{1-t}{2}\right) t f'\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)dt \right. \\ &\qquad \left. - \int_0^1 M_{p-1}\left(a,b;\frac{1+t}{2}\right) t f'\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)dt\right], \end{aligned} \tag{2.1}$$

where $p \in \mathbb{R}\backslash\{0\}$,

$$M_{p-1}\left(a,b;\frac{1-t}{2}\right) = \left[\left(\frac{1-t}{2}\right)a^p + \left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}-1}$$

and

$$M_{p-1}\left(a,b;\frac{1+t}{2}\right) = \left[\left(\frac{1+t}{2}\right)a^p + \left(\frac{1-t}{2}\right)b^p\right]^{\frac{1}{p}-1}.$$

Proof. By integration by parts, we observe that

$$\begin{aligned}(2.2)\quad &\left(\frac{b^p-a^p}{4p}\right)\int_0^1 M_{p-1}\left(a,b;\frac{1-t}{2}\right)tf'\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)dt\\ &=\frac{1}{2}\int_0^1 td\left[f\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right]\\ &=\frac{1}{2}\,tf\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\Big|_0^1-\frac{1}{2}\int_0^1 f\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)dt\\ &=\frac{f(b)}{2}-\frac{p}{b^p-a^p}\int_{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}}^{b}\frac{f(x)}{x^{1-p}}dx\end{aligned}$$

and

$$\begin{aligned}(2.3)\quad &\left(\frac{b^p-a^p}{4p}\right)\int_0^1 M_{p-1}\left(a,b;\frac{1+t}{2}\right)tf'\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)dt\\ &=-\frac{1}{2}\int_0^1 td\left[f\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right]\\ &=-\frac{1}{2}\,tf\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\Big|_0^1+\frac{1}{2}\int_0^1 f\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)dt\\ &=-\frac{f(a)}{2}+\frac{p}{b^p-a^p}\int_a^{\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}}\frac{f(x)}{x^{1-p}}dx.\end{aligned}$$

From the equalities (2.2) and (2.3), we get the equality (2.1). □

Remark 3. *If $p=1$, Lemma 1 becomes Lemma 2.1 proved in* [1, page 226] *with restricted domain $(0,\infty)$ of the function f.*

If $p=-1$, Lemma 1 gives the following new identity for harmonaically-convex functions.

Lemma 2. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° (the interior of the interval I) and let $a, b\in I^\circ$ with $a<b$. If $f'\in L[a,b]$, the following equality holds*

$$\begin{aligned}(2.4)\quad &\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\\ &=4ab(b-a)\left[\int_0^1\frac{t}{[(1-t)b+(1+t)a]^2}f'\left(\frac{2ab}{(1-t)b+(1+t)a}\right)dt\right.\\ &\qquad\left.-\int_0^1\frac{t}{[(1+t)b+(1-t)a]^2}f'\left(\frac{2ab}{(1+t)b+(1-t)a}\right)dt\right].\end{aligned}$$

We can now begin to establish the main results of our paper by using Lemma 1.

Theorem 5. *Let $f:[c,d]\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on (c,d) and $a, b\in(c,d)$ with $a<b$. If $f'\in L[a,b]$ and $\left|f'\right|^q$ is p-convex function for $q\geq1$ and*

$p\in\mathbb{R}\backslash\left\{0,-1,\frac{1}{2}\right\}$, *then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \tag{2.5}$$
$$\leq\left(\frac{b^p-a^p}{4p}\right)\left[(\vartheta(a,b;p))^{1-\frac{1}{q}}\left\{\eta_1(a,b;p)\left|f^{'}(a)\right|^q+\eta_2(a,b;p)\left|f^{'}(b)\right|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+(\vartheta(b,a;p))^{1-\frac{1}{q}}\left\{\eta_2(b,a;p)\left|f^{'}(a)\right|^q+\eta_1(b,a;p)\left|f^{'}(b)\right|^q\right\}^{\frac{1}{q}}\right],$$

where

$$\eta_1(a,b;p)=\frac{p^2\left[2b^{p+1}\left((2p-1)b^p+(2p+1)a^p\right)+2^{-\frac{1}{p}}(a^p+b^p)^{1+\frac{1}{p}}\left(a^p-b^p(4p+1)\right)\right]}{(a^p-b^p)^3(p+1)(2p+1)},$$

$$\eta_2(a,b;p)=\frac{p}{(a^p-b^p)^3(p+1)(2p+1)}\left[2^{-\frac{1}{p}}(a^p+b^p)^{1+\frac{1}{p}}p\left(-b^p+a^p(4p+1)\right)\right.$$
$$\left.-2b\left(b^{2p}+a^pb^p\left(2p^2-3p-2\right)+\left(2p^2+3p+1\right)a^{2p}\right)\right]$$

and

$$\vartheta(a,b;p)=\frac{2p\left[2^{-\frac{1}{p}}\cdot p(a^p+b^p)^{1+\frac{1}{p}}-b\left((p-1)b^p+(p+1)a^p\right)\right]}{(a^p-b^p)^2(p+1)}.$$

Proof. From Lemma 1 and using the power-mean inequality, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \tag{2.6}$$
$$\leq\left(\frac{b^p-a^p}{4p}\right)\left[\left(\int_0^1 tM_{p-1}\left(a,b;\frac{1-t}{2}\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 tM_{p-1}\left(a,b;\frac{1-t}{2}\right)\right.\right.$$
$$\left.\times\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 tM_{p-1}\left(a,b;\frac{1+t}{2}\right)dt\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_0^1 tM_{p-1}\left(a,b;\frac{1+t}{2}\right)\left|f^{'}\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By using the p-convexity of $\left|f^{'}\right|^q$ for $q\geq 1$, we have

$$\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q=\left|f^{'}\left(\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}}\right)\right|^q$$
$$\leq\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|^q.$$

and

$$\left|f^{'}\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q=\left|f^{'}\left(\left[\left(\frac{1+t}{2}\right)a^p+\left(\frac{1-t}{2}\right)b^p\right]^{\frac{1}{p}}\right)\right|^q$$
$$\leq\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|^q+\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|^q.$$

Hence

$$(2.7)\quad \int_0^1 M_{p-1}\left(a,b;\frac{1-t}{2}\right)t\left|f'\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt$$

$$\leq \int_0^1 \left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}-1} t\left[\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right]dt$$

$$=\left|f'(a)\right|^q\int_0^1\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}-1} t\left(\frac{1-t}{2}\right)dt$$

$$+\left|f'(b)\right|^q\int_0^1\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}-1} t\left(\frac{1+t}{2}\right)dt$$

$$=\left|f'(a)\right|^q\frac{p^2\left[2b^{p+1}\left(\left(2p-1\right)b^p+\left(2p+1\right)a^p\right)+2^{-\frac{1}{p}}\left(a^p+b^p\right)^{1+\frac{1}{p}}\left(a^p-b^p\left(4p+1\right)\right)\right]}{\left(a^p-b^p\right)^3\left(p+1\right)\left(2p+1\right)}$$

$$+\frac{\left|f'(b)\right|^q}{\left(a^p-b^p\right)^3\left(p+1\right)\left(2p+1\right)}p\Big[2^{-\frac{1}{p}}\left(a^p+b^p\right)^{1+\frac{1}{p}}p\left(a^p\left(4p+1\right)-b^p\right)$$

$$-2b\left(b^{2p}+a^pb^p\left(2p^2-3p-2\right)+\left(2p^2+3p+1\right)a^{2p}\right)\Big].$$

Similarly, one can have

$$(2.8)\quad \int_0^1 M_{p-1}\left(a,b;\frac{1+t}{2}\right)t\left|f'\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q dt$$

$$=\frac{\left|f'(a)\right|^q}{\left(a^p-b^p\right)^3\left(p+1\right)\left(2p+1\right)}p\Big[2^{-\frac{1}{p}}\left(a^p+b^p\right)^{1+\frac{1}{p}}p\left(a^p-b^p\left(4p+1\right)\right)$$

$$+2a\left(a^{2p}+a^pb^p\left(2p^2-3p-2\right)+\left(2p^2+3p+1\right)b^{2p}\right)\Big]$$

$$+\left|f'(b)\right|^q\frac{p^2\left[-2a^{p+1}\left(\left(2p-1\right)a^p+\left(2p+1\right)b^p\right)+2^{-\frac{1}{p}}\left(a^p+b^p\right)^{1+\frac{1}{p}}\left(a^p\left(4p+1\right)-b^p\right)\right]}{\left(a^p-b^p\right)^3\left(p+1\right)\left(2p+1\right)}.$$

We also observe that

$$(2.9)\quad \int_0^1 tM_{p-1}\left(a,b;\frac{1-t}{2}\right)dt=\int_0^1 t\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{1}{p}-1}dt$$

$$=\frac{2p\left[2^{-\frac{1}{p}}\cdot p\left(a^p+b^p\right)^{1+\frac{1}{p}}-b\left(\left(p-1\right)b^p+\left(p+1\right)a^p\right)\right]}{\left(a^p-b^p\right)^2\left(p+1\right)}$$

and

$$(2.10)\quad \int_0^1 tM_{p-1}\left(a,b;\frac{1+t}{2}\right)dt=\int_0^1 t\left[\left(\frac{1+t}{2}\right)a^p+\left(\frac{1-t}{2}\right)b^p\right]^{\frac{1}{p}-1}dt$$

$$=\frac{2p\left[2^{-\frac{1}{p}}\cdot p\left(a^p+b^p\right)^{1+\frac{1}{p}}-a\left(\left(p-1\right)a^p+\left(p+1\right)b^p\right)\right]}{\left(a^p-b^p\right)^2\left(p+1\right)}.$$

The result follows by applying the inequalities (2.7)-(2.10) in (2.6). $\square$

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied and if* $p=1$*, the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times\left[\left(\frac{\left|f'(a)\right|^q+5\left|f'(b)\right|^q}{12}\right)^{\frac{1}{q}}+\left(\frac{5\left|f'(a)\right|^q+\left|f'(b)\right|^q}{12}\right)^{\frac{1}{q}}\right]. \tag{2.11}$$

Theorem 6. *Let* $f:[c,d]\subset\mathbb{R}\rightarrow\mathbb{R}$ *be a differentiable function on* (c,d) *and* a, $b\in(c,d)$ *with* $a<b$. *If* $f'\in L[a,b]$ *and* $\left|f'\right|^q$ *is p-convex function for* $q>1$ *and* $p\in\mathbb{R}\backslash\{0,q\}$, *the following inequality holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \\ &\leq\left(\frac{b^p-a^p}{4p}\right)\left[(\theta(a,b;p,q))^{1-\frac{1}{q}}\left(\frac{\left|f'(a)\right|^q+(2q+3)\left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+(\theta(b,a;p,q))^{1-\frac{1}{q}}\left(\frac{(2q+3)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}}\right],\end{aligned}\tag{2.12}$$

where

$$\theta(a,b;p,q)=\frac{p(q-1)\left(2b^{\frac{p-q}{1-q}}-2^{\frac{q(1-p)}{p(1-q)}}(a^p+b^p)^{\frac{p-q}{p(1-q)}}\right)}{(p-q)(a^p-b^p)}.$$

Proof. Using Lemma 1 and by using the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\left(\frac{b^p-a^p}{4p}\right) \\ &\times\left[\left(\int_0^1\left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^q\left|f'\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_0^1\left(M_{p-1}\left(a,b;\frac{1+t}{2}\right)\right)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^q\left|f'\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.13}$$

The integrals involved can be calculated by using the p-convexity of $\left|f'\right|^q$, $q>1$ as follows

$$\int_0^1 t^q\left|f'\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt\leq\frac{\left|f'(a)\right|^q+(2q+3)\left|f'(b)\right|^q}{2(q+1)(q+2)}$$

and

$$\int_0^1 t^q \left|f'\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q \le \frac{\left|f'(a)\right|^q + (2q+3)\left|f'(b)\right|^q}{2(q+1)(q+2)}.$$

Moreover, we also observe that

$$\int_0^1 \left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{\frac{q}{q-1}} dt = \frac{p(q-1)\left(2b^{\frac{p-q}{1-q}} - 2^{\frac{q(1-p)}{p(1-q)}}\left(a^p+b^p\right)^{\frac{p-q}{p(1-q)}}\right)}{(p-q)\left(a^p-b^p\right)}$$

and

$$\int_0^1 \left(M_{p-1}\left(a,b;\frac{1+t}{2}\right)\right)^{\frac{q}{q-1}} dt = \frac{p(q-1)\left(-2a^{\frac{p-q}{1-q}} + 2^{\frac{q(1-p)}{p(1-q)}}\left(a^p+b^p\right)^{\frac{p-q}{p(1-q)}}\right)}{(p-q)\left(a^p-b^p\right)}.$$

Utilizing the above observations in (2.13), we get the inequality (2.12). □

Some important results for convex functions and harmonically-convex functions defined on $(0,\infty)$ can be deduced from Theorem 6 which are summarized in the following corollaries.

Corollary 2. *Suppose that the conditions of Theorem 6 are fulfilled and $p=1$, the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \left(\frac{b-a}{4}\right)\left[\left(\frac{\left|f'(a)\right|^q + (2q+3)\left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}} + \left(\frac{(2q+3)\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}}\right]. \tag{2.14}$$

Corollary 3. *Suppose that the suppositions of Theorem 6 are met and $p=-1$, the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \left(\frac{b-a}{4ab}\right)\left[\left(\theta(a,b;-1,q)\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(a)\right|^q + (2q+3)\left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}} + \left(\theta(b,a;-1,q)\right)^{1-\frac{1}{q}}\left(\frac{(2q+3)\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}}\right], \tag{2.15}$$

where

$$\theta(a,b;-1,q) = \frac{(q-1)\left(2b^{\frac{q+1}{q-1}} - 2^{\frac{2q}{q-1}}\left(a^{-1}+b^{-1}\right)^{\frac{1+q}{1-q}}\right)}{(q+1)\left(a^{-1}-b^{-1}\right)}.$$

Theorem 7. *Let $f:[c,d]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (c,d) and a, $b\in(c,d)$ with $a<b$. If $f^{'}\in L[a,b]$ and $\left|f^{'}\right|^{q}$ is p-convex function for $q>1$, $q\neq 2$ and $p\in\mathbb{R}\backslash\left\{0,\frac{q}{q-1},\frac{q}{q-2}\right\}$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\left(\frac{b^p-a^p}{4p}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \tag{2.16}$$
$$\times\left[\left(\lambda_1(a,b;p,q)\left|f^{'}(a)\right|^{q}+\lambda_2(a,b;p,q)\left|f^{'}(b)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\lambda_1(b,a;p,q)\left|f^{'}(a)\right|^{q}+\lambda_2(b,a;p,q)\left|f^{'}(b)\right|^{q}\right)^{\frac{1}{q}}\right],$$

where

$$\lambda_1(a,b;p,q)=\frac{4p^2b^{2p-pq+q}-p\cdot 2^{\frac{pq-q}{p}}\left(a^p+b^p\right)^{1-q+\frac{q}{p}}\left[(3p-pq+q)b^p-(p-pq+q)a^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}$$

and

$$\lambda_1(b,a;p,q)=\frac{4pb^{p-pq+q}\left[(p-pq+q)b^p-(2p-pq+q)a^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}$$
$$+\frac{p\cdot 2^{\frac{pq-q}{p}}\left(a^p+b^p\right)^{1-q+\frac{q}{p}}\left[(3p-pq+q)a^p-(p-pq+q)b^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}.$$

Proof. Taking absolute value on both sides of the equality in Lemma 1 and by using the Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq\left(\frac{b^p-a^p}{4p}\right)\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}} \tag{2.17}$$
$$\times\left[\left(\int_0^1\left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{q}\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left(M_{p-1}\left(a,b;\frac{1+t}{2}\right)\right)^{q}\left|f^{'}\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f^{'}\right|^{q}$, $q>1$ is p-convex, we have

$$\int_0^1\left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{q}\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^{q}dt$$
$$\leq\left|f^{'}(a)\right|^{q}\int_0^1\left(\frac{1-t}{2}\right)\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{q}{p}-q}dt$$
$$+\left|f^{'}(b)\right|^{q}\int_0^1\left(\frac{1+t}{2}\right)\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{q}{p}-q}dt.$$

By making the suitable substitution, we have

$$\begin{aligned}(2.18)\quad &\int_0^1 \left(\frac{1-t}{2}\right)\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{q}{p}-q}dt\\ &=\frac{4p^2b^{2p-pq+q}-p\cdot 2^{\frac{pq-q}{p}}\left(a^p+b^p\right)^{1-q+\frac{q}{p}}\left[(3p-pq+q)\,b^p-(p-pq+q)\,a^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}\end{aligned}$$

and

$$\begin{aligned}(2.19)\quad &\int_0^1 \left(\frac{1+t}{2}\right)\left[\left(\frac{1-t}{2}\right)a^p+\left(\frac{1+t}{2}\right)b^p\right]^{\frac{q}{p}-q}dt\\ &=\frac{4pb^{p-pq+q}\left[(p-pq+q)\,b^p-(2p-pq+q)\,a^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}\\ &\quad+\frac{p\cdot 2^{\frac{pq-q}{p}}\left(a^p+b^p\right)^{1-q+\frac{q}{p}}\left[(3p-pq+q)\,a^p-(p-pq+q)\,b^p\right]}{2\left(a^p-b^p\right)^2(2p-pq+q)(p-pq+q)}.\end{aligned}$$

Using (2.18) and (2.19) in (2.17), we get the desired result. □

The following corollaries are the direct consequences of Theorem 7.

Corollary 4. *According to the conditions mentioned in Theorem 7, the following inequality holds for* $p=1$*:*

$$\begin{aligned}(2.20)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 5. *Let the assumptions of Theorem 7 be satisfied and* $p=-1$*, the following inequality holds*

$$\begin{aligned}(2.21)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\le\left(\frac{b-a}{4ab}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\lambda_1(a,b;-1,q)\left|f'(a)\right|^q+\lambda_2(a,b;-1,q)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\lambda_1(b,a;-1,q)\left|f'(a)\right|^q+\lambda_2(b,a;-1,q)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\lambda_1(a,b;-1,q)=\frac{b^{2q-2}+2^{2q-2}\left(a^{-1}+b^{-1}\right)^{1-2q}\left[(2q-3)\,b^{-1}-(2q-1)\,a^{-1}\right]}{\left(a^{-1}-b^{-1}\right)^2(q-1)(2q-1)}$$

and

$$\lambda_1(b,a;p,q) = \frac{b^{2q-1}\left[(2q-2)a^{-1}-(2q-1)b^{-1}\right]}{\left(a^{-1}-b^{-1}\right)^2(q-1)(2q-1)} + \frac{2^{2q-2}\left(a^{-1}+b^{-1}\right)^{1-2q}\left[(2q-1)b^{-1}-(2q-3)a^{-1}\right]}{\left(a^{-1}-b^{-1}\right)^2(q-1)(2q-1)}.$$

Theorem 8. *Let $f:[c,d]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (c,d) and a, $b\in(c,d)$ with $a<b$. If $f^{'}\in L[a,b]$ and $\left|f^{'}\right|^q$ is p-convex function for $q>1$ and $p\in\mathbb{R}\backslash\{0\}$, the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \leq \left(\frac{b^p-a^p}{4p}\right)\left[(\mu(a,b;p,q))^{1-\frac{1}{q}}\left(\frac{\left|f^{'}(a)\right|^q+3\left|f^{'}(b)\right|^q}{4}\right)^{\frac{1}{q}} + (\mu(b,a;p,q))^{1-\frac{1}{q}}\left(\frac{3\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right], \tag{2.22}$$

where

$$\mu(a,b;p,q) = \left(\frac{a^p+b^p}{2}\right)^{\frac{q(1-p)}{p(q-1)}}\left(\frac{q-1}{2q-1}\right)\ {}_2F_1\left(2+\frac{1}{q-1},\frac{q(p-1)}{p(q-1)};3+\frac{1}{q-1};1-\frac{2a^p}{a^p+b^p}\right).$$

Proof. Taking absolute value on both sides of the equality in Lemma 1 and by using the Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right| \leq \left(\frac{b^p-a^p}{4p}\right) \tag{2.23}$$
$$\times\left[\left(\int_0^1 t^{\frac{q}{q-1}}\left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{\frac{q}{q-1}}\left(M_{p-1}\left(a,b;\frac{1+t}{2}\right)\right)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f^{'}\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $\left|f^{'}\right|^q$, $q>1$ is p-convex, we have

$$\int_0^1\left|f^{'}\left(M_p\left(a,b;\frac{1-t}{2}\right)\right)\right|^q dt \leq \frac{1}{4}\left|f^{'}(a)\right|^q+\frac{3}{4}\left|f^{'}(b)\right|^q$$

and

$$\int_0^1\left|f^{'}\left(M_p\left(a,b;\frac{1+t}{2}\right)\right)\right|^q dt \leq \frac{3}{4}\left|f^{'}(a)\right|^q+\frac{1}{4}\left|f^{'}(b)\right|^q.$$

Moreover

$$\int_0^1 t^{\frac{q}{q-1}} \left(M_{p-1}\left(a,b;\frac{1-t}{2}\right)\right)^{\frac{q}{q-1}} dt \tag{2.24}$$
$$= \int_0^1 t^{\frac{q}{q-1}} \left[\left(\frac{1-t}{2}\right) a^p + \left(\frac{1+t}{2}\right) b^p \right]^{\frac{q(1-p)}{p(q-1)}} dt$$
$$= \left(\frac{a^p+b^p}{2}\right)^{\frac{q(1-p)}{p(q-1)}} \left(\frac{q-1}{2q-1}\right) {}_2F_1\left(2+\frac{1}{q-1}, \frac{q(p-1)}{p(q-1)}; 3+\frac{1}{q-1}; -1+\frac{2a^p}{a^p+b^p}\right)$$
$$= \mu(a,b;p,q)$$

and likewise, we also have

$$\int_0^1 t^{\frac{q}{q-1}} \left(M_{p-1}\left(a,b;\frac{1+t}{2}\right)\right)^{\frac{q}{q-1}} dt = \mu(b,a;p,q). \tag{2.25}$$

Combining (2.23), (2.24) and (2.25) give the required result. □

3. Comparison of the Results

In this section we compare our results with results proved in [7].

Let

$$E_1(a,b;p,q) = \left(\frac{b^p-a^p}{2p}\right) C_1^{1-\frac{1}{q}} \left[C_2 \left|f'(a)\right|^q + C_3 \left|f'(b)\right|^q \right]^{\frac{1}{q}}$$

and

$$E_2(a,b;p,q) = \left(\frac{b^p-a^p}{4p}\right) \left[(\vartheta(a,b;p))^{1-\frac{1}{q}} \left\{ \eta_1(a,b;p) \left|f'(a)\right|^q + \eta_2(a,b;p) \left|f'(b)\right|^q \right\}^{\frac{1}{q}} \right.$$
$$\left. + (\vartheta(b,a;p))^{1-\frac{1}{q}} \left\{ \eta_2(b,a;p) \left|f'(a)\right|^q + \eta_1(b,a;p) \left|f'(b)\right|^q \right\}^{\frac{1}{q}} \right],$$

where C_1, C_2 and C_3 are defined in Theorem 2, and $\vartheta(a,b;p)$, $\eta_1(a,b;p)$ and $\eta_2(a,b;p)$ are defined in Theorem 5.

Let the function $f:(0,\infty)\to\mathbb{R}$ be defined as $f(x) = \frac{q}{q-p} x^{-\frac{p}{q}+1}$, $q \geq 1$. Then $\left|f'(x)\right|^q = x^{-p}$ is p-convex function for $p \geq 1$. By using the software Mathematica the following table is obtained:

Table 1

	$E_1(a,b;p,q)$	$E_2(a,b;p,q)$
$a=1, b=2, p=2, q=2$	0.208936	0.58386
$a=2, b=5, p=3, q=4$	0.450634	131.391
$a=2, b=5, p=15, q=15$	0.640252	6.57839×10^{18}
$a=10, b=15, p=10, q=50$	1.03967	4.57516×10^{20}

It is obvious from Table 1 that $E_1(a,b;p,q)$ gives better results than $E_2(a,b;p,q)$. Hence the result of Theorem 5 actually gives the improved bound.

Now we compare the results of Theorem 6 and Theorem 4. Let us denote the error bounds in Theorem 6 and Theorem 4 by $E_3(a,b;p,q)$ and $E_4(a,b;p,q)$ respectively.

That is

$$E_3(a,b;p,q) = \left(\frac{b^p-a^p}{4p}\right)\left[(\theta(a,b;p,q))^{1-\frac{1}{q}}\left(\frac{\left|f'(a)\right|^q+(2q+3)\left|f'(b)\right|^q}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\theta(b,a;p,q))^{1-\frac{1}{q}}\left(\frac{(2q+3)\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2(q+1)(q+2)}\right)^{\frac{1}{q}}\right]$$

and

$$E_4(a,b;p,q) = \frac{b^p-a^p}{2p}C_6^{\frac{1}{r}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

where $\theta(a,b;p,q)$ id defined in Theorem 6 and C_6 is given in Theorem 4. By using the software Mathematica, we obtain the following table

Table 2

	$E_3(a,b;p,q)$	$E_4(a,b;p,q)$
$a=1, b=2, p=3, q=\frac{3}{2}, r=3$	0.21354	0.215794
$a=1, b=10, p=10, q=\frac{5}{4}, r=5$	1.02222×10^{6}	1.16839×10^{10}
$a=1, b=10, p=20, q=\frac{5}{4}, r=5$	5.04108×10^{13}	5.76187×10^{13}
$a=10, b=15, p=20, q=\frac{5}{4}, r=5$	3.31374×10^{-15}	3.78463×10^{-15}

From Table 2, it reveals that the result of Theorem 6 gives better error bound as compared to the result of Theorem 4.

A similar comparison can be made with the other results of this paper with those of given in [7].

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